モース理論入門　第15回　15-1

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§24 Periodicity theorem for orthogonal groups

$\mathbb{R}^n$: n-dimensional Euclidean vector space
orthogonal group

$$O(n) := \{ T : \mathbb{R}^n \to \mathbb{R}^n \text{ linear}, \ \forall u, v \in \mathbb{R}^n \ {}^t(Tu) \cdot Tv = {}^t u \cdot v \}$$
$$= \{ T \in M_n(\mathbb{R}) \mid {}^tT \cdot T = I \}$$

Lie group, $\dim O(n) = \frac{n(n-1)}{2}$

Lie algebra of $O(n)$

$$\mathfrak{g} = o(n) := \{ A \in M_n(\mathbb{R}) \mid {}^tA = -A \}$$ (交代行列 (歪対称行列) 全体)

$$\exp : o(n) \to O(n), \ \left(\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \right)$$

The inner product
$$\langle A, B \rangle := \mathrm{trace}(A \cdot {}^tB) = \sum_{i,j} a_{ij} b_{ij}$$
induces a left-right invariant Riemann metric on $O(n)$.

Let n even, $n = 2m$

$$\Omega_1(2m) := \{ J \in O(2m) \mid J^2 = -I \}$$

space of complex structures on $\mathbb{R}^{2m}$

Example $J_1 : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$

$$J_1 \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{2m-1} \\ v_{2m} \end{pmatrix} := \begin{pmatrix} -v_2 \\ v_1 \\ \vdots \\ -v_{2m} \\ v_{2m-1} \end{pmatrix} \qquad (\mathbb{R}^{2m}, J_1) \cong (\mathbb{C}^m, \sqrt{-1}\times)$$

Lemma 24.1 The space of minimal geodesics $\Omega^d \subset \Omega(O(2m), I, -I)$ is homeomorphic to $\Omega_1(2m)$.

(!) Any geodesic on $O(2m)$ through I is given by
$\gamma(t) = \exp(\pi t A)$, $\gamma(0) = I$

$\gamma(1) = -I \iff \exp(\pi A) = -I$

$\forall A \in \mathfrak{o}(2m)$, $({}^tA = -A)$, $\exists T \in O(2m)$, $\exists a_1, \cdots, a_m \in \mathbb{R}_{\geq 0}$

$$TAT^{-1} = \begin{pmatrix} \boxed{\begin{matrix} 0 & a_1 \\ -a_1 & 0 \end{matrix}} & & & O \\ & \boxed{\begin{matrix} 0 & a_2 \\ -a_2 & 0 \end{matrix}} & & \\ & & \ddots & \\ O & & & \boxed{\begin{matrix} 0 & a_m \\ -a_m & 0 \end{matrix}} \end{pmatrix}$$

（線形代数から
- 実行列の固有値は実数または共役な複素数が対で現れる
- 交代行列の固有値は純虚数（実部がゼロ）
- 正規行列（$A^*A = AA^*$）はユニタリ行列で対角化できる
- Aの固有値 ia に対する固有空間の（エルミート計量 ${}^tu\bar{v}$ に関する）正規直交基底 $w_1, \cdots, w_s$ をとり，実正規直交ベクトル $\frac{1}{\sqrt{2}}(w_1 + \bar{w}_1)$, $\frac{1}{\sqrt{2}}(w_1 - \bar{w}_1)$, $\cdots$ にとり直す

たとえば 佐武一郎「線型代数学」p172 例3.）

$$\exp(T(\pi A)T^{-1}) = \begin{pmatrix} \begin{matrix} \cos\pi a_1 & \sin\pi a_1 \\ -\sin\pi a_1 & \cos\pi a_1 \end{matrix} & & \\ & \boxed{\begin{matrix} \cos\pi a_2 & \sin\pi a_2 \\ -\sin\pi a_2 & \cos\pi a_2 \end{matrix}} & \\ & & \ddots \end{pmatrix}$$

$$\exp(T(\pi A)T^{-1}) = T\exp(\pi A)T^{-1}$$

$\gamma(1) = -I \iff \exp(\pi A) = -I \iff a_1, \cdots, a_m$ odd integer

$\langle \pi A, \pi A \rangle = \mathrm{trace}(\pi A \cdot {}^t(\pi A)) = 2\pi^2(a_1^2 + a_2^2 + \cdots + a_m^2)$

$\| \pi A \| = \pi\sqrt{2(a_1^2 + a_2^2 + \cdots + a_m^2)}$

$a_i \geq 0$ とをっておいた

$L(\gamma|_{[0,1]}) = \pi\sqrt{2(a_1^2 + a_2^2 + \cdots + a_m^2}$

$\gamma|_{[0,1]}$ minimal $\iff a_1 = 1, a_2 = 1, \cdots, a_m = 1$

$\Big(\rho(I, -I) = \pi\sqrt{2m}$, minimum of $E = 2m\pi^2$ $\exists$!d

$E : \Omega(O(2m), I, -I) \to \mathbb{R} \Big)$

$\therefore \gamma|_{[0,1]}$ minimal $\iff TAT^{-1} = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & \ddots \end{pmatrix}$

