モース理論入門　第12回

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§20　Symmetric spaces

M: connected Riemannian manifold

M is a <u>symmetric space</u>（対称空間）

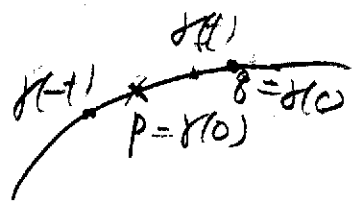
$\overset{\text{def.}}{\Longleftrightarrow}$ $\forall p \in M$, $\exists I_p : M \to M$ isometry s.t.
（等長写像）

- $I_p(p) = p$
- $\forall \gamma : (-\varepsilon, \varepsilon) \to M$ geodesic, $\gamma(0) = p$
  $I_p(\gamma(t)) = \gamma(-t)$ for sufficiently small $t \in \mathbb{R}$.

$\overset{\text{i.e.}}{\Longleftrightarrow}$ $\forall p \in M$, $\exists I_p : M \to M$ isometry,
$(I_p)_* : T_pM \to T_pM$, $(I_p)_*(v) = -v$

Lemma 20.1. $\gamma : (-\varepsilon, \varepsilon) \to M$ geodesic

$c \in (-\varepsilon, \varepsilon)$, $p = \gamma(0)$, $q = \gamma(c)$,
$\gamma(t)$, $\gamma(t+2c)$ are defined



$\Rightarrow$ ・$I_q I_p(\gamma(t)) = \gamma(t+2c)$

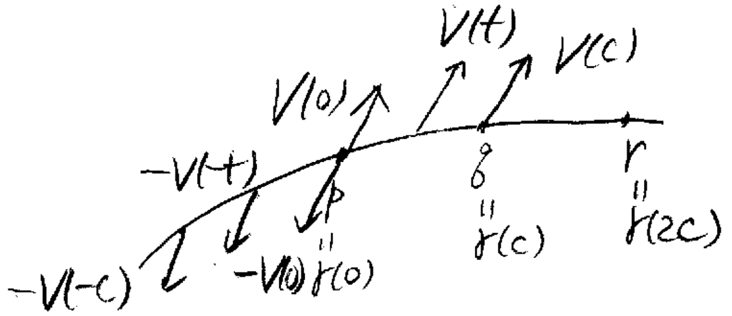
・$I_q I_p$ preserves parallel vector field along γ.

(!) $\tilde{\gamma}(t) := \gamma(t+c)$　$(t \in (-\varepsilon - c, \varepsilon - c))$

$\tilde{\gamma}$ geodesic, $\tilde{\gamma}(0) = \gamma(c) = q$

・$I_q I_p(\gamma(t)) = I_q(\gamma(-t)) = I_q(\tilde{\gamma}(-t-c)) = \tilde{\gamma}(t+c) = \gamma(t+2c)$

・V: parallel vector field along γ.
I_p: isometry $\therefore$ $(I_p)_*(V) = \tilde{V}$ parallel along $\gamma(-t)$
$(I_p)_*(V(0)) = -V(0)$ $\therefore$ $(I_{p*})(V(t)) = -(V(-t))$
$\therefore$ $(I_q I_p)_*(V(t)) = I_{q*}(-V(-t)) = V(t+2t)$ //



等長写像は平行なベクトルを平行にうつす。

Corollary 20.2 M: symmetric space $\Rightarrow$ M complete

(∵) Any geodesic γ is extended to $\gamma: \mathbb{R} \to M$ by Lemma 20.1 //

Corollary 20.3. $\forall p$, I_p is uniquely determined

(∵) Hopf-Rinow (Th. 10.9). $\forall p, q$ $\exists$ geodesic

$\gamma: \mathbb{R} \longrightarrow M$ $\quad \gamma(0) = p$, $\gamma(1) = q$.

