モース理論入門　第3回　3-1

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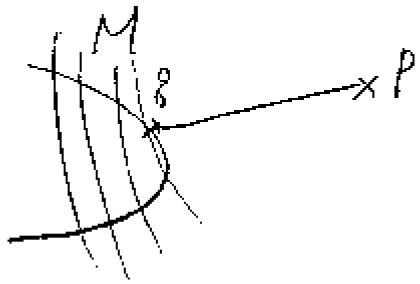
§6（モース関数の存在）

$M^k \subset \mathbb{R}^n$ submanifold $(k<n)$ $p \in \mathbb{R}^n$

$$L_p : M \to \mathbb{R}, \quad q \mapsto \|p-q\|^2$$ distance squared function

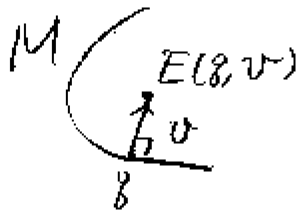


Claim: For almost all $p \in \mathbb{R}^n$, $L_p : M \to \mathbb{R}$ has only non-degenerate critical points (Morse function).

$N := \{(q, v) \in M \times \mathbb{R}^n \mid v \perp T_qM \text{ in } T_q\mathbb{R}^n\}$ normal bundle

N : n-dim. submanifold in $\mathbb{R}^n \times \mathbb{R}^n$.

$E : N \to \mathbb{R}^n \qquad E(q, v) = q + v$



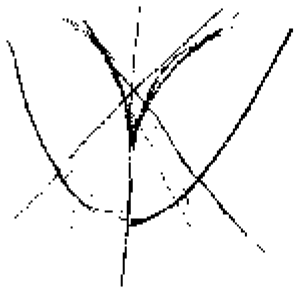
$e \in \mathbb{R}^n$ forcal point (焦点) of M

$\overset{\text{def}}{\Longleftrightarrow}$ e critical value of $E : N \to \mathbb{R}^n$

$(q, v) \in C(E)$, $e = E(q, v)$

$\mu := \dim \operatorname{Ker}(E_* : T_{(q,v)}N \to T_e\mathbb{R}^n)$

e is called a focal point with multiplicity(重複度) μ of (M, q)



Theorem 6.1 (Sard)

$f : M_1 \to M_2$ C^∞-map

$\Rightarrow$ $f(C(f)) \subset M_2$ the set of critical values is of measure 0.

Sard's theorem (1942)

N^n, M^m C^r-manifold, $f : N \to M$ C^r map

$r > \max\{n-m, 0\}$

$\Rightarrow$ $f(C(f))$ is of measure 0 in M

$x(u^1, \cdots, u^k) = (x_1(u^1, \cdots, u^k), \cdots, x_n(u^1, \cdots, u^k))$ local parametrization of M

$(g_{ij}) := \left(\left\langle \frac{\partial x}{\partial u^i}, \frac{\partial x}{\partial u^j} \right\rangle\right)$ 1st fundamental form

l_{ij} : normal component $\frac{\partial^2 x}{\partial u^i \partial u^j}$

<u>2nd fundamental form</u> at $q \in M$ of direction $v \in (T_q M)^\perp$ is given by $(\langle v, l_{ij} \rangle) = \left(\left\langle v, \frac{\partial^2 x}{\partial u^i \partial u^j} \right\rangle \right)$ $k \times k$ matrix

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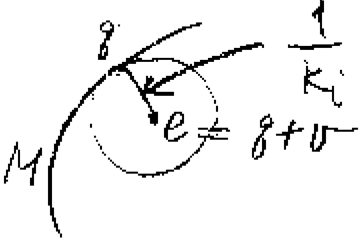
Suppose $(g_{ij}(q)) = I$ is the unit matrix.

Eigen values of 2nd fund. form $(\langle v, l_{ij} \rangle)$

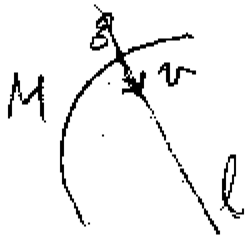
$K_1, \cdots, K_k$ are called <u>principal curvatures</u> (主曲率)

$\frac{1}{K_i}$: principal radius of curvature (主曲率半径)



Fix $q \in M$, $v \in (T_q M)^\perp$, $\|v\| = 1$ unit normal vector.