$\Rightarrow A^2 = -I, \quad A = T^{-1}\begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & \ddots \end{pmatrix} T \in O(2m)$

$\therefore \quad A \in \Omega_1(2m)$

Let $J \in \Omega_1(2m)$, $J^2 = -I$, $J\,{}^tJ = I \quad \therefore {}^tJ = -J$

J : skew-symmetric $J \in \mathfrak{o}(2m)$

$\exists T \in O(2m), \quad TJT^{-1} = \left(\begin{array}{cc|c} 0 & a_1 & \\ -a_1 & 0 & \\ \hline & & \ddots \end{array}\right)$

$J^2 = -I \quad \therefore a_1 = 1, a_2 = 1, \cdots, a_m = 1$, $\exp(\pi t J)|_{[0,1]}$
$\in \Omega(O(2m), I, -I)$ minimal

$\exp(\pi t \cdot) \, \Omega_1(2m) \to \Omega^d$ well-defined surjective
continuous

By §23 (4), it is injective.

$\Omega_1(2m)$ compact, Ω^d Hausdorff

$\therefore$ homeo. //

1つの複素構造

Remark $U(m) := \{ T \in O(2m) \mid J_1 T = T J_1 \}$

$$O(2m) \longrightarrow \Omega_1(2m)$$
$$T \longmapsto T J_1 T^{-1}$$
$$O(2m) / U(m) \cong \Omega_1(2m)$$

homogeneous space

($\Omega_1(2m)$ は $\mathbb{R}^{2m} = \mathbb{C}^m$ の "Lagrangian subspace" 全体の空間とも同一視できる)

Lemma 24.2 $\gamma \in \Omega(O(2m), I, -I)$
γ : non-minimal geodesic $\Rightarrow$ index $\gamma \geq 2m-2$
$(\gamma \in C(E) \setminus \Omega^d)$

(!) By Index theorem (Th. 15.1), Th. 20.5, Proof of Th. 21.7, index γ is estimated by studying eigen values of $K_A = -\frac{1}{4}(\mathrm{Ad}\,A)^2$ for

$$A = \begin{pmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & 0 & a_2 & \\ & & -a_2 & 0 & \\ & & & & \ddots \end{pmatrix} \qquad \left(K_A(W) = -\frac{1}{4}[A[A,W]] \right)$$

$a_1 \geq a_2 \geq \cdots \geq a_m > 0$, integers

non-zero eigen values of K_A are given by
1) For $i<j$, $e = \frac{1}{4}(a_i + a_j)^2$ multiplicity 2
2) $i<j$, $a_i \neq a_j$, $e = \frac{1}{4}(a_i - a_j)^2$

(たとえば $A = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}$ $W = \begin{pmatrix} 0 & 0 & W_{13} & W_{14} \\ 0 & 0 & W_{23} & W_{24} \\ -W_{13} & -W_{23} & 0 & 0 \\ -W_{14} & -W_{24} & 0 & 0 \end{pmatrix}$

に対して $[A[A,W]]$ と計算してみると
$\parallel$
$(\mathrm{Ad}\,A)^2(W)$)

$$(\mathrm{Ad}\,A)^2(W) = \left(\begin{array}{c|c} 0 & \begin{matrix} (-a^2-b^2)W_{13}+2abW_{24}, & (-a^2-b^2)W_{14}-2abW_{23} \\ (-a^2-b^2)W_{23}-2abW_{14}, & (-a^2-b^2)W_{24}+2abW_{13} \end{matrix} \\ \hline * & 0 \end{array}\right)$$

交代行列 $\Rightarrow$

$(\mathrm{Ad}\,A)^2$ の表現行列（の1つのブロック）は

$$\begin{bmatrix} -a^2-b^2 & 0 & 0 & 2ab \\ 0 & -a^2-b^2 & -2ab & 0 \\ 0 & -2ab & -a^2-b^2 & 0 \\ 2ab & 0 & 0 & -a^2-b^2 \end{bmatrix}$$

固有多項式は

$$(x+a^2+b^2+2ab)^2(x+a^2+b^2-2ab)^2$$
$$= (x+(a+b)^2)^2(x+(a-b)^2)^2$$

固有値は $x = -(a+b)^2, -(a-b)^2$ （重複度2）

K_A の固有値は $x = \frac{\pi^2}{4}(a+b)^2, \frac{\pi^2}{4}(a-b)^2$ //

$K_{\pi A}$ 〃

conjugate points corresponding $e = \frac{1}{4}(a_i+a_j)^2$

$$t = \frac{1}{\sqrt{e}}, \frac{2}{\sqrt{e}}, \frac{3}{\sqrt{e}}, \cdots$$
$$= \frac{2}{a_i+a_j}, \frac{4}{a_i+a_j}, \frac{6}{a_i+a_j} \cdots$$