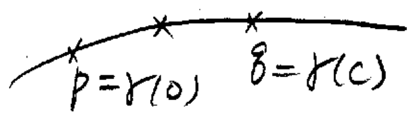
別かもしれないもの

$I_p(q) = I_p(\gamma(1)) = \gamma(-1) = \tilde{I}_p(\gamma(1)) = \tilde{I}_p(q)$

Corollary 20.4. U, V, W parallel vector fields along a geodesic γ

$\Rightarrow R(U, V)W$: parallel vector field along γ

(∵)



$T := I_{\gamma(\frac{c}{2})} I_p : M \to M$ isometry $T(p) = q$

X: parallel vector field along γ

By Lemma 20.1, $T_* U_p = U_q$, $T_* V_p = V_q$, $T_* W_p = W_q$, $T_* X_p = X_q$,

$$\langle R(U_q, V_q)W_q, X_q \rangle = \langle R(T_* U_p, T_* V_p) T_* W_p, T_* X_p \rangle \underset{T:\text{ isometry}}{=} \langle R(U_p, V_p)W_p, X_p \rangle$$

$\therefore \langle R(U, V)W, X \rangle$ constant along γ.

$$0 = \frac{d}{dt}\langle R(U,V)W, X \rangle = \langle D_{\dot\gamma} R(U,V)W, X \rangle + \langle R(U,V)W, \underbrace{D_{\dot\gamma} X}_{=0} \rangle$$

$\therefore \langle D_{\dot\gamma} R(U,V)W, X\rangle = 0$ for $\forall$ parallel vector field X
(By taking $X = P_1, \cdots, P_n$ parallel frame along γ)

$$D_{\dot\gamma} R(U,V)W = 0 \qquad //$$

◎ M : locally symmetric
$\overset{\text{def}}{\Longleftrightarrow}$ (U, V, W parallel vect. field along geodesic
$\Rightarrow R(U,V)W$ parallel

Cartan : complete, simply connected, locally symmetric
$\Rightarrow$ symmetric space.
(Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, p222, Th. 5.6)

Let M be locally symmetric
$\gamma : \mathbb{R} \to M$ a geodesic. $V := \dot\gamma(0) \in T_pM$

Define $K_V : T_pM \to T_pM$ by

$$K_V(W) := R(V,W)V$$

表現行列が対称行列

◎ K_V : self adjoint i.e. $\forall W, W' \in T_pM$

$$\langle K_V(W), W'\rangle = \langle W, K_V(W')\rangle$$

($\leftarrow$ Lemma 9.3 (4))

$\exists U_1, \cdots, U_n$: orthonormal bases of T_pM
s.t. U_i is an eigen vector of K_V
with eigen value $e_i \in \mathbb{R}$
(対称行列は直交行列により（実数上で）対角化できる)

Theorem 20.5

M : locally symmetric Riemann manifold

$\gamma : \mathbb{R} \to M$ geodesic, $p = \gamma(0)$, $V = \gamma'(0) \in T_pM$

$e_1, \cdots, e_n$ eigen values of K_V, $(K_V(W) = R(V,W)V)$.

$\Rightarrow$ The <u>conjugate</u> points of p along γ are given by

$$q = \gamma\left(\frac{\pi k}{\sqrt{e_i}}\right) \quad \text{for } k \in \mathbb{Z},\ k \neq 0,\ e_i > 0$$

(重複度こみ) (正固有値だけ)

$$\text{multiplicity of } \gamma(t) = \#\left\{e_i \;\middle|\; {}^{\exists}k \in \mathbb{Z}\setminus\{0\},\ t = \frac{\pi k}{\sqrt{e_i}}\right\}$$

(∵) Extend $U_1, \cdots, U_n \in T_pM$ to parallel frame along γ

$V := \gamma'(t)$ parallel along γ (同じ記号で紛らわしいが)

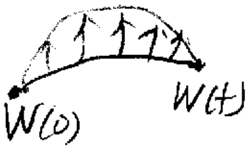
$$R(V, U_i)V = e_i U_i \quad \text{holds}$$

parallel (e_i : constant)

Let $W(t) = \sum_{i=1}^{n} w_i(t) U_i(t)$ be a vectorfield along γ

Jacobi equation : $D_{\gamma'}^2 W + \underset{R(V,W)V}{\underset{\shortparallel}{K_V(W)}} = 0$

$$\sum_i \left(\frac{d^2 w_i}{dt^2} U_i + e_i w_i U_i\right) = 0 \quad \text{i.e.}$$



$$\frac{d^2 w_i}{dt^2} + e_i w_i = 0 \qquad 1 \leq i \leq n$$

$\Big[$ $W \neq 0 \iff w_i$ are not all identically 0
$\ \ W(0) = 0 \iff w_i(0) = 0 \ (1 \leq i \leq n)$,

