モース理論入門　第2回　2-1

石川剛郎

§3

M smooth n-manifold, $f: M \to \mathbb{R}$ smooth function

$M^a := \{ p \in M \mid f(p) \leqq a \}$

$C(f) := \{ p \in M \mid f_* = 0 : T_pM \to T_{f(p)}\mathbb{R} \}$
set of critical points of f

Theorem 3.1 Let $a, b \in \mathbb{R}$ with $a < b$. Suppose that $C(f) \cap f^{-1}[a,b] = \phi$ and $f^{-1}[a,b]$ is compact. Then M^a is diffeomorphic to M^b and M^a is a deformation retract of M^b. In particular, $M^a \simeq M^b$ (ホモトピー同値).

(!) Choose a Riemannian metric g on M.

$g_p : T_pM \times T_pM \to \mathbb{R}$ $(p \in M)$

For $u, v \in T_pM$, write

$\langle u, v \rangle := g_p(u, v)$

$\mathrm{grad} f$: gradient vector field of f

内積 $\langle X, \mathrm{grad} f \rangle = X(f)$ 方向微分

for $\forall$ vector field X over M.

From $\mathrm{grad} f$, construct a vector field X s.t.

$$\begin{cases} \langle X, \mathrm{grad} f \rangle = 1 & \text{on } f^{-1}[a,b] \\ \mathrm{grad} f = 0 & \text{on } M \setminus K \text{ for some compact neighborhood of } f^{-1}[a,b] \end{cases}$$

Then X generates a 1-param. gr. of diffeom. $\varphi_t : M \to M$, and $\frac{d}{dt} f(\varphi_t(q)) = 1$ if $\varphi_t(q) \in f^{-1}[a,b]$.

Consider $\varphi_{b-a} : M \to M$. Then $\varphi_{b-a} : M^a \cong M^b$.

Define $r_t : M^b \to M^b$ by

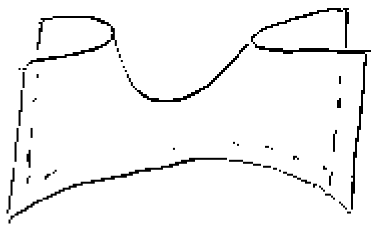
$$r_t(q) = \begin{cases} q & (q \in M^a) \\ \varphi_{t(a-f(q))}(q) & (q \in f^{-1}[a,b]) \end{cases}$$

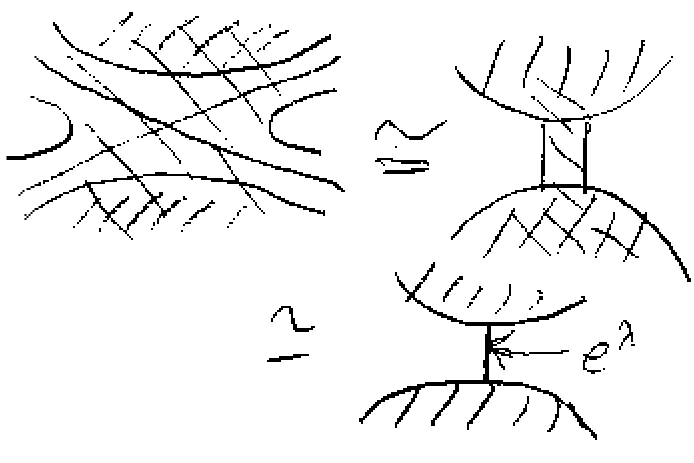
$r_0 = id_{M^b}$, $r_t|_{M^a} = id_{M^a}$ $r_1(M^b) = M^a$

Therefor M^a is a deform. retract of M^b. //

Using Morse's lemma, we have

Theorem 3.2 $f: M \to \mathbb{R}$ smooth, p: non-degen. crit. pt. of f of index λ, $f(p) = c$.
$f^{-1}[c-\varepsilon, c+\varepsilon]$ compact, $C(f) \cap f^{-1}[c-\varepsilon, c+\varepsilon] = \{p\}$
$\Rightarrow M^{c+\varepsilon} \simeq M^{c-\varepsilon} \cup e^{\lambda} = M^{c-\varepsilon} \cup_{\varphi} D^{\lambda}$





Remark 3.4 $M^c \simeq M^{c+\varepsilon}$
$M^c \simeq M^{c-\varepsilon} \cup e^{\lambda}$

"Morse function" 2-3

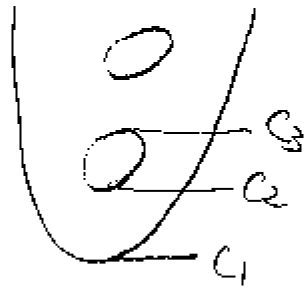
Theorem 3.5 $f: M \to \mathbb{R}$ smooth, f has only non-degenerate critical points. $\forall a \in \mathbb{R}$ M^a compact

$\Rightarrow$ $M \underset{\text{homot. eq.}}{\simeq} \exists$ CW complex $\bigcup_{p \in C(f)} e^{\operatorname{ind} p}$

(!) $c_1 < c_2 < c_3 < \cdots$ crit. values of f

$c_{i-1} < a < c_i = c$

$M^{c+\varepsilon} \simeq M^{c-\varepsilon} \cup_{\varphi_1} e^{\lambda_1} \cup \cdots \cup_{\varphi_{j(c)}} e^{\lambda_{j(c)}}$



if $f^{-1}(c)$ has crit. points $p_1, \cdots, p_{j(c)}$ of index $\lambda_1, \cdots, \lambda_{j(c)}$.

