モース理論入門 第11回

11-1

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§19 Some retations between topology and curvatures

(M, g) Riemannian manifold, $g(X,Y) = \langle X, Y \rangle$ inner product

$R(X,Y)Z$: curvature tensor

Lemma 19.1 $\forall p \in M$, $\forall A, B \in T_pM$
$\langle R(A,B)A, B \rangle \leq 0$
$\Rightarrow$ $\forall p, q \in M$, p, q are <u>not conjugate</u> along any geodesic from p to q.

(証) γ : geodesic from p to q ($\gamma : [0,1] \to M$)
J : Jacobi field along γ

$$D_{\dot\gamma}^2 J + R(\dot\gamma, J)\dot\gamma = 0$$

$$\therefore \langle D_{\dot\gamma}^2 J, J \rangle + \langle R(\dot\gamma, J)\dot\gamma, J \rangle = 0$$

$$\therefore \langle D_{\dot\gamma}^2 J, J \rangle = -\langle R(\dot\gamma, J)\dot\gamma, J \rangle \geq 0 \quad \text{by the assumption}$$

function on $[0,1]$

$$\frac{d}{dt}\langle D_{\dot\gamma} J, J \rangle = \langle D_{\dot\gamma}^2 J, J \rangle + \langle D_{\dot\gamma} J, D_{\dot\gamma} J \rangle \geq 0$$

$\therefore$ $\langle D_{\dot\gamma} J, J \rangle$ (weakly) monotonically increasing

Suppose $J(0) = 0$, $J(1) = 0$.

Then $\langle D_{\dot\gamma} J, J \rangle(0) = 0$, $\langle D_{\dot\gamma} J, J \rangle(1) = 0$

$\therefore$ $\langle D_{\dot\gamma} J, J \rangle \equiv 0$ on $[0,1]$ → $J(0)=0$, $D_{\dot\gamma}J(0)=0$ から $J\equiv 0$ を示してもよい

$\therefore$ $\frac{d}{dt}\langle J, J \rangle = 2\langle D_{\dot\gamma} J, J \rangle \equiv 0$

$\therefore$ $J \equiv 0$.

$\therefore$ p & q are not conj. along γ

$n \geqq 2$

Remark $A, B \in T_pM$ unit orthogonal vectors

$\langle R(A,B)A, B \rangle \in \mathbb{R}$ <u>sectional curvature</u>

$=$ Gaussian curvature of $(u_1, u_2) \mapsto \exp_p(u_1 A + u_2 B)$ at p

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Theorem 19.2 (Cartan) connected, $\pi_1(M) = \{e\}$

M: simply connected, complete Riemann. manifold

sectional curvature $\langle R(A,B)A, B \rangle \leq 0$ at $\forall p \in M$ 断面曲率が非正

$\Rightarrow$ • $\forall p, q \in M$ $\exists 1$ geodesic from p to q

• $M \cong \mathbb{R}^{\dim M}$ (diffeom.)

(証) By Lemma 19.1 and the Morse's index theorem (Th.15.1)

$\forall$ geodesic has index 0

$\Omega = \Omega(M; p, q) \underset{\text{homot. equiv.}}{\simeq} {}^{\exists} 0$-dim CW-complex (each cell corresponds to a geodesic from p to q) Th.17.3 基本定理

M: simply connected

$(\forall \omega, \omega' \in \Omega \quad \omega \simeq \omega' \text{ rel } 0 \,\&\, 1)$

$\therefore$ Ω connected $\quad \therefore \Omega \simeq 1$-point

$\therefore$ $\exists 1$ geodesic from p to q.

Now take $p \in M$

$\exp_p : T_pM \to M$ injective, surjective $\exists 1$

By Th.18.1, $\exp_p$ has no critical point.

$(\exp_p)^{-1} : M \to T_pM$ smooth

$\therefore \exp_p : T_pM \xrightarrow{\cong} M$ diffeo

$T_pM \cong \mathbb{R}^n$ if $\dim M = n$. //

Corollary 19.3. M complete Riemann manifold
sectional curvature $\langle R(A,B)A, B\rangle \leq 0$
$\Rightarrow$ i-th homotopy group $\pi_i(M) = 0$ $(i \geq 2)$

(証) Take the universal covering $pr : \tilde{M} \to M$ (普遍被覆)
$\tilde{M}$: complete Riemann manifold,
sectional curvature ≤ 0

By Th. 19.2 $\tilde{M} \cong \mathbb{R}^n$ diffeo. $(n = \dim M)$

$$pr_* : \pi_i(\tilde{M}) \longrightarrow \pi_i(M)$$
$$[f] \longmapsto [pr \circ f]$$
$$f : (D^i, \partial D^i) \longrightarrow (\tilde{M}, *) \quad \text{base point}$$

