モース理論入門 第10回

10-1

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§17

M : Riemannian manifold, connected, p, q

$\Omega = \Omega(M; p, q) := \{ \omega : [0,1] \to M$ piecewise smooth, $\omega(0) = p,\ \omega(1) = q \}$

$\Omega^a = E^{-1}([0,a]) \simeq$ CW complex under some conditions of M, p, q ($\leftarrow$ Th. 16.3 ノート 9-6)

($E : \Omega \to \mathbb{R}$ energy)

How about Ω?

$\Omega^* = \Omega^*(M, p, q) := \{ \omega : [0,1] \to M$ continuous, $\omega(0) = p,\ \omega(1) = q \}$

$\cup$

Ω metric distance ρ on M について

$\rho : M \times M \to \mathbb{R}$ $\quad \rho(p,q) := \inf \{ L(\omega) \mid \omega \in \Omega(M, p, q) \}$ length

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$d^* : \Omega^* \times \Omega^* \longrightarrow \mathbb{R}$

$d^*(\omega, \omega') := \max \{ \rho(\omega(t), \omega'(t)) \mid t \in [0,1] \}$

- d^* : distance on Ω^*

(The topology on Ω^* given by d^* coincides with the "compact-open topology" (C^0-topology))

- $d^* \leq d$ on Ω (§16 ノート 9-1)

  ((Ω, d) の方が (Ω, d^*) より細かいトポロジー; ε 近傍がより小さい)

- $i : (\Omega, d) \to (\Omega^*, d^*)$

  inclusion is continous.

Theorem 17.1 $i:\Omega\to\Omega^*$ is a homotopy equivalence

i.e. $\exists j:\Omega^*\to\Omega$ conti. s.t. $j\circ i\simeq id_\Omega$, $i\circ j\simeq id_{\Omega^*}$
でも直接 j を作らず

① $U\subset(M,g)$ ← Riemann. open set
U: <u>geodesically convex</u> (測地的凸)

最短 → minimal

$\overset{\Longleftrightarrow}{\text{def}}$ $\forall x,y\in U$, $\exists_1\gamma=\gamma_{x,y}:[0,1]\to U$ minimal geodesic from x to y, $\Gamma: U\times U\times[0,1]\to U$, $\Gamma(x,y,t)=\gamma_{x,y}(t)$, is smooth.

・(J.H.C. Whitehead: $\forall z\in M$, $z\in{}^\exists U$ geodesically convex open neighborhood)

$M=\bigcup_{\alpha\in A}U_\alpha$: an open covering of M by geodesic. convex open sets.

$k=1,2,3,\cdots$

$$\Omega_k^*:=\left\{\omega\in\Omega^*\;\middle|\;{}^\forall j\,(1\le j\le 2^k),\ \exists\alpha\in A,\ \omega\left(\left[\tfrac{j-1}{2^k},\tfrac{j}{2^k}\right]\right)\subset U_\alpha\right\}$$

$\Omega_k^*\subset\Omega^*$ open subset

$$\Omega_1^*\subset\Omega_2^*\subset\Omega_3^*\subset\cdots\subset\Omega^*$$

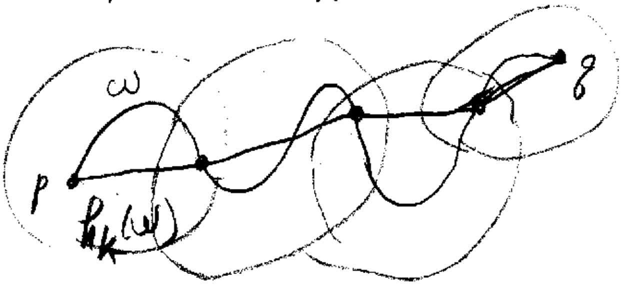
$$\Omega^*=\bigcup_{k=1}^\infty\Omega_k^*$$

$\Omega_k:=\Omega_k^*\cap\Omega=\{\omega\in\Omega\mid$ ——— $\}$ open in Ω

$$\Omega=\bigcup_{k=1}^\infty\Omega_k$$

$i_k = i|\Omega_k : \Omega_k \to \Omega_k^*$ inclusion

$h_k : \Omega_k^* \to \Omega_k$



極小測地線で結んで$h_k(\omega)$を作る

- h_k : continuous
- $i_k \circ h_k \simeq id_{\Omega_k^*} : \Omega_k^* \to \Omega_k^*$

$H_u(\omega)$：途中のuまでは極小測地線で行く

チェックポイントは経由してショートカット

- $h_k \circ i_k \simeq id_{\Omega_k} : \Omega_k \to \Omega_k$

$\therefore$ i_k is a homotopy equivalence

$$\begin{array}{ccccccccc}
\Omega_1 & \subset & \Omega_2 & \subset & \Omega_3 & \subset & \cdots & \subset & \Omega \\
{\scriptstyle i_1}\downarrow & & {\scriptstyle i_2}\downarrow & & {\scriptstyle i_3}\downarrow & & & & \\
\Omega_1^* & \subset & \Omega_2^* & \subset & \Omega_3^* & \subset & \cdots & \subset & \Omega^*
\end{array}$$

