モース理論入門　第14回

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§23 The Bott periodicity for the unitary group

Let us apply Morse theory to $U(n)$ (unitary group) and $SU(n)$ (special unitary group).

$$U(n) := \{ \underbrace{S : \mathbb{C}^n \to \mathbb{C}^n}_{\mathbb{C}\text{-linear}} \mid {}^t(Su)\overline{(Sv)} = {}^t u \bar{v}, \text{ for } {}^\forall u, v \in \mathbb{C}^n \}$$

unitary group　　　転置　　Hermite inner product

$$= \{ S \in M_n(\mathbb{C}) \mid SS^* = I \} \qquad S^* = {}^t\bar{S}$$

$M_n(\mathbb{C})$: $n\times n$ complex matrices　　adjoint matrix (随伴行列)

$$SU(n) := \{ S \in U(n) \mid \det S = 1 \}$$

$A \in M_n(\mathbb{C})$

$$\exp A := I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

absolutely convergent

$$\langle A, B \rangle := \mathrm{Re}\,(\mathrm{trace}(AB^*)) = \mathrm{Re}\Big(\sum_{1\le i,j\le n} a_{ij}\overline{b_{ij}}\Big)$$

$$\|A\| := \sqrt{\langle A, A\rangle} = \sqrt{\sum_{1\le i,j\le n} |a_{ij}|^2}$$

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(0) $\exp(O) = I$

(1) $\exp(A^*) = (\exp A)^*$,
$\exp(TAT^{-1}) = T(\exp A)T^{-1}$ $(T \in GL(n,\mathbb{C}))$ 正則行列

(1′) $\det(\exp A) = e^{\mathrm{trace} A}$

(∵) 対角化

(2) $AB = BA \Rightarrow \exp(A+B) = (\exp A)(\exp B)$

(3) $(\exp A)(\exp(-A)) = I$, $\exp A \in GL(n,\mathbb{C})$

(∵) $(\exp A)(\exp(-A)) = \exp(A-A) = \exp O = I$

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(4) $\exp : M_n(\mathbb{C}) \to M_n(\mathbb{C})$

diffeomorphism on a neighborhood of O

($M_n(\mathbb{C}) \cong T_O M_n(\mathbb{C}) \xrightarrow[(\exp)_*]{} T_I M_n(\mathbb{C}) \cong M_n(\mathbb{C})$ が同型,
あるいは exp がリー群 $GL(n,\mathbb{C})$ の exp とみなされることから)

(4') $A+A^*=0 \Rightarrow \exp A \in U(n)$

(!!) $(\exp A)(\exp A)^* = (\exp A)(\exp A^*) = (\exp A)(\exp(-A)) = I$

($\exp A$ が I に十分近ければ逆も言える ← (4))

(5) $U(n) \subset M_n(\mathbb{C})$ smooth submanifold

$\mathfrak{u}(n) := \{ A \in M_n(\mathbb{C}) \mid A^* = -A \}$ ← 歪エルミート行列

(ドイツ文字小文字のユーのつもり)

(6) $T_I U(n) = \mathfrak{u}(n)$

Lie algebra of $U(n) \cong \mathfrak{u}(n)$

(Left-invariant vector fields)

$[A, B] = AB - BA$

$\dim_{\mathbb{R}} U(n) = n + 2\frac{(n-1)n}{2} = n^2$

(説明) $A \in \mathfrak{u}(A) \rightsquigarrow$ left invariant vector field X_A over $U(n)$

$$X_A(S) := \frac{d}{dt}\left(S\exp(tA)\right)\Big|_{t=0} \quad (S \in U(n))$$

($T_S U(n) \subset T_S GL(n,\mathbb{C}) = M_n(\mathbb{C})$ と同一視して)

f: function on $GL(n,\mathbb{C})$

$$(X_A f)(S) = \frac{d}{dt} f(S\exp(tA))\Big|_{t=0} \quad (X_A f \text{ の def})$$

$$(X_B(X_A f))(S) = \frac{d}{dt}\left((X_A f)(S\exp(tB))\right)\Big|_{t=0} \quad (X_B \text{ の def})$$

$$= \frac{d}{dt}\frac{d}{ds} f(S\exp(tB)\exp(sA))\Big|_{(s,t)=(0,0)} \quad (X_A f \text{ の def})$$

$$(X_A(X_B f))(S) = \frac{d}{dt}\frac{d}{ds} f(S\exp(tA)\exp(sB))\Big|_{(s,t)=(0,0)}$$