By the homotopy lifting property (of covering space)

$$\begin{tikzcd} & (\tilde{M}, *) \arrow[d] \\ (D^i, \partial D^i) \arrow[ur, dashed, "\tilde{g}"] \arrow[r, "g"'] & (M, *) \end{tikzcd}$$

$\partial D^i = S^{i-1}$ path connected

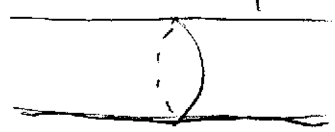
pr_* : surjective if $i \geq 2$

$\pi_i(\tilde{M}) \cong \pi_i(\mathbb{R}^n) = 0$

$\therefore$ $\pi_i(M) = 0$ if $i \geq 2$ //

※ Milnor先生の講義録では Cor. 19.3でさらに「$\pi_1(M)$は単位元以外には位数有限の元を持たない」ということも示しているが群のコホモロジーの概念が必要なので、ここでは省略する

例 M $\leftarrow$ $\tilde{M} = \mathbb{R}^2$



sectional curvature $= 0$, $\pi_1(M) \cong \mathbb{Z}$.

(M, g) Riemannian manifold, $(g(X,Y) = \langle X, Y \rangle)$
$R(X,Y)Z \left(= -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z\right)$ curvature tensor
$\langle R(A,B)A, B \rangle$: sectional curvature (of direction $\mathbb{R}A + \mathbb{R}B$)

Ricci tensor $K = K_p : T_pM \times T_pM \to \mathbb{R}$

$$K(U_1, U_2) := \mathrm{trace}\left(W \mapsto R(U_1, W)U_2 : T_pM \to T_pM\right)$$
$$(U_1, U_2 \in T_pM)$$

Claim: $U_1, \cdots, U_n$: orthonormal basis of T_pM.

$$\Rightarrow K(U_n, U_n) = \sum_{i=1}^{n-1} \langle R(U_n, U_i)U_n, U_i \rangle$$

(∵) $U_i \mapsto \underbrace{R(U_n, U_i)U_n}_{\in T_pM} = \sum_{k=1}^{n} r_{ik} U_k$ (とおく)

$$K(U_n, U_n) = \sum_{i=1}^{n} r_{ii} \quad \text{(表現行列の対角成分の和)}$$

$$U_n \mapsto R(U_n, U_n)U_n = 0 \quad (\leftarrow \text{Lemma 9.3})$$

$\therefore$ $r_{nn} = 0$

$$K(U_n, U_n) = \sum_{i=1}^{n-1} r_{ii}, \quad \langle R(U_n, U_i)U_n, U_i \rangle = r_{ii} \quad //$$

Theorem 19.4 (Myers) $r > 0$

$\forall p \in M$, $\forall U \in T_pM$, $\|U\| = 1$, $K(U, U) \geq \dfrac{n-1}{r^2}$

$\Rightarrow$ $\forall$ geodesic $\gamma : [0,1] \to M$ with length $> \pi r$,

- γ has a positive index
- $\exists \tau \in (0,1)$, $\gamma(0)$ & $\gamma(\tau)$ conjugate along γ
- γ is not minimal (最短でない)

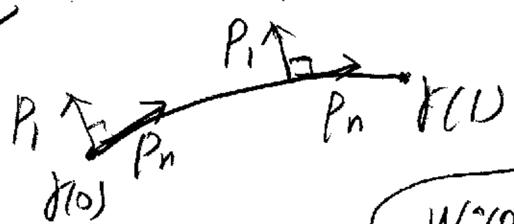
Proof of Th. 19.4

$\gamma : [0,1] \to M$ geodesic of length L

$P_1, \cdots, P_n$ parallel orthonormal frame along γ

P_n : tangent to γ

$\langle P_i, P_j \rangle = \delta_{ij}$



$W_i(0)=0, \ W_i(1)=0$

$\dot\gamma = L P_n, \quad \nabla_{\dot\gamma} P_i = 0$

Set $W_i(t) := (\sin \pi t) P_i(t) \in T_\gamma \Omega$ $\quad p=\gamma(0), \ q=\gamma(1)$

By 2nd variation formula Th 13.1,

$$\frac{1}{2}E_{**}(W_i, W_i) = -\int_0^1 \langle W_i, \nabla_{\dot\gamma}^2 W_i + R(\dot\gamma, W_i)\dot\gamma \rangle dt$$

$$= \int_0^1 \langle (\sin\pi t)P_i, \pi^2(\sin\pi t)P_i + R(LP_n, (\sin\pi t)P_i)LP_n \rangle dt$$

$$\left(\nabla_{\dot\gamma}^2((\sin\pi t)P_i) = -\pi^2(\sin\pi t)P_i \quad \because \ \nabla_{\dot\gamma}P_i = 0 \right)$$

$$= \int_0^1 (\sin\pi t)^2 \{ \pi^2 - L^2 \langle P_i, R(P_n, P_i)P_n \rangle \} dt$$

$$\frac{1}{2}\sum_{i=1}^{n-1} E_{**}(W_i, W_i)$$

$$= \int_0^1 (\sin\pi t)^2 \{ (n-1)\pi^2 - L^2 \overbrace{K(P_n, P_n)}^{\text{Ricci}} \} dt$$

