モース理論入門 第13回

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ホモトピー論の速習コース

$I = [0,1]$, X, Y topological spaces
$B \subset A \subset X$, $D \subset C \subset Y$

$f, g : (X, A, B) \to (Y, C, D)$ continuous $f(A) \subset C$, $f(B) \subset D$, $g(A) \subset C$, $g(B) \subset D$

$f \simeq g \overset{\text{def.}}{\iff} \exists H : (X \times I, A \times I, B \times I) \to (Y, C, D)$
homotopic $H(x, 0) = f(x)$, $H(x, 1) = g(x)$

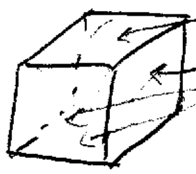
$[(X, A, B), (Y, C, D)]$: set of homotopy classes

$(X, A, B) \simeq (Y, C, D) \overset{\text{def}}{\iff} \exists f : (X, A, B) \to (Y, C, D)$
homotopy equivalent $\exists g : (Y, C, D) \to (X, A, B)$
s.t. $g \circ f \simeq id_X$, $f \circ g \simeq id_Y$

$[(X, A), (Y, C)] := [(X, A, *), (Y, C, *)]$ $*$ base point
$[X, Y] := [(X, *, *), (Y, *, *)]$

$I^n = \overbrace{I \times I \times \cdots \times I}^{n}$, $\partial I^n \supset J^{n-1} := I \times \partial I^{n-1} \cup \{1\} \times I^{n-1}$

I^2  $\leftarrow J^1$

I^3  J^2

∂I^n フタアリハコ
J^{n-1} フタナシハコ

$X \supset A \ni *$

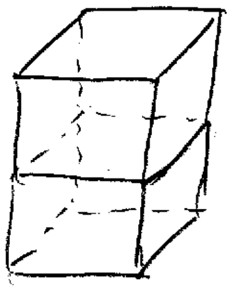
$n \geq 1$

$\pi_n(X, A) := [(I^n, \partial I^n, J^{n-1}), (X, A, *)]$
$\underset{\text{as set}}{\cong} [(D^n, \partial D^n, *), (X, A, *)]$

$\pi_1(X,A)$ set

$n \geq 2$ $\pi_n(X,A)$ group

$$(g*f)(t_1,\cdots,t_{n-1},t_n) := \begin{cases} f(t_1,\cdots,t_{n-1},2t_n) & (0 \leq t_n \leq \frac{1}{2}) \\ g(t_1,\cdots,t_{n-1},1-2t_n) & (\frac{1}{2} \leq t_n \leq 1) \end{cases}$$



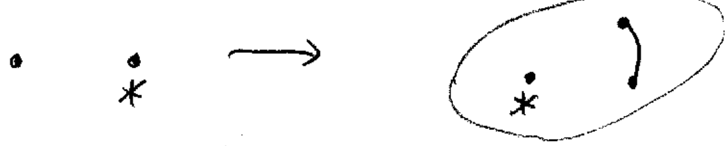
$n \geq 3$ $\pi_n(X,A)$ abel group

$$\pi_n(X) := \pi_n(X,*,*) \quad (n \geq 1) \quad \text{group}$$
$$= [(I^n, \partial I^n), (X,*)]$$
$$\underset{\text{as set}}{\cong} [(S^n,*),(X,*)]$$

$n \geq 2$ $\pi_n(X)$ abel group.

$\pi_0(X) := [(S^0,*),(X,*)]$

set of path components of X



$$\pi_0(X,A) := \pi_0(X) / \underbrace{\mathrm{Im}(i_* : \pi_0(A) \to \pi_0(X))}_{1点につぶす}$$

◎ homotopy exact sequence

$$\cdots \to \pi_{i+1}(X,A) \to \pi_i(A) \to \pi_i(X) \to \pi_i(X,A) \to$$
$$\cdots \to \pi_1(A) \to \pi_1(X) \to \pi_1(X,A) \to \pi_0(A) \to \pi_0(X) \to \pi_0(X,A) \to 0$$

exact.