$M^{c-\varepsilon} \underset{h}{\xrightarrow{\sim}} M^a \underset{h'}{\xrightarrow{\sim}} K$ CW-complex (by induction).

$h' \circ h \circ \varphi_i : \partial D^{\lambda_i} \to K$

"cell appoximation theorem"

$h' \circ h \circ \varphi_i \simeq \exists \psi_i : \partial D^{\lambda_i} \to K$ s.t.

$\psi_i(\partial D^{\lambda_i}) \subseteq K^{(\lambda_i - 1)}$ (λ_i-1)-skeleton

$M^c \simeq K \cup_{\psi_1} e^{\lambda_1} \cup \cdots \cup_{\psi_{j(c)}} e^{\lambda_{j(c)}}$

Repeat this procedure.

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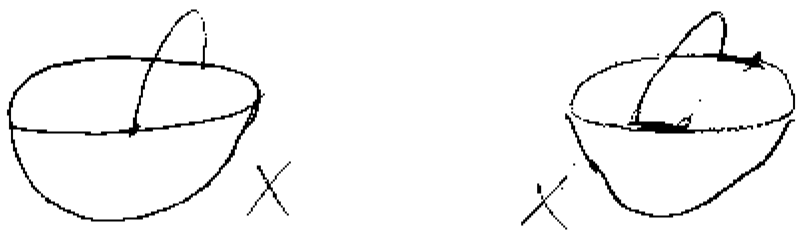
We used:

Lemma 3.6 (Whitehead) $\varphi_0, \varphi_1 : \overset{S^{\lambda-1}}{\overset{\parallel}{\partial D^\lambda}} \to X$, $\varphi_0 \simeq \varphi_1$

$\Rightarrow$ $id_X : X \to X$ is extended to a homotopy equivalence

$$k : X \cup_{\varphi_0} D^\lambda \longrightarrow X \cup_{\varphi_1} D^\lambda$$

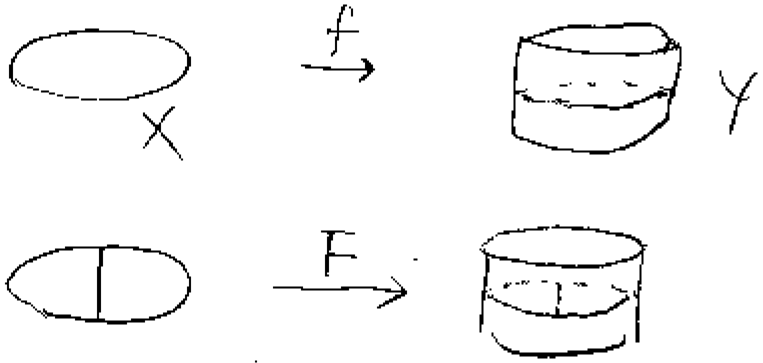


Lemma 3.7 $\varphi : \overset{S^{\lambda-1}}{\overset{\parallel}{\partial D^\lambda}} \longrightarrow X$ conti.

$f : X \longrightarrow Y$ homotopy equivalence

$\Rightarrow$ f is extended to a homotopy equivalence

$$F : X \cup_{\varphi} D^\lambda \longrightarrow Y \cup_{f\circ\varphi} D^\lambda$$



$\underset{\text{homot. eq.}}{\simeq}$ $\underset{\text{homeo.}}{\approx}$ $\underset{\text{diffeo}}{\cong}$

§4

Theorem 4.1 (Reeb) M: compact manifold (without) boundary of dim. n.
$\exists f : M \to \mathbb{R}$ has only two non-degenerate crit. pts.
$\Rightarrow$ $M \underset{\text{homeo.}}{\approx} S^n$

Remark Not necessarily $M \underset{\text{diffeo}}{\cong} S^n$
$\exists$ example M^7 $M^7 \approx S^7$ but $M^7 \not\cong S^7$
(Milnor's exotic sphere)

Example If M has only three non-degen. crit. points, then n is even and $M \simeq S^{\frac{n}{2}} \cup e^n$.
$\mathbb{R}P^2 \cong$ Möbius $\cup D^2$, $\mathbb{C}P^2 \cong \mathbb{C}P^1 \cup D^4$