The number of t with $0<t<1$ is $\frac{a_i+a_j}{2}-1$

$\frac{1}{4}(a_i+a_j)^2$ contributes to index λ by a_i+a_j-2

$\frac{1}{4}(a_i-a_j)^2$ 〃 a_i-a_j-2

$$\therefore \text{index}\,\gamma = \sum_{i<j}(a_i+a_j-2) + \sum_{i<j}(a_i-a_j-2)$$

odd, positive ≥ 1. $a_i > a_j$

γ: non-minimal $\therefore a_1 \geq 3$

$$\text{index}\,\gamma \geq \sum_{j=2}^{m}(3+a_j-2) \geq \sum_{j=2}^{m}(3+1-2) = 2m-2 \quad //$$

Theorem 24.3 (Bott) The inclusion $\Omega^d \hookrightarrow \Omega(O(2m), I, -I)$ induces isomorphisms $\pi_i(\Omega^d) \xrightarrow{\cong} \pi_i(\Omega(O(2m)), I, -I)$ for $i \leq 2m-4$.

$$\pi_i(\Omega_1(2m)) \cong \pi_{i+1}(O(2m)) \quad (i \leq 2m-4)$$

(!) By Th. 22.1 and Lem. 24.2, we have $\pi_i(\Omega, \Omega^d) = 0$ for $0 \leq i < 2m-2$. Then by the exact sequence

$$\to \pi_{i+1}(\Omega, \Omega^d) \to \pi_i(\Omega^d) \xrightarrow{i_*} \pi_i(\Omega) \to \pi_i(\Omega, \Omega^d) \to$$

we have $\pi_i(\Omega^d) \cong \pi_i(\Omega)$ for $0 \leq i \leq 2m-4$

Since $\Omega^d \approx \Omega_1(2m)$, $\Omega(O(2m), I, -I) \simeq \Omega(O(2m))$

$\pi_i(\Omega_1(2m)) \cong \pi_i(\Omega^d) \cong \pi_i(\Omega(O(2m))) \cong \pi_{i+1}(O(2m))$ for $0 \leq i \leq 2m-4$.

Let $J_1, \cdots, J_{k-1}$ be complex structures on $\mathbb{R}^{2m}$ with $J_r J_s = -J_s J_r$ for $r \neq s$, $1 \leq r, s \leq k-1$

Let $\Omega_k(2m)$ be complex structures J on $\mathbb{R}^{2m}$ with $J J_r = -J_r J$ for $1 \leq r \leq k-1$.

Suppose $\Omega_k(2m) \neq \emptyset$.

Then $\Omega_k(2m) \subset \Omega_{k-1}(2m) \subset \cdots \subset \Omega_1(2m) \subset \Omega_0(2m) = O(2m)$.

$\Omega_k(2m)$ is closed in $O(2m)$. $\therefore$ $\Omega_k(2m)$ compact.

Lemma 24.4 $\Omega_k(2m) \subset O(2m)$: totally geodesic submanifold. Every connected component of $\Omega_k(2m)$ are symmetric spaces.

The space of minimal geodesics in $\Omega(\Omega_\ell(2m), J_1, -J_1)$ is homeomorphic to $\Omega_{\ell+1}(2m)$ $(0 \leq \ell \leq k-1)$

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submanifold $N \subset (M, g)$ is totally geodesic (全測地的)

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall$ geodesic γ of $(N, g|_N)$ is a geodesic of (M, g).

Proof of Lemma 24.4 $n=2m$

$\exp: o(n) \longrightarrow O(n)$ 〃, $O \in {}^{\exists}U \subset o(n)$ open, $I \in {}^{\exists}V \subset O(n)$ open

(交代対称行列全体)

s.t. $\exp: U \xrightarrow{\approx} V$ (ノート 14-2 (4))

Claim 1: Let $J \in \Omega_1(n)$, Then $\{J\exp A \mid A \in U\}$ is a neighborhood of J in $O(n)$

If $AJ = -JA$, $A \in U$, then $J\exp A \in \Omega_1(n)$.

If U sufficiently small, $J\exp A \in \Omega_1(n)$, $A \in U$, then $AJ = -JA$.

(∵) Let $AJ = -JA$. $J^{-1}(\exp A)J = \exp(J^{-1}AJ) = \exp(-A)$
$= (\exp A)^{-1}$, $J^{-1} = -J$. ∴ $(J\exp A)^2 = -I$, $J\exp A \in \Omega_1(n)$.

Let $J\exp A \in \Omega_1(n)$. $(J\exp A)^2 = -I$ $J^{-1}(\exp A)J(\exp A) = I$

$\exp(J^{-1}AJ + A) = I$ U: suffi small, $J^{-1}AJ + A = O$

∴ $AJ = -JA$ //

Claim 2 Let $J \in \Omega_k(n)$. Then

$(J\exp A)J_\ell = -J_\ell(J\exp A) \Leftrightarrow AJ_\ell = J_\ell A$ $(1 \le \ell \le k-1)$

(∵) $(J\exp A)J_\ell = -J_\ell(J\exp A) \Leftrightarrow J(\exp A)J_\ell = JJ_\ell(\exp A)$ $(J_\ell J = -JJ_\ell)$