Ω is a <u>homotopy direct limit</u> of $\{\Omega_k\}$

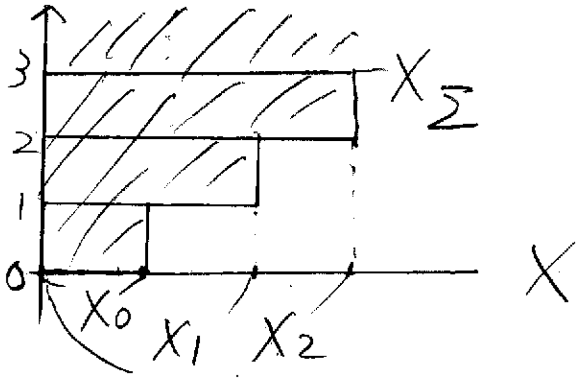
Ω^* ———————— $\{\Omega_k^*\}$

By the following Theorem A, we have that $i : \Omega \to \Omega^*$ is a homotopy equivalence //

X : topological space

$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$

$X_\Sigma := \bigcup_{i=1}^{\infty} X_i \times [i, i+1] \subset X \times \mathbb{R}$



Def. X is a homotopy direct limit of $\{X_i\}$

$\overset{\text{def}}{\iff}$ $p : X_\Sigma \to X$, $p(x, \tau) = x$, is a homotopy equivalence.

Example 1. $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$

$\forall x \in X$, $\exists i$ $x \in \mathrm{Int}\, X_i$.

X paracompact $\Rightarrow$ X : homot. direct limit of $\{X_i\}$

(◎ metric space (距離空間) $\Rightarrow$ paracompact (Stone's theorem)

$\therefore$ Ω, Ω^* : paracompact)

Example 2. X CW-complex, $X_0 \subset X_1 \subset \cdots \subset X$ subcomplexes. $\bigcup_{i=0}^{\infty} X_i = X$

$\Rightarrow$ X is a homotopy direct limit.

Theorem A : X : homotopy direct limit of $\{X_i\}$, Y : —— of $\{Y_i\}$, $f : X \to Y$ continuous, $f(X_i) \subset Y_i$,

$f|_{X_i} : X_i \to Y_i$ homotopy equivalence

$\Rightarrow$ f is a homotopy equivalence // (証明省)

◎ $\Omega^* \simeq$ CW-complex
(① 小松・中岡・菅原「位相幾何学I」岩波書店
p.256, 定理3.13 定理3.16 等による)

Corollary 17.2 $\Omega \simeq$ CW-complex
① Th. 17.1

Theorem 17.3 (Fundamental theorem of Morse theory)

M complete connected Riemann manifold,
$p, q \in M$ not conjugate through any geodesic from p to q
$\Rightarrow$ Ω (resp. Ω^*) has a homotopy type of a <u>countable</u> CW-complex having one λ-cell e^λ correspondingly to each geodesic from p to q of index λ.

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① $a_0 < a_1 < a_2 < \cdots$ regular values of $E: \Omega \to \mathbb{R}$
s.t. (a_{i-1}, a_i) contains just one critical <u>value</u>
(By Th. 16.2, the set of critical values $E(C(E))$ is discrete.)

$$\underset{\phi''}{\Omega^{a_0}} \subset \Omega^{a_1} \subset \Omega^{a_2} \subset \Omega^{a_3} \subset \cdots \subset \Omega$$

$$\Omega^{a_i} \simeq \Omega^{a_{i-1}} \cup e^{\lambda_{i1}} \cup \cdots \cup e^{\lambda_{ij(i)}}$$

$$\begin{array}{ccccccccc} \Omega^{a_1} & \subset & \Omega^{a_2} & \subset & \Omega^{a_3} & \subset & \cdots & \subset & \Omega \\ f_1 \downarrow \wr\!\wr & & \downarrow f_2 \wr\!\wr & & \downarrow f_3 \wr\!\wr & & \cdots & & \downarrow f \\ K_1 & \subset & K_2 & \subset & K_3 & \subset & \cdots & \subset & K \end{array}$$ sequence of CW-complexes

f_i: homotopy equivalence

$f : \Omega \to K$ — homotopy group of deg. n $\left(\forall I^{n+1} \xrightarrow{h} \Omega \ \exists k \ h(I^{n+1}) \subset \Omega_k \right)$

$f_* : \pi_n(\Omega) \longrightarrow \pi_n(K)$ isomorphism for $\forall n$

(i.e., f : a weak homotopy equivalence)

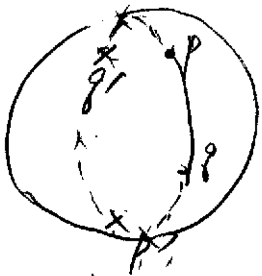
$\Omega \simeq$ CW-complex, K : CW-complex

($\Leftarrow$ Cor. 17.2)

By Whitehead's theorem, f is a homotopy equivalence.