Lie bracket

$([X_A, X_B]f)(S)$

$$= \frac{d}{dt}\frac{d}{ds}\Big[f(S\exp(tA)(sB)) - f(S\exp(tB)\exp(sA))\Big]_{(s,t)=(0,0)}$$

Take the coordinate function "a_{ij}" on $GL(n,\mathbb{C})$,
(i,j)-component, linear function on $M_n(\mathbb{C})$.

Then

$$([X_A, X_B]a_{ij})(S) = \frac{d}{dt}\frac{d}{ds}\Big[a_{ij}\big(S\exp(tA)(sB) - S(tB)(sA)\big)\Big]_{(s,t)=(0,0)}$$

$$= a_{ij}(SAB - SBA) = a_{ij}(S[A,B])$$ (行列の bracket)

$$= a_{ij}(S\exp(t[A,B])|_{t=0})$$

$$= (X_{[A,B]}\,a_{ij})(S) \qquad (X_{[A,B]} \text{ の def})$$

$\therefore\ [X_A, X_B]a_{ij} = X_{[A,B]}a_{ij}$ (as function on GL) for $\forall ij$

$\therefore\ [X_A, X_B] = X_{[A,B]}$ (as vector field on GL) //

$\mathfrak{g}$ ||

$\langle A, B\rangle := \mathrm{Re}(\mathrm{trace}\, AB^*)$
positive definite inner product on $\mathfrak{u}(n)$
$\{A \mid A + A^* = 0\}$

$\langle\ ,\ \rangle$ induces a left-invariant metric g on $U(n)$

◎ g is right invariant

(∵) $\mathrm{ad}(S) : U(n) \to U(n)$ $\quad (S \in U(n))$

$$X \longmapsto SXS^{-1} = (L_S R_{S^{-1}})(X)$$

$$\mathrm{Ad}(S) := (L_S R_{S^{-1}})_* : \underset{\mathfrak{u}(n)}{\overset{\shortparallel}{T_I U(n)}} \to \underset{\mathfrak{u}(n)}{\overset{\shortparallel}{T_I U(n)}}$$

$\langle Ad(S)A, Ad(S)B \rangle = \langle SAS^{-1}, SBS^{-1} \rangle$
$= \mathrm{Re}\, \mathrm{trace}\, (SAS^{-1}(SBS^{-1})^*)$
$= \mathrm{Re}\, \mathrm{trace}\, (SAS^{-1}(S^{-1})^* B^* S^*) = \mathrm{Re}\, \mathrm{trace}\, (SAB^*S^{-1})$
$= \mathrm{Re}\, \mathrm{trace}\, (AB^*) = \langle A, B \rangle.$ for $\forall S \in U(n)$

The left invariant metric g is defined by
$g_S(u,v) := \langle (L_{S^{-1}})_* u, (L_{S^{-1}})_* v \rangle$ $(u, v \in T_S U(n))$

The right invariant metric $\tilde{g}$ "

$\tilde{g}_S(u,v) := \langle (R_{S^{-1}})_* u, (R_{S^{-1}})_* v \rangle$
$= \langle Ad(S^{-1})(R_{S^{-1}})_* u, Ad(S^{-1})(R_{S^{-1}})_* v \rangle$
$= \langle (L_{S^{-1}})_* u, (L_{S^{-1}})_* v \rangle$
$= g_S(u,v)$

$\therefore \tilde{g}_S = g_S$ $\quad \therefore g$: left-right invariant //

Ⓒ $\exp = \exp_I : T_I U(n) \longrightarrow U(n)$
$A \longmapsto \exp(A)$, $A \in T_I U(n)$