$\Leftrightarrow (\exp A)J_\ell = J_\ell(\exp A)$

$\Leftrightarrow \exp(J_\ell^{-1}AJ_\ell - A) = I \underset{U:\text{small}}{\Leftrightarrow} J_\ell^{-1}AJ_\ell - A = O \Leftrightarrow AJ_\ell = J_\ell A$ //

$\underbrace{U \cap \{A \in o(n) \mid AJ = -JA,\ AJ_\ell = J_\ell A\ (1 \le \ell \le k-1)\}}_{T} \xrightarrow[\approx]{\exp} V \cap \Omega_k(n)$

$A \in T$, $t \mapsto J\exp(tA)$: geodesic through J (in $O(n)$)

is contained in $\Omega_k(n)$

∴ $\Omega_k(n)$: totally geodesic

By the isometry $J\exp tA \mapsto J(J\exp tA)^{-1}J = J\exp(-tA)$, $\Omega_k(n)$ turns to be a symmetric space (cf. Lemma 21.2).

Finally let us show

$$\Omega_{\ell+1}(2m) \cong \{\text{minimal geodesics}\} \subset \Omega(\Omega_\ell(2m), J_\ell, -J_\ell)$$

Let $J \in \Omega_{\ell+1}(2m)$. $A := J_\ell^{-1}J$. Then $A \in \Omega_1(2m)$,

$AJ_1 = J_1A, \cdots, AJ_{\ell-1} = J_{\ell-1}A, \quad AJ_\ell = -J_\ell A$

$$\left(\begin{array}{l} AJ_1 = J_\ell^{-1}JJ_1 = -J_\ell^{-1}J_1J = J_\ell J_1 J = -J_1J_\ell J = J_1A, \cdots \\ AJ_\ell = J_\ell^{-1}JJ_\ell = -J = -J_\ell A, \quad A^2 = J_\ell^{-1}JJ_\ell^{-1}J = J_\ell JJ_\ell J = -I \end{array}\right)$$

$\gamma(t) := J_\ell\exp(\pi tA)$ geodesic in $\Omega_\ell(2m)$ ($\Leftarrow$ Claim 2)

$\gamma(0) = J_\ell, \quad \gamma(1) = J_\ell\exp(\pi A)$. $\exists \overset{\in O(2m)}{T}, \; T^{-1}AT = \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \\ & & \ddots \end{pmatrix}$

(15.4 Remark)

$\therefore \gamma(1) = -J_\ell$, γ: minimal in $O(2m)$ ($\Leftarrow$ 15.3) $\therefore \gamma$: minimal in $\Omega_\ell(2m)$.

Conversely let $\gamma(t) = J_\ell\exp(\pi tA) \in \Omega(\Omega_\ell(2m), J_\ell, -J_\ell)$ be a minimal geodesic, $A \in o(2m)$.

$\exp(\pi A) = -I \quad \therefore \exists T \in O(2m), \; T^{-1}AT = \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \\ & \ddots \end{pmatrix}$

$A \in \Omega_1(2m), \quad J := J_\ell A \in \Omega_{\ell+1}(2m)$

(End of Proof of Lemma 24.4) //

$$\Omega_k(2m) \hookrightarrow \Omega_k(2m+2m')$$

$J_1', \cdots, J_k'$: complex structures, anti-commutative on $\mathbb{R}^{2m'}$

$$\overset{\in}{J} \longmapsto J \oplus J_k'$$

anti-commutative to $J \oplus J_\alpha'$ $(1 \le \alpha \le k-1)$

Definition: $\Omega_k := \bigcup_m \Omega_k(2m)$

(ただし anti-commutative な $J_1, \cdots, J_k$ が $\mathbb{R}^{2m}$ 上に存在するような m をとる)

$\mathbb{O} := \bigcup_n O(n)$ infinite orthogonal group

(direct limit topology, i.e. $\Omega_k \supset U$ open $\Leftrightarrow \Omega_k(2m) \cap U$ open in $\Omega_k(2m)$)

$$\Omega_{k+1}(2m) \hookrightarrow \Omega(\Omega_k(2m), J_k, -J_k) \simeq \Omega(\Omega_k(2m))$$

induces $\Omega_{k+1} \to \Omega(\Omega_k)$

Theorem 24.5 $\Omega_{k+1} \simeq \Omega\Omega_k \quad (k \geq 0)$

$$\pi_h \underset{\Omega_0}{\mathbb{O}} \cong \pi_{h-1}\Omega_1 \cong \pi_{h-2}\Omega_2 \cong \cdots \cong \pi_1\Omega_{h-1}$$

$(h > 0)$

(証明はモース理論を応用、あとで概略を説明)

Descriptions of $\Omega_k(n)$ $\quad k = 0, 1, 2, \cdots, 8$

$k=0$: $\Omega_0(n) = O(n)$ orthogonal group

$k=1$: $\Omega_1(n)$: complex structures on $\mathbb{R}^n$

By a fixed $J_1 \in \Omega_1(n)$, $\mathbb{R}^n \cong \mathbb{C}^m$ $(n=2m)$.