$e_i > 0 \Rightarrow w_i(t) = c_i \sin(\sqrt{e_i}\, t)$ ※

$$\underset{(t \neq 0)}{w_i(t) = 0} \iff \sqrt{e_i}\, t = {}^{\exists}k\pi$$

$e_i = 0 \Rightarrow w_i(t) = c_i t$, $e_i < 0 \Rightarrow w_i(t) = c_i \sinh(\sqrt{|e_i|}\, t)$

$$w_i(t) = 0 \iff t = 0$$

§21 Lie groups as symmetric spaces

G : Lie group $\quad \tau \in G$

$L_\tau : G \to G$, $L_\tau(\sigma) := \tau\sigma$,

$R_\tau : G \to G$, $R_\tau(\sigma) := \sigma\tau$

g : Riemann metric on G. $\quad g(V, W) = \langle V, W \rangle$

g : left-invariant (right-invariant)

$\overset{\text{def}}{\Longleftrightarrow}$ $g((L_\tau)_* V, (L_\tau)_* W) = g(V, W)$

for $\forall \tau \in G$, V, W vector field over G

$(g((R_\tau)_* V, (R_\tau)_* W) = g(V, W))$

Example G : compact Lie group

$\Rightarrow$ $\exists$ left & right invariant metric on G

(∵) $\exists 1$ μ : invariant measure (Haar measure) on G :

$\int \{\alpha f(x) + \beta g(x)\} d\mu(x) = \alpha \int f(x) d\mu(x) + \beta \int g(x) d\mu(x)$

$f(x) \geq 0 \Rightarrow \int f(x) d\mu(x) \geq 0$, $= 0 \Leftrightarrow f(x) \equiv 0$

$\int 1 \, d\mu(x) = 1$,

$\int f(x\tau) d\mu(x) = \int f(\tau x) d\mu(x) = \int f(x^{-1}) d\mu(x) = \int f(x) d\mu(x)$

$(f, g : G \to \mathbb{R}$ conti, $\alpha, \beta \in \mathbb{R}$, $\tau \in G)$

cf. ポントリャーギン「連続群論」上, 第5章

$\langle\langle V, W \rangle\rangle := \int_{G \times G} \langle L_{\sigma *} R_{\tau *}(V), L_{\sigma *} R_{\tau *}(W) \rangle d\mu(\sigma) d\mu(\tau)$

: left & right invariant

//

Lemma 21.1 G: Lie group with a left & right invariant metric $\Rightarrow$ G: symmetric space

$$I_\tau(\sigma) = \tau\sigma^{-1}\tau$$

(:) L_τ, R_τ : isometry $I_e : G \to G$ $I_e(\sigma) := \sigma^{-1}$.

$I_{e*} : T_eG \to T_eG$ $I_{e*}(v) = -v$ isometry at e.

($c(t)$, $\dot{c}(0) = v$, $\hat{c}(t) := c(t)^{-1}$ $c(t)\hat{c}(t) = e$ $\dot{c}(0) + \dot{\hat{c}}(0) = 0$)

$I_e = R_{\sigma^{-1}} I_e L_{\sigma^{-1}}$ $((I_{\sigma^{-1}} I_e L_{\sigma^{-1}})(x) = (\sigma^{-1}x)^{-1}\sigma^{-1} = x^{-1})$

: isometry at $\sigma \in G$

$\therefore I_e : G \to G$ isometry, $I_e(\gamma(t)) = \gamma(-t)$.

$I_\tau : G \to G$, $I_\tau(\sigma) := \tau\sigma^{-1}\tau$: isometry

$I_\tau(\gamma(t)) = \gamma(-t)$ for geodesic γ with $\gamma(0) = \tau$. //

Lemma 21.2 G: Lie group with left-right invariant metric. The geodesics γ in G with $\gamma(0) = e$ are precisely the one-parameter subgroups $\gamma : \mathbb{R} \to G$ (smooth group homomorphisms).

(:) 省略

X: vector field over G.

$[X, Y] = -[Y, X]$
alternative

X: <u>left invariant</u>

$\overset{def}{\Longleftrightarrow}$ $(L_a)_* X_b = X_{ab}$ $(a, b \in G)$

$\mathfrak{g} := $ (left invariant vector field over G)

Lie algebra by Lie bracket $[\ ,\ ]$

$\cong$ as vector space

T_eG

Jacobi identity

$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

Theorem 21.3 $X, Y, Z, W \in \mathfrak{g}$, Then

a) $\langle [X,Y], Z \rangle = \langle X, [Y,Z] \rangle$

b) $R(X,Y)Z = \frac{1}{4}[[X,Y],Z]$

c) $\langle R(X,Y)Z, W \rangle = \frac{1}{4}\langle [X,Y],[Z,W] \rangle$

(Gの曲率がリー環で調べられる)