Ⓟ $\exp(tA)$: 1-parameter subgroup of $U(n)$
geodesic, $\frac{d}{dt}\exp(tA)|_{t=0} = A$. //

Ⓓ $SU(n) := \{ S \in U(n) \mid \det S = 1 \}$
$su(n) := \{ A \in M_n(\mathbb{C}) \mid A + A^* = 0,\ \mathrm{trace}\, A = 0 \}$
$\cong T_I SU(n)$
((1)′ $\det(\exp A) = e^{\mathrm{trace}\, A}$)

Ⓒ $\exp : u(n) \to U(n)$, $\exp : su(n) \longrightarrow SU(n)$ surjective

Ⓟ $\forall S \in U(n)$, $\exists T \in U(n)$, $\exists R = \begin{pmatrix} e^{ia_1} & & 0 \\ & \ddots & \\ 0 & & e^{ia_n} \end{pmatrix}$

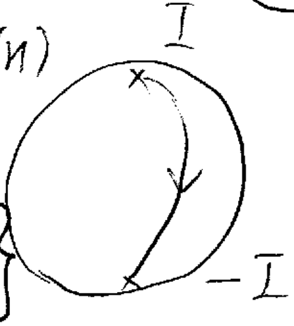
s.t. $R = TST^{-1}$.

$A := \begin{pmatrix} ia_1 & & 0 \\ & \ddots & \\ 0 & & ia_n \end{pmatrix}$. Then $\exp(T^{-1}AT) = T^{-1}(\exp A)T$
$= T^{-1}RT = S$ //

($\det S = 1 \Leftrightarrow \det R = 1 \Leftrightarrow \mathrm{trace}\, A \equiv 0\ (2\pi)$)

$$\Omega(U(n), \overset{p}{I}, \overset{q}{-I})$$

$$= \{\omega : [0,1] \longrightarrow U(n) \text{ piecewise smooth}, \ \omega(0) = I,\ \omega(1) = -I\}$$



$$\supset \{\text{geodesics } \gamma(t) = \exp(tA),\ \exp(A) = -I\}$$
$$A \in u(n) = T_I U(n)$$

Let $A \in u(n)$ with $\exp(A) = -I$

$\exists T \in U(n)$

$$TAT^{-1} = \begin{pmatrix} ia_1 & & & 0 \\ & ia_2 & & \\ & & \ddots & \\ 0 & & & ia_n \end{pmatrix}$$

$$\exp(TAT^{-1}) = \begin{pmatrix} e^{ia_1} & & & 0 \\ & e^{ia_2} & & \\ & & \ddots & \\ 0 & & & e^{ia_n} \end{pmatrix}$$

$$\exp A = -I \iff e^{ia_j} = -1 \quad (1 \leq j \leq n)$$

$$\iff TAT^{-1} = \begin{pmatrix} k_1 i\pi & & & 0 \\ & k_2 i\pi & & \\ & & \ddots & \\ 0 & & & k_n i\pi \end{pmatrix}$$

$k_1, \cdots, k_n$ odd integers.

$$\|A\|^2 = \mathrm{Re}\,\mathrm{tr}\,AA^* \left(= \mathrm{Re}\,\mathrm{tr}(TAT^{-1})(TAT^{-1})^*\right)$$

$$= \pi^2(k_1^2 + k_2^2 + \cdots + k_n^2)$$

$$\|A\| = \pi\sqrt{k_1^2 + k_2^2 + \cdots + k_n^2}$$

$\gamma(t) = \exp tA$: minimal

$$\iff k_i = \pm 1 \quad (1 \leq i \leq n) \qquad \left(\begin{array}{c} A\text{の固有値は} \\ \pm i\pi \end{array}\right)$$

$$\left(\rho(I, -I) = \pi\sqrt{n}\right)$$

$A \longleftrightarrow \{x \in \mathbb{C}^n \mid Ax = i\pi x\} \oplus \{x \in \mathbb{C}^n \mid Ax = -i\pi x\}$