$k=2$ $\Omega_2(n)$

By a fixed $J_2 \in \Omega_2(n)$ $\mathbb{C}^m \cong \mathbb{H}^l$ $(n=4l)$

$(\mathbb{H}, i, j)$: quaternion numbers (四元数)

$$Sp(\ell) := \{ T : \overset{\mathbb{R}^{4\ell}}{\overset{\parallel}{\mathbb{H}^\ell}} \to \overset{\mathbb{R}^{4\ell}}{\overset{\parallel}{\mathbb{H}^\ell}} \mid T : \mathbb{H}\text{-linear isometry} \}$$

$$\Omega_2(4\ell) \cong U(2\ell)/Sp(\ell)$$

$k=3$, $n=16r$.

Lemma 24.6 ($k=3$).

$$\Omega_3(16r) \underset{\text{as set}}{\cong} \{\mathbb{H}\text{-subspaces of } \mathbb{H}^{4r}\}$$

(!) $J_1 \in \Omega_1(16r)$, $J_2 \in \Omega_2(16r)$; $J_3 \in \Omega_3(16r)$

$J_1J_2J_3 \in O(16r)$, $(J_1J_2J_3)^2 = I$ $\therefore$ Eigen values of $J_1J_2J_3$ are ± 1. $V_1 := \{x \in \mathbb{R}^{16r} \mid J_1J_2J_3 x = x\}$

$V_2 := V_1^\perp = \{x \in \mathbb{R}^{16r} \mid J_1J_2J_3 x = -x\}$

$\mathbb{R}^{16r} = V_1 \oplus V_2$, $V_1, V_2 : J_1, J_2$-invariant

In particular $V_1 : \mathbb{H}$-subspace of $\mathbb{H}^{4r}$.

(V_2 is determined by V_1).

Conversely, given $\mathbb{R}^{16r} = V_1 \oplus V_2$, define $J_3 \in \Omega_4(16r)$

$J_3|V_1 := -J_1J_2|V_1$, $J_3|V_2 := J_1J_2|V_2$

Then $J_3 \in \Omega_3(16r)$ //

Fix $J_3 \in \Omega_3(16r)$ such that $\dim_{\mathbb{H}} V_1 = \dim_{\mathbb{H}} V_2 = 2r$

Lemma 24.6 ($k=4$)

$$\Omega_4(16r) \cong \{\varphi : V_1 \to V_2 \text{ isometry, } \mathbb{H}\text{-linear}\}$$

$$\cong Sp(2r)$$

Proof of Lemma 24.6 ($k=4$)

J_1, J_2, J_3 fixed, Let $J_4 \in \Omega_4(16r)$

$(J_3 J_4)(J_1 J_2 J_3) = -(J_1 J_2 J_3)(J_3 J_4)$, $J_3 J_4(V_1) \subset V_2$

$(J_3 J_4) J_1 = J_1 (J_3 J_4)$, $(J_3 J_4) J_2 = J_2 (J_3 J_4)$

$J_3 J_4 | V_1 : V_1 \to V_2$: $\mathbb{H}$-isom., isometry

Conversely, let $\varphi : V_1 \to V_2$ $\mathbb{H}$-isom, isometry

Define $J_4 : \mathbb{R}^{16r} \to \mathbb{R}^{16r}$ by $J_4 | V_1 = J_3^{-1} \varphi$, $J_4 | V_2 = -\varphi^{-1} J_3$

Then $J_4 \in \Omega_4(16r)$

Lemma 24.6 ($k=5$)

J_1-invariant

$$\Omega_5(16r) \underset{\text{as set}}{\cong} \{ W \subset V_1 \mid W : \mathbb{C}\text{-subspace } V_1 = W \oplus J_2 W \}$$

(!) Let $J_5 \in \Omega_5(16r)$ $(J_1 J_4 J_5)(J_1 J_2 J_3) = (J_1 J_2 J_3)(J_1 J_4 J_5)$

$(J_1 J_4 J_5)^2 = I$, $(J_1 J_2 J_3)^2 = I$ $\therefore$ $J_1 J_4 J_5 (V_1) \subset V_1$

$W := \{ x \in V_1 \mid J_1 J_4 J_5 x = x \}$, $J_2 (J_1 J_4 J_5) = -(J_1 J_4 J_5) J_2$

$J_2 W = W^{\perp}$ in V_1, $J_1 J_4 J_5 | (J_2 W) = -I$, $J_1 W = W$

Conversely given W, define J_5 by

$J_5 | W = -J_1 J_4$, $J_5 | J_2 W = J_1 J_4$, $J_5 | V_2 = J_1 J_4$ 上の構成から自然に定まる

Then $J_5 \in \Omega_5(16r)$. //

Remark : $U(2r) := \{ \varphi : V_1 \to V_1$ $\mathbb{H}$-isom, $\varphi(W) \subset W \}$

$\hookrightarrow Sp(2r)$

$$\Omega_5(16r) \cong Sp(2r) / U(2r)$$

Lemma 24.6 ($k=6$)

$$\Omega_6(16r) \cong \{ X \subset W \ \mathbb{R}\text{-subspace}, \ W = X \oplus J_1 X \}$$

(!) $J_6 \in \Omega_6(16r)$. $J_2 J_4 J_6 (W) \subset W$.