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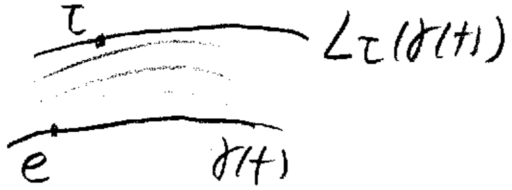
① $X \in \mathfrak{g}$ left invariant $\therefore \nabla_X X = 0$

(integral curves of X are left translations of 1-parameter subgroups



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$$0 = \nabla_{X+Y}(X+Y) = \nabla_X X + \nabla_X Y + \nabla_Y X + \nabla_Y Y$$
$$= \nabla_X Y + \nabla_Y X$$

$\nabla_X Y - \nabla_Y X = [X,Y]$ (Levi-Civita connection torsion free) Lemma 8.6

$\therefore$ d) $2\nabla_X Y = [X,Y]$

$\langle X, Z \rangle$: constant function

$$0 = Y\langle X,Z \rangle = \langle \nabla_Y X, Z \rangle + \langle X, \nabla_Y Z \rangle$$
$$\underset{\text{d)}}{\therefore} \langle [Y,X], Z \rangle + \langle X, [Y,Z] \rangle = 0$$
$$\therefore \langle [X,Y], Z \rangle = -\langle [Y,X], Z \rangle = \langle X, [Y,Z] \rangle \quad \cdots \text{a)}$$

$$R(X,Y)Z := -\nabla_X(\nabla_Y Z) + \nabla_Y(\nabla_X Z) + \nabla_{[X,Y]} Z$$
$$\underset{(d)}{=} -\frac{1}{4}[X,[Y,Z]] + \frac{1}{4}[Y,[X,Z]] + \frac{1}{2}[[X,Y],Z]$$
$$= \frac{1}{4}[[Y,Z],X] - \frac{1}{4}[[X,Z],Y] + \frac{1}{2}[[X,Y],Z]$$
$$\underset{\text{Jacobi}}{=} \frac{1}{4}[[X,Y],Z] \quad \cdots \text{b)}$$

$$\langle R(X,Y)Z, W\rangle \overset{(b)}{=} \frac{1}{4}\langle [[X,Y],Z], W\rangle$$
$$\underset{(a)}{=} \frac{1}{4}\langle [X,Y],[Z,W]\rangle \quad \cdots (c)$$

Corollary 21.4 sectional curvature ≥ 0

$$\langle R(X,Y)X,Y\rangle = \frac{1}{4}\langle [X,Y],[X,Y]\rangle$$

is non-negative, $= 0 \Leftrightarrow [X,Y] = 0$

$\mathfrak{g}$: Lie algebra

$\mathfrak{C} := \{X \in \mathfrak{g} \mid \forall Y \in \mathfrak{g},\ [X,Y] = 0\}$

center of $\mathfrak{g}$

Corollary 21.5 G : Lie group with left & right invariant metric, $\mathfrak{C} = 0$

$\Rightarrow G$: compact, $\pi_1(G)$: finite group

(!) $X_1 \in \mathfrak{g}$, $\|X_1\| = 1$

$X_1, X_2, \cdots, X_n$ orthonormal frame of $\mathfrak{g}$

$\exists i\ [X_1, X_i] \neq 0 \quad (\leftarrow \mathfrak{C} = 0)$ $M = G$

(Ricci curvature $K : T_pM \times T_pM \to \mathbb{R}$

$K(U_1, U_2) := \mathrm{trace}\,(W \mapsto R(U_1, W)U_2)$)

$$K(X_1, X_1) = \mathrm{trace}\,(W \mapsto R(X_1, W)X_1)$$
$$= \sum_{i=1}^{n} \langle R(X_1, X_i)X_1, X_i\rangle$$
$$= \frac{1}{4}\sum_{i=2}^{n} \langle [X_1, X_i], [X_1, X_i]\rangle > 0$$

$\{X_1 \mid X_1 \in \mathfrak{g}, \|X_1\| = 1\}$ compact

$\exists r>0 \quad K(X_1, X_1) \geq \frac{n-1}{r^2} > 0$ ($n \geq 2$)