Ω^d : minimal geodesics $\underset{\text{as set}}{\simeq} \{W \subset \mathbb{C}^n,\ \mathbb{C}\text{-subspaces}\}$

$d = \pi^2 n$ (エネルギーの最小値)

$G_\ell(\mathbb{C}^n) := \{W \subset \mathbb{C}^n \mid W : \mathbb{C}\text{-subspace},\ \dim_{\mathbb{C}} W = \ell\}$

$$\Omega^d \cong \bigcup_{\ell=0}^{n} G_\ell(\mathbb{C}^n)$$

$n = 2m$, $\Omega(SU(2m), I, -I)$ $\quad I, -I \in SU(2m)$

$\exp : su(2m) \to SU(2m)$

$k_1 i\pi, \cdots, k_{2m} i\pi$, $k_j = \pm 1$

$\sum_{j=1}^{2m} k_j i\pi = 0 \Rightarrow \#\{j \mid k_j = 1\} = m$

Lemma 23.1

$$\Omega(SU(2m), I, -I)^d \underset{\text{homeo}}{\approx} G_m(\mathbb{C}^{2m})$$

$d = \pi^2 d$

Lemma 23.2 $\gamma : [0,1] \to SU(2m)$ geodesic from I to $-I$.

γ : non-minimal $\Rightarrow$ index of $\gamma \geq 2m+2$ (証明はあとで)

By Th. 22.1, $\pi_i(\Omega, \Omega^d) = 0 \quad (0 \leq i \leq 2m+1)$

By homotopy exact sequence

$$\pi_i(\Omega) \cong \pi_i(\Omega^d) \cong \pi_i(G_m(\mathbb{C}^{2m}))$$
$$0 \leq i \leq 2m$$

$\Omega = \Omega(SU(2m), I, -I) \simeq \Omega(SU(2m))$ (loop space)

$\therefore \pi_i(\Omega) \cong \pi_i(\Omega(SU(2m))) \cong \pi_{i+1}(SU(2m))$

Theorem 23.3 (Bott)

(1) $\pi_i(G_m(\mathbb{C}^{2m})) \cong \pi_{i+1}(SU(2m)) \quad (0 \leq i \leq 2m)$

Lemma 23.4 $\quad i \leq 2m$

(2) $\pi_i(G_m(\mathbb{C}^{2m})) \cong \pi_{i-1}(U(m))$

(3) $\qquad \cong \pi_{i-1}(U(m+1)) \cong \pi_{i-1}(U(m+2)) \cong \cdots$

(4) $j \neq 1 \Rightarrow \pi_j(U(m)) \cong \pi_j(SU(m))$

Proof of Lemma 23.4

$U(m+1)$ acts on S^{2m+1} : unit sphere in $\mathbb{C}^{m+1} = \mathbb{R}^{2m+2}$

$(T, x) \longmapsto Tx \quad$ transitive

$U(m) = \left\{ T \in U(m+1) \;\middle|\; T = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & * \end{array}\right) \right\}$

$U(m+1) \to S^{2m+1}$ fibration with fiber $U(m)$

Homotopy exact sequence ← スチーンロッド「ファイバー束のトポロジー」 吉岡書店 P.113

$\to \pi_i(S^{2m+1}) \to \pi_{i-1}(U(m)) \to \pi_{i-1}(U(m+1))$

$i \leq 2m \quad \pi_i(S^{2m+1}) = 0$

$\to \pi_{i-1}(S^{2m+1}) \to \cdots, \quad \pi_{i-1}(S^{2m+1}) = 0$

$\therefore \pi_{i-1}(U(m)) \cong \pi_{i-1}(U(m+1)) \quad \therefore (3)$

$i = 2m+1$

$\pi_{2m}(U(m)) \xrightarrow[\text{onto}]{} \pi_{2m}(U(m+1)) \cong \pi_{2m}(U(m+2)) \cong \cdots$

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$V_{2m,m} := \{(v_1, \cdots, v_m) \mid v_1, \cdots, v_m \text{ orthonormal in } \mathbb{C}^{2m}\}$