$X := \{ x \in W \mid J_2 J_4 J_6 x = x \}$ $J_1 X = X^{\perp}$ in W //

Remark $O(2r) = \{\varphi : W \to W \text{ } \mathbb{C}\text{-linear isom}, \varphi(X) \subset X\}$

$$\Omega_6(16r) \cong U(2r)/O(2r)$$

Lemma 24.6 $(k=7)$

$$\Omega_7(16r) \cong \{X_1 \subset X = \mathbb{R}^{2r}, \ \mathbb{R}\text{-linear subspaces}\}$$

Fix $J_7 \in \Omega_7(16r)$ such that $\dim_{\mathbb{R}} X_1 = r$, $X_2 = X_1^{\perp}$ in X, $\dim_{\mathbb{R}} X_2 = r$

Lemma 24.6 $(k=8)$

$$\Omega_8(16r) \cong \{\psi : X_1 \to X_2, \ \mathbb{R}\text{-linear isometry}\}$$

$$\cong O(r)$$

$$\Omega_8 = \bigcup_r \Omega_8(16r) \underset{\text{homeo}}{\approx} \mathbb{O} = \bigcup_r O(r)$$

Theorem 24.7 (Bott)

$$\mathbb{O} \simeq \Omega\Omega\Omega\Omega\Omega\Omega\Omega\Omega(\mathbb{O})$$

$$\pi_i(\mathbb{O}) \cong \pi_{i+8}(\mathbb{O})$$

(!) By Th. 24.5

$$\mathbb{O} \approx \Omega_8 \simeq \Omega\Omega_7 \simeq \Omega\Omega\Omega_6 \simeq \cdots \simeq \Omega\Omega\Omega\Omega\Omega\Omega\Omega\Omega(\Omega_0)$$

$$= \Omega\Omega\Omega\Omega\Omega\Omega\Omega\Omega\,\mathbb{O}$$

$$\pi_i(\mathbb{O}) \cong \pi_{i+1}(\overbrace{\Omega\cdots\Omega}^{7}\mathbb{O}) \cong \pi_{i+2}(\underbrace{\Omega\cdots\Omega}_{6}\mathbb{O})$$

$$\cdots \cong \pi_{i+8}(\mathbb{O})$$

//

$$Sp := \Omega_4 = \bigcup_m \Omega_4(16r) \simeq \Omega\Omega\Omega\Omega\Omega(\mathbb{O})$$

$i \pmod 8$	0	1	2	3	4	5	6	7
$\pi_i(\mathbb{O})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\pi_i(Sp)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$
$\pi_i(\mathbb{U})$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

Proof of Th. 24.5

Let V be a Euclidean vector space, $J_1, \cdots, J_k$ anti-commuting complex structures on V.

Definition: $(V, J_1, \cdots, J_k)$ is a <u>minimal $(J_1, \cdots, J_k)$-space</u>

$$\overset{\text{def}}{\Longleftrightarrow} \left(\begin{array}{l} W \subset V \text{ linear subspace } \; J_\ell(W) \subset W \quad (1 \le \ell \le k) \\ \qquad \Rightarrow \; W = \{0\} \text{ or } V \end{array} \right)$$

$(V, J_1, \cdots, J_k) \cong_{\text{isom}} (V', J_1', \cdots, J_k')$ isomorphic

$\overset{\text{def}}{\Longleftrightarrow} \exists \varphi : V \to V'$ isometric isom,

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "J_\ell"'] \arrow[dr, phantom, "\circlearrowright"] & V' \arrow[d, "J_\ell'"] \\ V \arrow[r, "\varphi"'] & V' \end{tikzcd} \qquad (1 \le \ell \le k)$$

Lemma 24.8 *Let $k \not\equiv 3 \pmod 4$. Then all minimal $(J_1, \cdots, J_k)$-spaces are isomorphic. For any k, the dimension of minimal $(J_1, \cdots, J_k)$-space m_k is uniquely determined:* $m_0 = 1$, $m_1 = 2$, $m_2 = m_3 = 4$, $m_4 = m_5 = m_6 = m_7 = 8$, $m_8 = 16$, $m_k = 16\, m_{k-8}$ $(k > 8)$. （証明略） //

Proof of Th. 24.5 when $k \not\equiv 2 \pmod 4$.

$J \in \Omega_k(n) = \{J : \text{complex structures on } \mathbb{R}^n (\mathbb{R}^n; J_1, \cdots, J_{k-1}),\ JJ_l = -J_l J,\ (1 \le l \le k-1)\}$

$T_J \Omega_k(n) \cong T := \{A \in o(n) \mid AJ = -JA,\ AJ_l = J_l A\ (1 \le l \le k-1)\}$

$A \in T$, $\gamma(t) = J\exp(\pi t A)$

γ geodesic in $\Omega(\Omega_k(n); J, -J) \Leftrightarrow \forall$ eigen value of A ia, a: odd integer.