$l = \{ (q + tv) \mid t \in \mathbb{R} \}$



Lemma 6.3 The set of focal points of (M, q) on l is given by $\{ q + \frac{1}{K_i} v \mid 1 \le i \le k, \ K_i \ne 0 \}$

Proof of Lemma 6.3.

互いに直交, 長さ1 　　uに依存した基底

$\vec{w}_1(u), \cdots, \vec{w}_{n-k}(u)$; orthonormal frame of the normal bundle N of M

$$\left(u = (u^1, \cdots, u^k) \longmapsto \underbrace{\vec{x}(u) = (x_1(u), \cdots, x_n(u)) \in M \subset \mathbb{R}^n}_{\text{parametrization of } M} \right)$$

$$E : (u,t) \longmapsto \vec{x}(u) + \sum_{\alpha=1}^{n-k} t^\alpha \vec{w}_\alpha(u) =: \vec{e}(u,t)$$

Consider the Jacobian matrix of E.

$$J = \left(\frac{\partial \vec{e}}{\partial u^1} \cdots \frac{\partial \vec{e}}{\partial u^k} \ \frac{\partial \vec{e}}{\partial t^1} \cdots \frac{\partial \vec{e}}{\partial t^{n-k}} \right) \quad n \times n\text{-matrix}$$

$$\frac{\partial \vec{e}}{\partial u^i}(u,t) = \frac{\partial \vec{x}}{\partial u^i} + \sum_\alpha t^\alpha \frac{\partial \vec{w}_\alpha}{\partial u^i}, \quad \frac{\partial \vec{e}}{\partial t^\beta}(u,t) = w_\beta$$

From linear algebra :

V ; metric vector space $\langle \ , \ \rangle$ inner product (内積)

$v_1, \cdots, v_n$; basis (基底) of V, $a_1, \cdots, a_n \in V$

Lemma ＊ ; $\operatorname{rank}(a_1, \cdots, a_n) = \operatorname{rank}(\langle v_i, a_j \rangle)$

(ij 成分が $\langle v_i, a_j \rangle$ の行列の階数)

$\frac{\partial \vec{x}}{\partial u^1}, \cdots, \frac{\partial \vec{x}}{\partial u^k}, \vec{w}_1, \cdots, \vec{w}_{n-k}$; basis

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$$\operatorname{rank} J = \operatorname{rank} \begin{array}{c} \\ i \\ \beta' \end{array} \begin{array}{cc} j & \beta \\ \left(\begin{array}{c} \left\langle \frac{\partial \vec{x}}{\partial u^i}, \frac{\partial \vec{x}}{\partial u^j} \right\rangle + \sum t^\alpha \left\langle \frac{\partial \vec{w}_\alpha}{\partial u^i}, \frac{\partial \vec{x}}{\partial u^j} \right\rangle \\ \underbrace{\left\langle \vec{w}_{\beta'}, \frac{\partial \vec{x}}{\partial u^j} \right\rangle}_{=0} \end{array} \right. & \left. \begin{array}{c} \underbrace{\left\langle \frac{\partial \vec{x}}{\partial u^i}, w_\beta \right\rangle}_{0=} + \sum t^\alpha \left\langle \frac{\partial w_\alpha}{\partial u^i}, w_\beta \right\rangle \\ \langle \vec{w}_{\beta'}, \vec{w}_\beta \rangle \end{array} \right) \end{array}$$

$$= \operatorname{rank} \begin{pmatrix} g_{ij}(u) + \sum t^\alpha \left\langle \frac{\partial \vec{w}_\alpha}{\partial u^i}, \frac{\partial \vec{x}}{\partial u^j} \right\rangle & \sum t^\alpha \left\langle \frac{\partial \vec{w}_\alpha}{\partial u^i}, w_\beta \right\rangle \\ 0 & \underbrace{\delta_{\beta'\beta}}_{\text{クロネッカー}} \end{pmatrix}$$