Stiefel manifold

$U(2m)$ acts on $V_{2m,m}$ transitively with isotropy group $U(m)$

$$V_{2m,m} \cong U(2m)/U(m) \quad \text{homogeneous space}$$

fibration $U(m) \to U(2m) \to U(2m)/U(m)$

Homotopy exact sequence

$$\pi_i(U(m)) \underset{\substack{\uparrow \\ \text{surjective}}}{\to} \pi_i(U(2m)) \underset{\substack{\uparrow \\ 0\text{-map}}}{\to} \pi_i(U(2m)/U(m)) \underset{\substack{\uparrow \\ 0\text{-map}}}{\to} \pi_{i-1}(U(m)) \underset{\cong}{\to} \pi_{i-1}(U(2m))$$

if $i \leq 2m$.

$$\therefore \quad \pi_i(U(2m)/U(m)) = 0 \quad (0 \leq i \leq 2m)$$

$$\underset{\substack{\uparrow \\ \text{isotropy group}}}{U(m)} \to U(2m)/U(m) = V_{2m,m} \to G_m(\mathbb{C}^{2m})$$

$$(v_1, \cdots, v_m) \mapsto \langle v_1, \cdots, v_m \rangle_{\mathbb{C}}$$

Homotopy exact sequence

$$\underset{\substack{\| \\ 0}}{\pi_i\left(\frac{U(2m)}{U(m)}\right)} \to \pi_i(G_m(\mathbb{C}^{2m})) \to \pi_{i-1}(U(m)) \to \underset{\substack{\| \\ 0}}{\pi_{i-1}\left(\frac{U(2m)}{U(m)}\right)}$$

if $i \leq 2m$

$$\pi_i(G_m(\mathbb{C}^{2m})) \cong \pi_{i-1}(U(m)) \quad \therefore (2)$$

$$(0 \leq i < 2m)$$

fibration

$$SU(m) \to U(m) \xrightarrow[\det]{} S^1$$

$$\pi_{j+1}(S^1) \to \pi_j(SU(m)) \to \pi_j(U(m)) \to \pi_j(S^1) \to \cdots$$

$j \geq 2$ or $j = 0$

$$\pi_j(SU(m)) \cong \pi_j(U(m))$$

Stable homotopy group

$$U := \bigcup_{n=1}^{\infty} U(n) \quad \text{(direct limit)}$$

By Lemma 23.4, Theorem 23.3,

$$\pi_{i-1}(U(m)) \cong \pi_{i-1}(U(m+1)) \cong \cdots \cong \pi_{i-1}(U) \quad (0 \le i \le 2m)$$

$$\begin{array}{c} \pi_{i-1}(U(m)) \\ \wr\| \\ \pi_i(G_m(\mathbb{C}^{2m})) \end{array} \underset{(\text{Th. }23.3)}{\cong} \pi_{i+1}(SU(2m)) \cong \pi_{i+1}(U(2m)) \cong \pi_{i+1}(U) \quad (1 \le i \le 2m)$$

★ Bott Periodicity Theorem, $1 \le i$

$$\pi_{i-1}(U) \cong \pi_{i+1}(U)$$

$$\pi_0(U) \cong \pi_0(U(1)) \cong \pi_0(S^1) \cong 0$$
$$\pi_1(U) \cong \pi_1(U(1)) \cong \pi_1(S^1) \cong \mathbb{Z}$$
$$\pi_2(U) \cong \pi_2(SU(2)) \cong \pi_2(S^3) = 0$$
$$\pi_3(U) \cong \pi_3(SU(2)) \cong \pi_3(S^3) \cong \mathbb{Z}$$
$$\pi_0(U) \cong \pi_2(U) \cong \pi_4(U) \cong \cdots = 0$$
$$\pi_1(U) \cong \pi_3(U) \cong \pi_5(U) \cong \cdots \cong \mathbb{Z}$$