$\left(\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \text{ eigen value } \pm ia \leftarrow \text{15-2 15-3 15-7} \right)$

$K_A : T \to T$, $K_A(B) := -\frac{1}{4}[A, [A, B]]$

Try to find a lower estimate of index γ.

Decompose $\mathbb{R}^n = M_1 \oplus M_2 \oplus \cdots \oplus M_s$ into closed & minimal subspaces under the actions of $J_1, \cdots, J_{k-1}, J, A$, $M_i \perp M_j$.

Eigen values of $A|M_h$ are all equal up to sign. $\pm i a_h$

$a_1, \cdots, a_s$: positive odd integers.

M_h is $(J_1, \cdots, J_{k-1}, J, J')$-minimal, where $J' = \frac{1}{a_h} JA|M_h$

$\therefore \dim_{\mathbb{R}} M_h = m_{k+1}$ ($\leftarrow$ Lemma 24.8). $k+1 \not\equiv 3 \pmod 4$.

$\therefore M_1 \cong M_2 \cong \cdots \cong M_s$ ($\leftarrow$ Lemma 24.8).

For $h \ne j$, define $B : \mathbb{R}^n \to \mathbb{R}^n$ by $B|M_l = 0$ $(l \ne h, j)$

$B|M_h : M_h \to M_j$ isometry such that $BJ_\alpha = J_\alpha B$ $(1 \le \alpha \le k-1)$

$BJ = -JB$, $BJ' = J'B$, $B|M_j : M_j \to M_h$, $B|M_j = -(B|M_h)^*$,

Then $B \in T$.

Moreover B is an eigen vector of K_A with eigen value $\frac{1}{4}(a_h + a_j)^2$.

$s = \frac{n}{m_{k+1}}$, γ is not minimal $\Leftrightarrow \exists a_h \ge 3$.

Claim : K_A has at least $s - 1 = \frac{n}{m_{k+1}} - 1$ eigen values which are greater than or equal to $\frac{1}{4}(3+1)^2 = 4$. (a_h & a_1, $\cdots$ a_h & a_{h-1}, a_h & a_{h+1}, $\cdots$, a_h & a_s)

For a non minimal geodesic $\gamma(t) = J\exp(\pi t A)$,

$\exists$ at least one conjugate point for each eigenvalue ≥ 4.

Claim $\forall$ geodesic in $\Omega(\Omega_k(n), J, -J)$ has index $\geq \frac{n}{m_{k+1}} - 1$.

By Th. 22.1, $\pi_i(\Omega_{k+1}(n)) \xrightarrow{\cong} \pi_i(\Omega(\Omega_k(n)))$
for $0 \leq i \leq \frac{n}{m_{k+1}} - 3$. $\quad \frac{n}{m_{k+1}} \to \infty$ (kできる)

By taking the limit $n \to \infty$, the inclusion $i : \Omega_{k+1} \to \Omega\Omega_k$ induces isomorphisms $\pi_i(\Omega_{k+1}) \to \pi_i(\Omega\Omega_k)$ for any i.
$\Omega_{k+1} \simeq$ CW complex, $\Omega\Omega_k \simeq$ CW complex ($\leftarrow$ Th. 17.3 等)
$\therefore$ $\Omega_{k+1} \simeq \Omega\Omega_k$ (Whiteheadの定理, Th17.3の証明でも使用)
under the condition $k \not\equiv 2 \pmod 4$.

Remark:
By the above proof, we have in particular the analogue of Th. 24.3, that the inclusion
$$\Omega_2(n) \hookrightarrow \Omega(\Omega_1(n), J_1, -J_1)$$
induces isomorphisms
$$\pi_i(\Omega_2(n)) \cong \pi_i(\Omega(\Omega_1(n), J_1, -J_1))$$
$$(\cong \pi_{i+1}(\Omega_1(n)))$$
for $0 \leq i \leq \frac{n}{m_2} - 3 = \frac{n}{4} - 3$

However the inclusion
$$\Omega_3(n) \hookrightarrow \Omega(\Omega_2(n), J_2, -J_2)$$
<u>does not</u> induce an isomorphism, as sets,
$\pi_0(\Omega_3(n))$ and $\pi_0(\Omega(\Omega_2(n), J_2, -J_2))$
$\qquad \cong \pi_1(\Omega_2(n))$

連結成分が有限 $\qquad (\cong \mathbb{Z})$

Therefore, to show Th 24.5, we have to look at each connected component of $\Omega(\Omega_2(n), J_2, -J_2)$ (when n is sufficiently large).

Proof of Th. 24.5 on the case $k \equiv 2 \pmod 4$

To describe $\pi_0(\Omega(\Omega_k(n), J, -J)) \cong \pi_1(\Omega_k(n))$,
define $f : \Omega_k(n) \to S^1 \subset \mathbb{C}$ as follows:
$i := J_1 J_2 \cdots J_{k-1}$ is a complex structure on $\mathbb{R}^n$
i commutes with each $J_1, J_2, \cdots, J_{k-1}$. ($\Leftarrow k \equiv 2 \pmod 4$)
Fix $J \in \Omega_k(n)$. For any $J' \in \Omega_k(n)$, $J^{-1}J'$ commutes with i. Then $J^{-1}J' : \mathbb{R}^n \underset{i}{=} \mathbb{C}^{\frac{n}{2}} \to \mathbb{C}^{\frac{n}{2}}$ is a $\mathbb{C}$-linear unitary transformation. Define $f(J') := \det(J^{-1}J') \in S^1$.