By Cor. 19.5, G : compact

$\tilde{G}$: universal covering. $\tilde{G}$: compact
$\therefore$ $\pi_1(G)$: finite. //

Corollary 21.6 G : Lie group with left & right invariant metric, simply connected,
$\Rightarrow G \cong G' \times \mathbb{R}^k$ as Lie group
G' : compact Lie group $\mathfrak{c}' = 0$ (center of Lie alg of G')

(∵) $\mathfrak{g}' := \{X \in \mathfrak{g} \mid \forall C \in \mathfrak{c},\ \langle X, C\rangle = 0\}$ (G の center)
$= \mathfrak{c}^\perp \subset \mathfrak{g}$

$\mathfrak{g}'$; Lie subalgebra,
($X, Y \in \mathfrak{g}',\ C \in \mathfrak{c}$ ← Th. 21.3 (a)
$\langle [X, Y], C\rangle = \langle X, [Y, C]\rangle = 0$)
$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{c}$

G ; simply connected $G = G' \times G''$
G' compact ($\mathfrak{g}'$ の center $= 0$, Cor 21.5)
G'' simply connected, Abelian (← $\mathfrak{c}$: commutative)
$G'' \cong {}^\exists \mathbb{R}^k$

//

Theorem 21.7 (Bott)
G: compact, simply connected Lie group
loop space $\Omega(G) = \Omega(G; e, e)$ has the homotopy type of a CW-complex with no odd dimensional cells and with only finitely manny λ-cells for $\forall \lambda$.

$$\Omega(G) \simeq \bigcup_{k=0}^{\infty} \left(e^{2k_1} \cup \cdots \cup e^{2k_{l(k)}}\right)$$

e^{λ}
λ-cell

Example $\Omega(S^3) \simeq e^0 \cup e^2 \cup e^4 \cup e^6 \cup \cdots$ (Cor 17.4)

Proof of Th. 21.7 $\Omega(G) \simeq \Omega(G; p, q)$,
p and q are not conjugate along any geodesic from p to q.
By Th. 17.3, $\Omega(G; p, q) \simeq \bigcup_{\mathrm{ind}(\gamma)=\lambda} e^{\lambda}$: CW-complex consisting of λ-cells corresponding to geodesics of index λ. By Th. 19.6, there are finite number of λ-cells for each λ.
Claim: For each geodesic $\gamma: [0,1] \to G$, $\gamma(0) = p$, $\gamma(1) = q$, index of γ is even.
(:) $\gamma'(0) = V \in T_pG$

$K_V : T_pG \to T_pG$

$K_V(W) = R(V, W)V = \frac{1}{4}[[V, W], V]$

Define $\mathrm{Ad}\,V : \mathfrak{g} \to \mathfrak{g}$

$\mathrm{Ad}\,V(W) := [V, W]$ regarding V as a left invariant vect. field $\in \mathfrak{g}$

Then
$K_V = -\frac{1}{4}(\mathrm{Ad}\,V) \circ (\mathrm{Ad}\,V)$

$\left(K_V(W) = -\frac{1}{4}[V, [V, W]] \right)$

$\langle \mathrm{Ad}\,V(W), W' \rangle = -\langle W, \mathrm{Ad}\,V(W') \rangle$

($\leftarrow$ Th. 21.3 (a))

$\therefore$ $\mathrm{Ad}\,V$; skew-symmetric.

(表現行列が ${}^tA = -A$)

For an orthonormal basis of $\mathfrak{g}$

$$\mathrm{Ad}\,V \sim \begin{pmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & 0 & a_2 & \\ & & -a_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

$$(\mathrm{Ad}\,V)\circ(\mathrm{Ad}\,V) \sim \begin{pmatrix} -a_1^2 & & & & \\ & -a_1^2 & & & \\ & & -a_2^2 & & \\ & & & -a_2^2 & \\ & & & & \ddots \end{pmatrix}$$

$$K_V = -\frac{1}{4}(\mathrm{Ad}\,V)\circ(\mathrm{Ad}\,V) \sim \frac{1}{4}\begin{pmatrix} a_1^2 & & & & \\ & a_1^2 & & & \\ & & a_2^2 & & \\ & & & a_2^2 & \\ & & & & \ddots \end{pmatrix}$$

Each eigenvalue of K_V has even multiplicity.

By Th. 20.5, the multiplicity of $\gamma(t)$ along $\gamma|_{[0,t]}$ must be even.

By index theorem (Th. 15.1), index of γ is even //