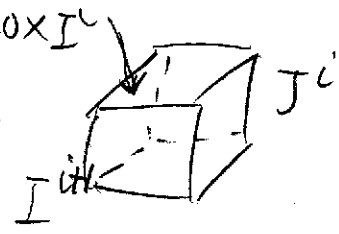
ココに制限

$[f] \in \pi_{i+1}(X, A)$

$f : (I^{i+1}, \partial I^{i+1}, J^i) \to (X, A, *)$

$f|_{\{0\} \times I^i} : (I^i, \partial I^i) \to (A, *)$

$[f|_{\{0\} \times I^i}] \in \pi_i(A)$



X : topological space $\quad p, q \in X$

$\Omega^*(X, p, q) = \{ \omega : [0,1] \to X \mid \omega \text{ conti.}, \ \omega(0) = p \ \ \omega(1) = q \}$

$\pi_0(X) = 0 \Rightarrow \forall p, p', q, q' \in X$

$\qquad \Omega^*(X; p, q) \simeq \Omega^*(X, p', q')$

$p = q = * \quad \Omega^*(X)$ loop space

(M : smooth manifold , $\Omega^*(M, p, q) \simeq \Omega(M, p, q) \simeq \Omega(M)$)

● $\pi_i(\Omega^* X) \cong \pi_{i+1}(X)$

$\quad \underset{[f]}{\Psi} \qquad f : (I^n, \partial I^n) \to (\Omega^* X, c_*)$ $\quad$ constant $*$ loop

Define

$F : (I^{n+1}, \partial I^{n+1}) \to (X, *)$ by

$F(t_1, \cdots, t_n, t_{n+1}) := f(t_1, \cdots, t_n)(t_{n+1})$

$[F] \in \pi_{n+1}(X)$.

Lemma : $g : (I^r, \partial I^r, J^{r-1}) \to (X, A, *)$

$X = A \cup e^\lambda \quad e^\lambda : \lambda\text{-cell} \quad , \quad r < \lambda$

(with $\varphi : I^\lambda \to X$ conti. $\varphi(I^\lambda) = \overline{e^\lambda}$, $\varphi(\partial I^\lambda) \subset A$,

$\varphi|_{I^\lambda \setminus \partial I^\lambda} : I^\lambda \setminus \partial I^\lambda \to e^\lambda$ homeo.)

$\Rightarrow g \simeq \exists g' : (I^r, \partial I^r, J^{r-1}) \to (X, A, *)$

$\qquad g'(I^r) \subset A \quad , \quad [g] = 0 \in \pi_r(X, A)$ //

食パンの耳の一辺だけ残って、中身を のこりの耳にあたためむ

§22

M: connected smooth manifold

$\omega : [0,1] \to M$ piecewise smooth path

$\overset{\text{def}}{\Longleftrightarrow}$ 0) ω continuous

1) $\exists$ subdivision of $[0,1]$, $0 = t_0 < t_1 < \cdots < t_k = 1$

$\omega|[t_{i-1}, t_i]$ C^∞ $(i = 1, 2, \cdots, k)$

$p, q \in M$

$\Omega(M; p, q) = \{\omega \mid \omega : [0,1] \to M \text{ piecewise smooth } \omega(0) = p, \omega(1) = q\}$

$\omega \in \Omega(M; p, q)$

arclength $S_\omega(t) := \int_0^t \|\dot{\omega}(t)\| dt$

$\omega, \eta \in \Omega(M; p, q)$,

$$d(\omega, \eta) := \max_{0 \le t \le 1} \underbrace{\rho(\omega(t), \eta(t))}_{\rho:\ \text{metric distance on } M} + \left[\int_0^1 \left(\frac{dS_\omega(t)}{dt} - \frac{dS_\eta(t)}{dt} \right)^2 dt \right]^{\frac{1}{2}}$$

$\Omega(M; p, q)$ topological space induced by d.