Let $\gamma \in \Omega(\Omega_k(n), J, -J)$ be a geodesic from J to $-J$,
$\exists A \in o(n)$, $\gamma(t) = J \exp(\pi t A)$.
Then A commutes with $i = J_1 J_2 \cdots J_{k-1}$.
$A : \mathbb{R}^n = \mathbb{C}^{\frac{n}{2}} \to \mathbb{C}^{\frac{n}{2}}$ is complex linear skew-Hermite
$(A^* = -A)$, $\forall$ eigenvalue of A is pure-imaginary.
$\left(z \mapsto iaz,\ z = x+iy.\ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},\ z = x - iy,\ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \right)$

$$f(J\exp(\pi t A)) = \det(\exp(\pi t A)) = e^{\pi t \,\mathrm{trace}\, A}$$

($\mathrm{trace}\, A = i a_1 + \cdots + i a_{\frac{n}{2}}$, a_j : odd integer, 重複あり, $\frac{n}{2}$ even
$\therefore$ $\mathrm{trace}\, A = i \times$ (even integer))
$\mathrm{trace}\, A$ is a homotopy invariant of γ.

To estimate index of γ, decompose $\mathbb{R}^n = M_1 \oplus \cdots \oplus M_r$ into minimal subspaces M_h invariant under $J_1, \cdots, J_{k-1}$, J and A. Then the restriction $A|M_h$ has only one eigen value $i a_h$ ($a_1, \cdots, a_r$: odd integers)
$A|M_h = a_h J_1 J_2 \cdots J_{k-1}|M_h$, $\dim_{\mathbb{C}} M_h = \frac{m_k}{2}$,
$\mathrm{trace}\, A = i a_1 \frac{m_k}{2} + \cdots + i a_r \frac{m_k}{2} = i(a_1 + \cdots + a_r)\frac{m_k}{2}$. $(r m_k = n)$.

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For $h \neq j$, define an isometry $B: \mathbb{R}^n \to \mathbb{R}^n$ s.t. $B|M_h : M_h \to M_j$ commutes with $J_1, \cdots, J_{k-1}$, anti-commutes with J, $B|M_j : M_j \to M_h$ negative conju. of $B|M_h$, and $B|M_\ell = 0$ $(\ell \neq h, j)$. Then B is an eigen vector of K_A with eigen value $\frac{1}{4}(a_h - a_j)^2$:

$$K_A(B) = \frac{1}{4}(a_h - a_j)^2 B.$$

By the index theorem, the index of γ

$$\lambda \geq \sum_{a_h > a_j} \left(\frac{a_h - a_j}{2} - 1 \right) \quad \cdots (*)$$

Take any connected component C of $\Omega\Omega_k(n)$ corresponding to trace $A = ic\frac{m_k}{2}$.
Then $a_1 + \cdots + a_r = c$, $a_1, \cdots, a_r$ or odd integers.

(もしγが minimal なら すべての $a_h = \pm 1$, (*)の右辺は0. しかしγが non-minimal なら $\exists h$, $|a_h| \geq 3$. $a_1 + \cdots + a_r = c$ という制限から(*)の右辺をr, cで下から評価)

$p := \sum_{a_h > 0} a_h$, $q = \sum_{a_h < 0} (-a_h)$. Then $p - q = c$, $p + q = r$

$p = \frac{1}{2}(r + c)$, $q = \frac{1}{2}(r - c)$, $r = \frac{n}{m_k}$.

Suppose $\exists j$, $a_j = -3$ for exemple. Then

$$\lambda \geq \sum_{a_h > a_j} \left(\frac{a_h - a_j}{2} - 1 \right) \underset{\text{1つのjだけみる}}{\geq} \sum_{a_h > 0} \left(\frac{a_h - (-3)}{2} - 1 \right) > \frac{p}{2} = \frac{1}{4}(r + c)$$

Suppose $\exists h$, $a_h = 3$. Then $\lambda > \frac{q}{2} = \frac{1}{4}(r - c)$.

Fix c. Take n sufficiently large.
Take a component $\tilde{C}$ of $\Omega_{k+1}(n)$ corresponding to C.

$$\pi_i(\tilde{C}) \cong \pi_i(C), \quad 0 \leq i \leq \min\left\{ \frac{1}{4}\left(\frac{n}{m_k} + c\right), \frac{1}{4}\left(\frac{n}{m_k} - c\right) \right\}$$

Taking $n \to \infty$, we have $\pi_i(\Omega_{k+1}) \cong \pi_i(\Omega\Omega_k)$ for any i.
$\Omega_{k+1} \simeq$ CW complex, $\Omega\Omega_k \simeq$ CW complex. By Whitehead's theorem $\Omega_{k+1} \simeq \Omega\Omega_k$ //