モース理論入門　第4回

(4-1)

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Part II A rapid course in Riemannian geometry

§8

M : smooth manifold of dimension n

$\mathfrak{X}(M)$: the set of smooth vector fields over M

Definition: <u>affine connection</u> on M (アフィン接続)

(1) For $^\forall X, Y \in \mathfrak{X}(M)$, $\nabla_X Y \in \mathfrak{X}(M)$ is given
(<u>covariant derivative</u>)

(2) $\nabla_X Y$ is linear on X and Y

(3) $\nabla_{fX} Y = f \nabla_X Y$ for $^\forall f : M \to \mathbb{R}$ smooth

(4) $\nabla_X (fY) = (Xf)Y + f(\nabla_X Y)$

Remark $(\nabla_X Y)_q$ is determined only X_q and Y

$(U, x^1, \cdots, x^n)$ local coordinates of M

$\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$ local frame of TM on U

(5) $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x^k}$　$\Gamma_{ij}^k : U \to \mathbb{R}$ smooth
(クリストッフェル記号, 接続係数)

◎ covariant derivative along curve

$c : (a, b) \to M$ smooth　$c(t) \in M$

$V : (a, b) \to TM$　$V_t = V(t) \in T_{c(t)}M$ vector field along c

$t \mapsto V(t)(f) \in \mathbb{R}$ C^∞ for $^\forall f : M \to \mathbb{R}$ C^∞

$\dot{c} = \frac{dc}{dt} := c_*\left(\frac{d}{dt}\right) : t \mapsto \frac{dc}{dt}(t) \in T_{c(t)}M$ velocity vector field

Lemma 8.1.

V : vector field along curve c

($V(t) = \sum_{j=1}^{n} v^j(t) \frac{\partial}{\partial x^j}\Big|_{c(t)}$ $v^j(t)$ smooth)

$\nabla_{\dot c(t)} V = \frac{DV}{dt}$: vector field along c is uniquely determined by the conditions:

a) $\nabla_{\dot c(t)}(V+W) = \nabla_{\dot c(t)}(V) + \nabla_{\dot c(t)}(W)$

b) $\nabla_{\dot c(t)}(fV) = \frac{df}{dt} V + f \nabla_{\dot c(t)}(V)$ ($f = f(t)$ funct.)

c) $V(t) = Y(c(t))$, $Y \in \mathfrak{X}(M) \Rightarrow \nabla_{\dot c(t_0)}(V) = \nabla_{\dot c(t_0)}(Y)$

for $\forall$ fixed $t_0 \in (a,b)$

(!) In fact, if $V(t) = \sum_{j=1}^{n} v^j(t) \frac{\partial}{\partial x^j}\Big|_{c(t)}$, $\dot c(t) = \sum_{i=1}^{n} \frac{dx^i}{dt} \frac{\partial}{\partial x^i}$

then

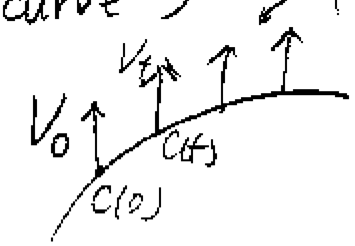
$$(\nabla_{\dot c(t)} V)(t) = \sum_{k=1}^{n} \left(\frac{dv^k}{dt} + \sum_{i,j} \frac{dx^i}{dt} \Gamma_{ij}^{k} v^j \right) \frac{\partial}{\partial x^k}\Big|_{c(t)}$$

$$\left(\nabla_{\dot c(t)} \frac{\partial}{\partial x^j} = \sum_i \frac{dx^i}{dt} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right)$$

Definition (parallel vector field along curve) "平行移動"

V is parallel (平行) along c

$\overset{\text{def}}{\Longleftrightarrow}$ $\nabla_{\dot c(t)}(V) = 0$



Lemma 8.2. $\forall V_0 \in T_{c(0)}M$ $\exists^1 V$ parallel vect. field along c (一意的に存在)

connection $\boxed{\begin{array}{ccc} \text{接続} & = & \text{平行移動} \\ \text{(を与える)} & & \text{(の仕方を与える)} \end{array}}$ parallel translation

(M, g) Riemannian manifold,

$\left(\begin{array}{l} p \in M \quad g_p : T_pM \times T_pM \to \mathbb{R} \text{ (pos. def) inner prod.} \\ u, v \in T_pM \;;\; g_p(u, v) = \langle u, v \rangle \in \mathbb{R} \\ \text{local} \quad g_x\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij}(x) \quad \text{smooth} \end{array} \right)$

Definition : connection ∇ on M is <u>compatible</u> with g

$\overset{\text{def}}{\Longleftrightarrow}$ P, P' parallel vector fields along a curve C

$(\nabla_{\dot{c}(t)} P = 0, \ \nabla_{\dot{c}(t)} P' = 0)$

$\Rightarrow$ $\langle P, P' \rangle$ is constant

(平行移動でベクトルの長さが変わらない)

Lemma 8.3. ∇ compatible, V, W vector fields along C

$$\Rightarrow \quad \frac{d}{dt} \underbrace{\langle V, W \rangle}_{\uparrow t\text{の関数}} = \langle \underbrace{\nabla_{\dot{c}(t)} V}_{\frac{DV}{dt}}, W \rangle + \langle V, \underbrace{\nabla_{\dot{c}(t)} W}_{\frac{DW}{dt}} \rangle$$