$E : \Omega(M; p, q) \to \mathbb{R}$ Energy functional

$\Omega =$

$$E(\omega) := \int_0^1 \|\dot{\omega}(t)\|^2 dt$$

p, q のとり方による

- $\Omega(M; p, q)$ のホモトピー型は p, q に依らない
- $\{E$ の臨界点$\} = \{M$ の p から q への測地線$\}$
- generic な（ほとんどすべての）p, q については、E は非退化な臨界点のみを持ち、非退化な臨界点のみをもつ関数で有限次元近似される（モース関数、§16）
  $\to$ Ω のホモトピー型が測地線の index でわかる
- 特別な p, q に対して、Energy E を調べると何がわかるか？（$\leftarrow$ Bott）

M complete, $p, q \in M$ $\rho(p, q) = \sqrt{d}$

最短 長さ $\sqrt{d}$ エネルギー d

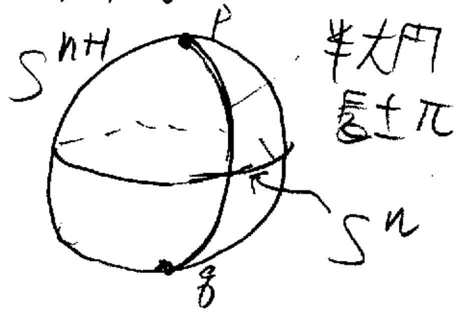
$\Omega^d = \{\omega \in \Omega \mid E(\omega) = d\}$ space of <u>minimal</u> geodesics from p to q

Example $M = S^{n+1}$ unit sphere in $\mathbb{R}^{n+2}$

p, q : anti-podal (対蹠点, 対心点) $\rho(p,q) = \pi$

space of minimal geodesics

$$\Omega^{\pi^2} = \Omega(M; p, q)^{\pi^2} \cong S^n$$



$E : \Omega(M, p, q) \to \mathbb{R}$ genenerate

(p, q を結ぶ測地線で, p, q が共役になるものが存在する. というか, p, q を結ぶすべての測地線について p, q は共役)

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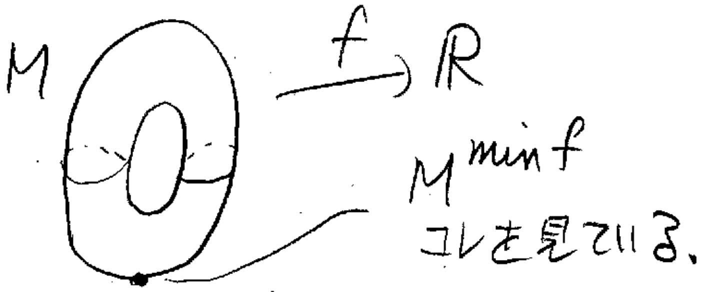
§1 の場合のアナロジーで言うと



Theorem 22.1 M complete connected Riemann. manifold

$\left(\begin{array}{l} p, q \in M \quad d := \rho(p,q)^2 \\ \Omega^d = \text{space of minimal geodesics from } p \text{ to } q \end{array} \right)$

Ω^d : topological manifold, λ_0 positive integer

$\forall$ non-minimal geodesic from p to q has index $\geq \lambda_0$

$$\Rightarrow \quad \pi_i(\Omega, \Omega^d) = 0 \qquad (0 \leq i < \lambda_0)$$

Cor. 22.2

$$\begin{aligned} \pi_i(\Omega^d) &\cong \pi_i(\Omega) \quad (0 \leq i < \lambda_0 - 1) \\ &\cong \pi_i(\Omega^*) \quad (\S 17) \\ &\cong \pi_{i+1}(M) \end{aligned}$$

Proof of Cor. 22.2

$$\to \pi_{i+1}(\Omega, \Omega^d) \to \pi_i(\Omega^d) \to \pi_i(\Omega) \to \pi_i(\Omega, \Omega^d) \to \cdots$$

exact

$i \le \lambda_0 - 2 \quad \therefore \; i+1 \le \lambda_0 - 1 < \lambda_0$ loopspace