$$\operatorname{rank} J = \operatorname{rank}\left(g_{ij}(u) + \sum_\alpha t^\alpha \left\langle \frac{\partial \vec{w}_\alpha}{\partial u^i}, \frac{\partial \vec{x}}{\partial u^j} \right\rangle\right)$$

$$0 \equiv \frac{\partial}{\partial u^i}\underbrace{\left\langle \vec{w}_\alpha, \frac{\partial \vec{x}}{\partial u^j} \right\rangle}_{0} = \left\langle \frac{\partial \vec{w}_\alpha}{\partial u^i}, \frac{\partial \vec{x}}{\partial u^j} \right\rangle + \left\langle w_\alpha, \frac{\partial^2 \vec{x}}{\partial u^i \partial u^j} \right\rangle$$

$$\therefore \left\langle \frac{\partial \vec{w}_\alpha}{\partial u^i}, \frac{\partial \vec{x}}{\partial u^j} \right\rangle = -\left\langle \vec{w}_\alpha, \frac{\partial^2 \vec{x}}{\partial u^i \partial u^j} \right\rangle = -\left\langle \vec{w}_\alpha, \vec{l}_{ij} \right\rangle$$

normal vector

$$\therefore \operatorname{rank} J = \operatorname{rank}\left(g_{ij}(u) - \sum t^\alpha \left\langle \vec{w}_\alpha, \vec{l}_{ij} \right\rangle\right)$$

$$= \operatorname{rank}\left(g_{ij}(u) - \left\langle \sum_\alpha t^\alpha \vec{w}_\alpha, \vec{l}_{ij} \right\rangle\right)$$

Given $\vec{g} = \vec{x}(u)$, unit normal vector $\vec{v}$,
Suppose the normal vector $\sum_\alpha t^\alpha \vec{w}_\alpha = t\vec{v}$, for some $t \in \mathbb{R}$.
($\vec{x}(u) + \sum_\alpha t^\alpha \vec{w}_\alpha(u)$ が l の上にある)

$$\operatorname{rank} J \text{ at } (\vec{g}, t\vec{v}) = \operatorname{rank}(g_{ij} - t\underbrace{\langle \vec{v}, \vec{l}_{ij} \rangle}_{\text{2nd fund. form}})$$

(6.4) $\vec{g} + t\vec{v}$ is a focal point of $(M, \vec{g})$ of multiplicity μ
$\iff$ nullity ($= n - \operatorname{rank} J$) of $(g_{ij} - t\langle v, \vec{l}_{ij}\rangle) = \mu$.

If $(g_{ij}) = I$ at g,
$\operatorname{rank} J < n \iff t \neq 0$ and $\frac{1}{t}$ is an eigen value (固有値) of $(\langle v, l_{ij} \rangle)$

2nd fund. form

$\frac{1}{t} = {}^\exists k_i \neq 0, \quad t = \frac{1}{k_i}$

This shows Th. 6.3.

Lemma 6.5 $M^k \subset \mathbb{R}^n$, $p \in \mathbb{R}^n$

$q \in M$ is a degenerate critical point of $L_{\vec{p}}$ distace squared funct.

$\Leftrightarrow$ p is a focal point of (M, q)

The nullity of q = The multiplicity of p.

(∵) $\vec{q} = \vec{x}(u)$,

fixed

$$\underset{f(u)}{\underset{\parallel}{L_{\vec{p}}(\vec{x}(u))}} := \|\vec{x}(u) - \vec{p}\|^2$$

$$= \langle \vec{x} - \vec{p}, \vec{x} - \vec{p} \rangle \left(= \langle \vec{x}, \vec{x} \rangle - 2\langle \vec{x}, \vec{p} \rangle + \langle \vec{p}, \vec{p} \rangle \right)$$

← ユークリッド内積

$$\frac{\partial f}{\partial u^i} = 2\left\langle \frac{\partial \vec{x}}{\partial u^i}, \vec{x} - \vec{p} \right\rangle$$

q : critical point of $L_p \Leftrightarrow \dfrac{\partial \vec{x}}{\partial u^i} \perp \vec{x} - \vec{p}$