Now $K(P_n, P_n) \geq \dfrac{n-1}{r^2}$ $\quad L > \pi r$

$\therefore \ (n-1)\pi^2 - L^2 K(P_n, P_n) < 0$

$\therefore \ \dfrac{1}{2}\sum_{i=1}^{n-1} E_{**}(W_i, W_i) < 0$

$\exists i$ $(1 \le i \le n-1)$ $E_{**}(W_i, W_i) < 0$

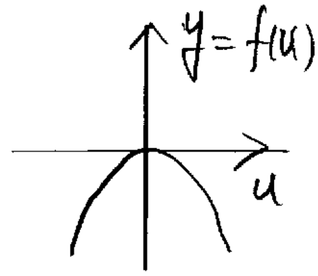
index of E_{**} at γ is positive.

By index theorem (Th.15.1), $\exists \tau$, $0 < \tau < 1$

$\gamma(0)$ and $\gamma(\tau)$ are conjugate along γ.

Let $\bar{\alpha} : (-\varepsilon, \varepsilon) \to \Omega$ be a variation with the variational vector field W_i

Set $f(u) := E(\bar{\alpha}(u))$.



Then $f'(0) = 0$ ($\because$ γ : geodesic)

$f''(0) = E_{**}(W_i, W_i) < 0$

$\therefore \exists u \neq 0$, $\underset{E(\bar{\alpha}(u))}{\underset{\shortparallel}{f(u)}} < f(0) = E(\gamma)$

$\therefore$ γ is not minimal. //

Example $M = S^n(r) = \{x \in \mathbb{R}^{n+1} \mid \|x\| = r\}$ $r > 0$

sectional curvature $= \frac{1}{r^2}$, $K(U, U) = \frac{n-1}{r^2}$

Any geodesic of length $> \pi r$ contains a conj. pt.

Corollary 19.5 $r > 0$, complete, $\forall U$: unit vector

$K(U, U) \ge \frac{n-1}{r^2} > 0$

$\Rightarrow$ M : compact, diameter of $M \le \pi r$

$\sup\{\rho(p, q) \mid p, q \in M\}$

($\because$) $p, q \in M$, γ : minimal geod. from p to q.

metric distance

By Th 19.4 length of $\gamma \le \pi r$ $\therefore \rho(p, q) \le \pi r$

M : closed bounded set in complete M. $\therefore M$ compact //

Theorem 19.6 M : compact Riemann manifold ($n \geq 2$)
Ricci tensor is <u>positive</u> definite
$\Rightarrow$ $\Omega(M; p, q) \simeq {}^{\exists}$ countable CW complex with finite cells for each dim.

(!) $K(U,U)$ positive continuous function on
$UTM := \{ U \mid U \in T_pM \text{ for } {}^{\exists} p \in M, \|U\| = 1 \}$
compact (sphere bundle over M)

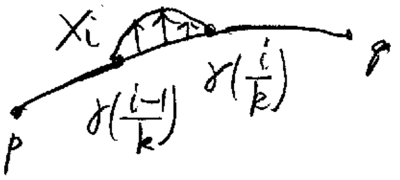
m : minimum of $K(U,U)$, $m > 0$

$$r := \sqrt{\frac{n-1}{m}}, \quad m = \frac{n-1}{r^2}$$

$\gamma \in \Omega(M; p, q)$ geodesic of length $> \pi r$
$\underset{\text{Th. 19.4}}{\Longrightarrow}$ index $\gamma > 0$

γ : of length $> k\pi r \Rightarrow$ index of $\gamma \geq k$

($\exists$ vector field $X_1, \cdots, X_k \in T_\gamma \Omega$



$E_{**}(X_i, X_i) < 0$
$E_{**}(X_i, X_j) = 0$ $(i \neq j)$

$W := \langle X_1, \cdots, X_k \rangle_{\mathbb{R}} \subset T_\gamma \Omega$
$\dim_{\mathbb{R}} W = k$
$E_{**}|_{W \times W}$ negative definite.
$\therefore$ index $\gamma \geq k$)

$\Omega(M;p,q)$ のホモトピー型は p, q によらない!

May suppose p & q are not conjugate along any geodesic.
By Th. 16.3, #(geodesics of length $\leq k\pi r$) $< \infty$
#(geodesics of index $< k$) $< \infty$. By Th. 13.3 we have the result. //