$\therefore \; \pi_i(\Omega^d) \cong \pi_i(\Omega) \cong \pi_i(\Omega(M)) \cong \pi_{i+1}(M)$ //

Corollary 22.3 $\quad \pi_i(S^n) \cong \pi_{i+1}(S^{n+1}) \quad (i \le 2n-2)$

(!) $M = S^{n+1}, \; \Omega^d \approx S^n$ //

Corollary $\quad H_i(\Omega^d, \mathbb{Z}) \cong H_i(\Omega(M), \mathbb{Z}) \quad (0 \le i < \lambda_0 - 1)$

(!) Relative Hurewicz Theorem (服部「位相幾何学」岩波 定理11.22 p.404 / 小松・中岡・菅原「位相幾何学I」岩波 定理1.4 p.550) + homology exact sequence //

To show Th. 22.1, prepare

Lemma 22.4 $\quad K \subset \mathbb{R}^n$ compact subset

$K \subset U \subset \mathbb{R}^n$ open neighborhood of K.

$f : U \to \mathbb{R}$ smooth function

$\forall p \in K \cap C(f)$ critical point has index $\ge \lambda_0$ とする

$g : U \to \mathbb{R}$ smooth function, $\exists \varepsilon > 0$

$$\left| \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \right| < \varepsilon, \quad \left| \frac{\partial^2 g}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_i \partial x_j} \right| < \varepsilon \quad \text{on } K$$

$(i, j = 1, 2, \cdots n)$

$\Rightarrow \forall q \in K \cap C(g)$ critical point of g in K

has index $\ge \lambda_0$

Proof of Lemma 22.4

$k_g(x) := \sum_{i=1}^{n} \left|\frac{\partial g}{\partial x_i}(x)\right|$. $k_g(x) = 0 \Leftrightarrow x \in C(g)$

$e^1_g(x) \le e^2_g(x) \le \cdots \le e^n_g(x)$: eigenvalues of $\underset{H_g}{\left(\frac{\partial^2 g}{\partial x_i \partial x_j}(x)\right)}$.

index of Hessian H_g at $x \ge \lambda \Leftrightarrow e^{\lambda}_g(x) < 0$

$e^1_g, e^2_g, \cdots, e^n_g : U \to \mathbb{R}$ continuous

Set $m_g(x) := \max\{k_g(x), -e_g^{\lambda_0}(x)\}$

for the given positive integer λ_0

$\forall f : U \to \mathbb{R}$

($\forall p \in K \cap C(f)$, index H_f at $p \ge \lambda_0$) $\Leftrightarrow m_f > 0$ on K

Let $\delta := \min_{x \in K} m_f(x) > 0$ (K : compact)

Take $\varepsilon > 0$ sufficiently small such that, for $\forall x \in K$

$|k_g(x) - k_f(x)| < \delta$, $|e_g^{\lambda_0}(x) - e_f^{\lambda_0}(x)| < \delta$

(gとfの1階と2階の偏導関数が近いと k_g と k_f は近い
H_g と H_f も近い．固有値も近い．$e_g^{\lambda_0}$ と $e_f^{\lambda_0}$ も近い)

$\therefore$ $k_g(x) > 0$, $e_g^{\lambda_0}(x) > 0$ on K

$\therefore$ $m_g > 0$ on K $\forall q \in K \cap C(g)$ ind H_g at $q \ge \lambda_0$ //

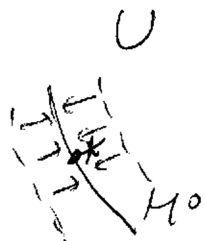
Let us show an analogue of Th. 22.1

Lemma 22.5 : M smooth manifold, $f : M \to \mathbb{R}$ smooth function
$\min(f) = 0$ (最小値が0), $M^c := f^{-1}[0, c]$ compact
M^0 topological manifold,
$\forall$ critical point in $M \setminus M^0$ has index $\ge \lambda_0$
$\Rightarrow$ $\pi_r(M, M^0) = 0$ ($0 \le r < \lambda_0$)