$\Leftrightarrow \vec{x} - \vec{p} \in (T_q M)^\perp$

$$\frac{\partial^2 f}{\partial u^i \partial u^j} = 2\left\langle \frac{\partial^2 \vec{x}}{\partial u^i \partial u^j}, \underset{-t\vec{v}}{\underset{\parallel}{\vec{x} - \vec{p}}} \right\rangle + 2 \underset{g_{ij}}{\underset{\parallel}{\left\langle \frac{\partial \vec{x}}{\partial u^i}, \frac{\partial \vec{x}}{\partial u^j} \right\rangle}}$$

Set $\vec{p} - \vec{x} = t\vec{v}$, $\vec{v}$: unit normal $\in (T_q M)^\perp$

$\vec{p} = \vec{x} + t\vec{v}$

$$\frac{\partial^2 f}{\partial u^i \partial u^j} = 2\left(g_{ij}(u) - t\langle \vec{v}, \vec{l}_{ij} \rangle\right)$$

2nd fund. form

By (6.4), we have the result. //

Theorem 6.6 For allmost all $p \in \mathbb{R}^n$, $L_p : M \to \mathbb{R}$ has only non-degen. critical points

(!) L_p has only non-deg. crit. pt

$\underset{(6.5)}{\Leftrightarrow}$ p is not a focal point of M

$\underset{\text{def}}{\Leftrightarrow}$ p is a regular value of $E : \overset{\text{normal bdle of } M}{N} \to \mathbb{R}^n$

By Sard Th (6.1), the set of regular values of E is dense (complement of a measure zero set). //

Corollary 6.7 $\forall M$: smooth manifold, $\exists f : M \to \mathbb{R}$ has only non-degenerate critical point s.t. $\forall a \in \mathbb{R}$, $M^a = \{p \in M \mid f(p) \leq a\}$ is compact.

(!) By Whitney embedding theorem, M^n is embedded in $\mathbb{R}^{2n+1}$ as a closed subset. Take a regular point p of E. Then $f = L_p$ has only non-degen. crit. pts.

$M^a = M \cap \{q \in M \mid \|q - p\|^2 \leq a\}$ bounded, closed in $\mathbb{R}^{2n+1}$
compact //

Application : Any smooth manifold is homotopy equivalent to a CW complex

(!) Th. 3,5

Lemma 6.9 (Index theorem on L_p)

For a non-degenerate critical point $q \in M$ of L_p $(p \neq q)$

index of L_p at q

$= \#$(focal points of (M, q) on $\overline{pq}$) counted with multiplicity.

(!) $\left(\frac{\partial^2 L_p}{\partial u^i \partial u^j}(u)\right) = 2\left(g_{ij}(u) - t\langle v, l_{ij}(u)\rangle\right)$ (Proof of Lemma 6,5)

where $p = q + tv$, $t > 0$, $q = x(u)$

$(g_{ij}(u)) = I$ and set $A = (\langle v, l_{ij}(u)\rangle)$

Then, for $\lambda \in \mathbb{R}$,

$\dim \operatorname{Ker}(\lambda I - 2(I - tA)) = \dim \operatorname{Ker}(A - \frac{2-\lambda}{2t} I)$

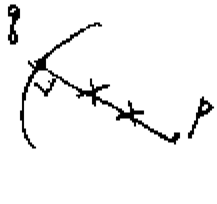
and

$\lambda < 0 \iff \underbrace{\frac{2-\lambda}{2t}}_{\exists\, = K_i} > \frac{1}{t}$ $\qquad 0 < \frac{1}{K_i} < t$

index of $\left(\frac{\partial^2 L_p}{\partial u^i \partial u^j}\right)$

$= \#\{$principal curvatures $> \frac{1}{t}\}$ counted with multip.

$= \#\{$focal points $p + \frac{1}{K_i} v$ $\quad 0 < \frac{1}{K_i} < t\}$ counted with multp. //



§7 (Application to topology of algebraic varieties,
Lefshetz theorem)
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