($\uparrow$ たとえば小松・中岡・菅原, p.254 定理3.6).

Exemple $M = S^n$. p, q, $p \neq q$, not anti-podal.



$\gamma \in \Omega(S^n; p, q)$

index $\exists k(n-1)$.

(ノート 7-8 Example 2, Th. 15.1 (ノート 8-1))

Cor. 17.4 $\Omega(S^n)$ $(:= \Omega(S^n; p, p))$

$\simeq e^0 \cup e^{n-1} \cup e^{2(n-1)} \cup \cdots$

($\Leftarrow$ Th. 17.3)

(!) The homotopy type of $\Omega(S^n; p, q)$ is independent of p & q.

Take p & q as in Th. 17.3. //

$n \geq 3 \Rightarrow$

$$H_*(\Omega(S^n), \mathbb{Z}) \cong H_*(e^0 \cup e^{n-1} \cup e^{2(n-1)} \cup \cdots)$$

$$\cong \underset{0}{\mathbb{Z}} \oplus \underset{2}{\mathbb{Z}} \oplus \underset{4}{\mathbb{Z}} \oplus \cdots$$

Corollary 17.5 $M \simeq S^n$ $(n \geqq 3)$
$p, q \in M$ not conjugate through any geodesic
$\Rightarrow$ p, q are connected by infinitely many geodesics.

(!!) $\Omega(M) \simeq \Omega^*(M) \simeq \Omega^*(S^{n-1}) \simeq \Omega(S^{n-1})$

$\therefore$ $H_*(\Omega(M), \mathbb{Z}) \cong \underset{0}{\mathbb{Z}} \oplus \underset{2}{\mathbb{Z}} \oplus \underset{4}{\mathbb{Z}} \oplus \cdots$

$\exists$ geodesics of index $0, 2, 4, \cdots$

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§18

Theorem 18.1 M complete Riemannian manifold, $p \in M$
$\exp = \exp_p : T_pM \to M$ exponential map
$(\exp_p(v) := \gamma_v(1)$, ${}^{\exists 1}\gamma_v : [0,1] \to M$ geodesic $\gamma_v(0) = p$, $\dot\gamma_v(0) = v)$
Then
p and $\exp_p(v)$ are conjugate along γ_v
$\Longleftrightarrow$ $(\exp_p)_* : T_v(T_pM) \to T_{\exp_p(v)}M$ not surjective
(i.e. v critical)

(!!) $\Leftarrow$
Suppose $\exp_p$ is critical at $v \in T_pM$.
$\exists X \neq 0$ $X \in T_v(T_pM)$ s.t. $(\exp_p)_*(X) = 0$
$v(u) := v + uX$ path in T_pM
$\alpha(u,t) := \exp_p(t(v + uX))$
variation of γ_v by geodesics.

$W(t) := \frac{\partial}{\partial u}(\exp_p(t(v+uX)))\Big|_{u=0}$

Jacobi field along γ_v (← Lemma 14.3)

$W(0)=0$, $W(1)=0$, $W(t) \not\equiv 0$

$$\left(\begin{array}{l} W(1) = \frac{\partial}{\partial u}(\exp_p v(u))\Big|_{u=0} = \exp_{p*} v'(u)\Big|_{u=0} = \exp_{p*} X = 0 \\ \frac{DW}{dt}(0) = \frac{D}{du}\frac{\partial}{\partial t}(\exp_p tv(u))\Big|_{(0,0)} = \frac{D}{du}v(u)\Big|_{u=0} = X \neq 0 \end{array} \right)$$

∴ p & $\exp_p(v)$ conjugate along γ_v

⇒ Suppose $\exp_p(v)$ is not critical.

$\exists X_1, \cdots, X_n \in T_v(T_pM)$ linear independent

$(\exp_p)_*(X_1), \cdots, (\exp_p)_*(X_n)$ lin. indep

$\alpha_i(u,t) := \exp_p(t(v+uX_i))$

$W_i(t) = \frac{\partial}{\partial u}\alpha_i(u,t)\Big|_{u=0}$

$W_1, \cdots, W_n$: Jacobi field along γ_v

$W_i(0)=0$, $W_i(1) = \exp_p X_i$.

$W_1(1), \cdots, W_n(1)$ <u>lin. indep.</u>

$\forall J$: Jacobi field along γ_v, $J(0)=0$, $J(1)=0$

$J = \sum {}^\exists c_i W_i$. $0 = J(1) = \sum c_i W_i(1)$

∴ $c_1 = 0, \cdots, c_n = 0$ ∴ $J=0$

∴ p & $\exp_p(v)$ are not conj. //

Corollary 18.2. $p \in M$, For almost all $q \in M$

p & q are <u>not</u> conjugate along any geodesic from p to q.

(!) Apply Sard's theorem to $\exp_p : T_pM \to M$. //