Proof of Lemma 22.5

$M^0 \subset^{\exists} U \subset M$ neighborhood of M^0 in M

U is a deformation retract of M^0



Fix a base point $* \in M^0$

Take any continuous map

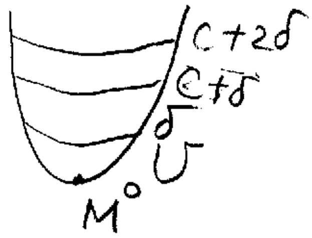
$h : (I^r, \partial I^r, J^{r-1}) \to (M, M^0, *)$, $(r < \lambda_0)$

Claim : $h \simeq {}^{\exists} h'' : (I^r, \partial I^r, J^{r-1}) \to (M^0, M^0, *)$

(!) $c := \max\{ f(x) \mid x \in h(I^r) \}$

$\delta := \frac{1}{3} \min\{ f(x) \mid x \in M^c \setminus U \} < c$

compact



Take $g : M^{c+2\delta} \to \mathbb{R}$ smooth s.t.

(0) critical points of g are non-degenerate (Cor 6.8 Part I)

(1) $|f(x) - g(x)| < \delta$ $\quad (x \in M^{c+2\delta})$

(2) $\forall$ critical point of g on $g^{-1}[2\delta, c+\delta]$ $(\subset f^{-1}[\delta, c+2\delta])$ has index $\geq \lambda_0$ (Lemma 22.4)

$g^{-1}[0, c+\delta] \simeq (g^{-1}[0, 2\delta]$ attached λ-cells, $\lambda \geq \lambda_0)$

(Th. 3.5, ParI)

Since $r < \lambda_0$

$h : (I^r, \partial I^r, J^{r-1}) \to (M, M^0, *)$, $h(I^r) \subset f^{-1}[0, c]$

$\simeq {}^{\exists} h' : (I^r, \partial I^r, J^{r-1}) \to (M, M^0, *)$ s.t.

$h'(I^r) \subset g^{-1}[0, 2\delta] \subset U$ ← deformation retract of M^0

$\simeq {}^{\exists} h'' : (I^r, \partial I^r, J^{r-1}) \to (M, M^0, *)$

$h''(I^r) \subset M^0$

//

fix a minimal geodesic

Proof of Th. 22.1

Take any $h : (I^r, \partial I^r, J^{r-1}) \to (\Omega, \Omega^d, *)$ conti.

Take $c > \min\{E(\omega) \mid \omega \in h(I^r)\}$

Then $h : (I^r, \partial I^r, J^{r-1}) \to (\underset{\overset{\|}{E^{-1}[0,c)}}{\mathrm{Int}\,\Omega^c}, \Omega^d, *)$

By §16, Part III

$\mathrm{Int}\,\Omega^c \supset \mathrm{Int}\,\Omega^c(t_0, t_1, \cdots, t_k)$ deformation retract

$(\supset \Omega^d)$

$\Omega \supset \Omega(t_0, t_1, \cdots, t_k) = \{\omega \in \Omega(M, p, q) \mid \omega \mid [t_{i-1}, t_i] \text{ geodesic}\}$

$\mathrm{Int}\,\Omega^c(t_0, t_1, \cdots, t_k) = (\mathrm{Int}\,\Omega^c) \cap \Omega(t_0, t_1, \cdots, t_k)$

B $\|$ finite dimensional manifold (Lemma 16.1)

E/B : $\forall$ critical point has index $\geq \lambda_0$

$F : [d, c) \to [0, \infty)$ diffeom.

$f = F \circ E : B \to \mathbb{R}$ satisfies the assumptions of Lemma 22.5

$h \simeq h' : (I^r, \partial I^r, J^{r-1}) \to (B, B^{min}, *)$

$\subset (\Omega, \Omega^d, *)$

$h' \simeq h'' : (I^r, \partial I^r, J^{r-1}) \to (\Omega, \Omega^d, *)$

$h''(I^r) \subset \Omega^d$

$\therefore \ \pi_r(\Omega, \Omega^d) = 0 \quad (r < \lambda_0)$ //