Proof of Lemma 23.2

$\mathfrak{g}' := su(2m) = \{A \in M_{2m}(\mathbb{C}) \mid A^* = -A, \operatorname{trace} A = 0\}$

non-minimal geodesic $\exp(tA)$

$\Leftrightarrow A \in \mathfrak{g}'$ eigen value $i\pi k_1, \cdots, i\pi k_{2m}$

k_j odd, $k_1 + \cdots + k_{2m} = 0$

Th. 20.5. $K_A : \mathfrak{g}' \to \mathfrak{g}'$

$K_A(W) := R(A, W)A = \frac{1}{4}[[A W], A]$

Suppose $A = \begin{pmatrix} i\pi k_1 & & 0 \\ & \ddots & \\ 0 & & i\pi k_{2m} \end{pmatrix}$

$k_1 \geq k_2 \geq \cdots \geq k_{2m}$

$W = (w_{j\ell}) \in \mathfrak{g}'$.

$[A, W] = (i\pi(k_j - k_\ell) w_{j\ell})$

$[A, [A, W]] = (-\pi^2 (k_j - k_\ell)^2 w_{j\ell})$

$K_A(W) = (\frac{\pi^2}{4}(k_j - k_\ell)^2 w_{j\ell})$

$\exists$ basis of $\mathfrak{g}'$ consisting of eigen vectors of K_A

1) $j < \ell$ eigen vector $E_{j\ell} = \begin{array}{c|c} & \begin{matrix} j & \ell \end{matrix} \\ \begin{matrix} j \\ \ell \end{matrix} & \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \end{array}$ eigen value $\frac{\pi^2}{4}(k_j - k_\ell)^2$

2) $j < \ell$ $E'_{j\ell} = \begin{array}{c|c} & \begin{matrix} j & \ell \end{matrix} \\ \begin{matrix} j \\ \ell \end{matrix} & \begin{pmatrix} & i \\ i & \end{pmatrix} \end{array}$ eigen value $\frac{\pi^2}{4}(k_j - k_\ell)^2$

3) diagonal matrices are eigen vectors with eigen value 0

eigen values of K_V are $\frac{\pi^2}{4}(k_j - k_\ell)^2 = e$

eigen space 2-dim.

$\gamma(t) = \exp(tA)$ geodesic $\gamma(0) = I$

contains conjugate point at

$t = \frac{\pi}{\sqrt{e}}, \frac{2\pi}{\sqrt{e}}, \frac{3\pi}{\sqrt{e}}, \cdots$ ($\leftarrow$ Th. 20.5)

$= \frac{2}{k_j - k_\ell}, \frac{4}{k_j - k_\ell}, \frac{6}{k_j - k_\ell}, \cdots$

$0 < t < 1$, $\exists$ $\frac{k_j - k_\ell}{2} - 1$ conj. pts with mutiplicity 2

$\operatorname{ind} \gamma = \sum_{k_j > k_\ell} (k_j - k_\ell - 2)$ ($\leftarrow$ index th.)

$k_1 \geq k_2 \geq \cdots \geq k_{2m}$

Case 1. $0 > k_m$

$k_1 \geq 3 \quad \therefore \ ind \geq \sum_{j=1}^{m+1}(3 - k_{m-1+j} - 2)$

$\geq \sum_{j=1}^{m+1}(3-(-1)-2) = 2m+2$

Case 1′ $k_{m+1} > 0$

$k_{2m} \leq -3 \quad \therefore \ ind \geq \sum_{j=1}^{m+1}(k_j - (-3) - 2)$

$\geq \sum_{j=1}^{m+1}(1-(-3)-2) = 2m+2$

Case 2 $k_m > 0 > k_{m+1}$

$k_1 \geq 3, \quad k_{2m} \leq -3$

index $\geq \sum_{j=1}^{m-1}(3 - k_{m+j} - 2) + \sum_{j=1}^{m-1}(k_{j+1} - (-3) - 2)$

$+ k_1 - k_{2m} - 2$

$\geq \sum_{j=1}^{m-1}(3-(-1)-2) + \sum_{j=1}^{m-1}(1-(-3)-2)$

$+ 3 - (-3) - 2$

$= 2(m-1) + 2(m-1) + 4 = 4m \geq 2m+2$

//