Example (Application) $\mathbb{C}P^n = \{ z \in \mathbb{C}^{n+1} \mid \sum_{j=0}^{n} |z_j|^2 = 1 \} / \sim$
$(z \sim z' \Leftrightarrow \exists \alpha \in \mathbb{C}, |\alpha| = 1 \quad z' = \alpha z)$
Set $f(z_0 : z_1 : \cdots : z_n) := \sum_{j=0}^{n} c_j |z_j|^2$
where $c_0, c_1, \cdots, c_n$ disj. real numbers,
$C(f) = \{ (0 : \cdots : 0 : 1 : 0 : \cdots : 0) \mid 0 \le i \le n \}$ non-deg. index $0, 2, 4, \cdots, 2n$
$\mathbb{C}P^n \simeq e^0 \cup e^2 \cup e^4 \cup \cdots \cup e^{2n}$
(Th. 3.5)

$$H_i(\mathbb{C}P^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & (i = 0, 2, 4, \cdots, 2n) \\ 0 & \text{otherwise} \end{cases}$$

$\square$

§5 Morse inequality

X topological space, $Y \subset X$ sub top. sp.

$\rightsquigarrow S(X,Y) \in \mathbb{Z}$

S is subadditive (additive) for $Z \subset Y \subset X$

$\overset{\text{def}}{\iff} S(X,Z) \underset{(=)}{\leq} S(X,Y) + S(Y,Z)$

Example F: field

$R_\lambda(X,Y) := \mathrm{rank}_F H_\lambda(X,Y;F)$ Betti number (ベッチ数)

subadditive

$\chi(X,Y) := \sum(-1)^\lambda R_\lambda(X,Y)$ Euler characteristic (オイラー標数)

χ is additive i.e

$\chi(X,Z) = \chi(X,Y) + \chi(Y,Z)$ for $Z \subset Y \subset X$

Lemma 5.1 S: subadditive, $X_0 \subset X_1 \subset \cdots \subset X_n$

$\Rightarrow S(X_n, X_0) \leq \sum_{i=1}^{n} S(X_i, X_{i-1})$

If S is additive, then $=$

Theorem 5.2 (Morse weak inequality)
M compact manifold $f: M \to \mathbb{R}$ smooth function with only non-degenerate critical points, Let C_λ be the number of crit. pts of index λ. Then

$$R_\lambda(M) \ (= R_\lambda(M,\emptyset)) \leq C_\lambda$$

$$\sum_{i=0}^{n}(-1)^\lambda R_\lambda(M) = \sum_{\lambda=0}^{n}(-1)^\lambda C_\lambda$$

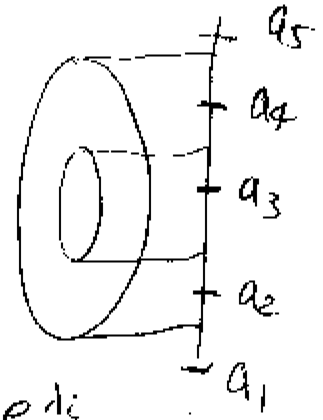
(1') of Th. 5.2

M compact $\therefore$ $\# C(f) < \infty$

$a_1 < \cdots < a_k$ regular values of f s.t.

$f^{-1}[a_{i-1}, a_i]$ contains only one critical point of index λ_i.

$M^{a_i} \simeq M^{a_{i-1}} \cup e^{\lambda_i}$



$$H_*(M^{a_i}, M^{a_{i-1}}) \cong H_*(M^{a_{i-1}} \cup e^{\lambda_i}, M^{a_{i-1}})$$

$$\cong H_*(\overline{e^{\lambda_i}}, \overline{e^{\lambda_i}} \setminus e^{\lambda_i})$$

(excision)

$$\cong H_*(D^{\lambda_i}, S^{\lambda_i - 1})$$

$$\cong \begin{cases} F & (* = \lambda_i) \\ 0 & (\text{otherwise}) \end{cases}$$

$$R_\lambda(M) \le \sum_{i=1}^{k} \underbrace{R_\lambda(M^{a_i}, M^{a_{i-1}})}_{0 \text{ or } 1,\ 1 \Leftrightarrow \lambda_i = \lambda} = C_\lambda$$

$$\chi(M) = \chi(M, \emptyset) = \sum_{i=1}^{k} \chi(M^{a_i}, M^{a_{i-1}})$$

$$= C_0 - C_1 + C_2 - \cdots + (-1)^n C_n \quad //$$

Moreover we have

Morse inequality

$$R_\lambda(M) - R_{\lambda-1}(M) + \cdots + (-1)^\lambda R_0(M)$$

$$\le C_\lambda - C_{\lambda-1} + \cdots + (-1)^\lambda C_0$$

for any $\lambda = 0, 1, \cdots, n$