Cor. 8.4. ∇ compatible, V, W vector fields along C

$X, Y, Y' \in \mathfrak{X}(M)$

$$\Rightarrow \quad X\langle Y, Y' \rangle = \langle \nabla_X Y, Y' \rangle + \langle Y, \nabla_X Y' \rangle$$

Theorem 8.5

$$\nabla_X Y - \nabla_Y X = [X, Y] \iff \Gamma_{ij}^k = \Gamma_{ji}^k \quad \left(\begin{array}{l}\text{locally} \\ \forall i, j, k\end{array}\right)$$

Lemma 8.6. (Fundamental lemma of Riemannian geometry)
(M, g) : Riemannian manifold.
$\exists 1$ symmetric connection ∇ (Levi-Civita connection) compatible with g Riemannian connection とも言う.

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smooth map $S : U \to M$: parametrized surface ($U \subset \mathbb{R}^2$ (x,y) plane)
V : vector field along S
$\overset{\text{def}}{\Leftrightarrow}$ $V : U \to TM$ smooth, $V_{(x,y)} \in T_{S(x,y)}M$

$S_*\frac{\partial}{\partial x} : (x,y) \mapsto S_*\frac{\partial}{\partial x}\Big|_{(x,y)}$, $S_*\frac{\partial}{\partial y} : (x,y) \mapsto S_*\frac{\partial}{\partial y}\Big|_{(x,y)}$ vector fields along S

$S_*\frac{\partial}{\partial x} = \frac{\partial S}{\partial x}$, $S_*\frac{\partial}{\partial y} = \frac{\partial S}{\partial y}$

$\frac{DV}{\partial x} = \nabla_{\frac{\partial S}{\partial x}} V$: vector field along S is defined.

Lemma 8.7 ∇ is symmetric

$$\Rightarrow \frac{\partial}{\partial x}\frac{\partial S}{\partial y} = \frac{\partial}{\partial y}\frac{\partial S}{\partial x}$$

$$\left(\nabla_{\frac{\partial S}{\partial x}} \frac{\partial S}{\partial y} = \nabla_{\frac{\partial S}{\partial y}} \frac{\partial S}{\partial x} \right)$$

Remark If (M, g) is Euclidean space, then $\forall \Gamma_{ij}^k = 0$

$$\frac{\partial}{\partial x}\frac{\partial S}{\partial y} = \frac{\partial^2 S}{\partial x \partial y} \qquad \frac{\partial}{\partial y}\frac{\partial S}{\partial x} = \frac{\partial^2 S}{\partial y \partial x}$$

$\lrcorner$

§9 curvature tensor (曲率テンソル)

Definition ∇: connection on a manifold

$X, Y, Z \in \mathfrak{X}(M)$ (vector fields over M)

$$R(X,Y)Z := -\nabla_X(\nabla_Y Z) + \nabla_Y(\nabla_X Z) + \nabla_{[X,Y]}Z$$
$$= -\{\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z\}$$
$$\in \mathfrak{X}(M)$$

Lemma 9.1 $p \in M$

The value $(R(X,Y)Z)_p$ depends only on X_p, Y_p, Z_p.

$$T_pM \times T_pM \times T_pM \longrightarrow T_pM \quad \forall\text{extension}$$
$$(X_p, Y_p, Z_p) \longmapsto (R(X,Y)Z)_p$$

is trilinear map (3重線形)

Definition R is called the curvature tensor of (M, ∇).

The followings hold:

$R(fX,Y)Z = fR(X,Y)Z$, $R(X,gY)Z = gR(X,Y)Z$

$R(X,Y)(hZ) = hR(X,Y)Z$.

If $X = \sum_i X^i \frac{\partial}{\partial x^i}$, $Y = \sum_j Y^j \frac{\partial}{\partial x^j}$, $Z = \sum_k Z^k \frac{\partial}{\partial x^k}$,

then $R(X,Y)Z = \sum_{i,j,k} X^i Y^j Z^k R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}$

Remark If (M, g) is Euclidean space, ∇: Levi-Civita

$\Rightarrow$ $R(X,Y)Z = 0$ for $\forall X, Y, Z$.

Lemma 9.2 (Variational formula)
$s: U(\subset \mathbb{R}^2) \to M$ parametrized surface
V : vector field along s

$$\Rightarrow \quad \frac{D}{\partial y}\left(\frac{D}{\partial x} V\right) - \frac{D}{\partial x}\left(\frac{D}{\partial y} V\right) = R\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right) V$$

テンソルだから意味あり

$$\nabla_{\frac{\partial s}{\partial y}}\left(\nabla_{\frac{\partial s}{\partial x}} V\right) - \nabla_{\frac{\partial s}{\partial x}}\left(\nabla_{\frac{\partial s}{\partial y}} V\right) = R\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right) V$$

(ノート 4-4 参照) 後で使います

Lemma 9.3 (M, g) Riemannian manifold,
∇ : Levi-Civita connection (compatible with the metric g)
R : curvature tensor of ∇. Then we have

(1) $R(X,Y)Z + R(Y,X)Z = 0$

(2) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

(3) $\langle R(X,Y)Z, W\rangle + \langle R(X,Y)W, Z\rangle = 0$

(4) $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$

for $\forall X, Y, Z, W \in \mathfrak{X}(M)$

後で使います