モース理論入門　第1回 (1-1)

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モース理論の意義：
特異点にかかわる大域理論
微分幾何とトポロジーの融合

空間の曲がり具合が決まれば測地線が決まる
測地線がわかれば空間の曲がり具合もわかる
トポロジーもわかる

ミルナーの講義録
ミルナー「モース理論」志賀浩二訳　吉岡書店
J. Milnor, Morse Theory, Princeton Univ. Press
を概観し，関連する現代数学の事項を解説する．

Overview

Part I　(有限次元)多様体上の"モース関数"から多様体の"ハンドル分解"を得てトポロジー(ホモトピー型)を調べる

Part II　リーマン幾何，特に測地線や曲率についてすばやく学ぶ

Part III　リーマン多様体上の曲線全体の空間上のエネルギー関数の特異点として測地線を捉え，多様体のトポロジーを調べる

Part IV　対称空間，リー群にモース理論を応用し，特にボットの周期性定理を得る．

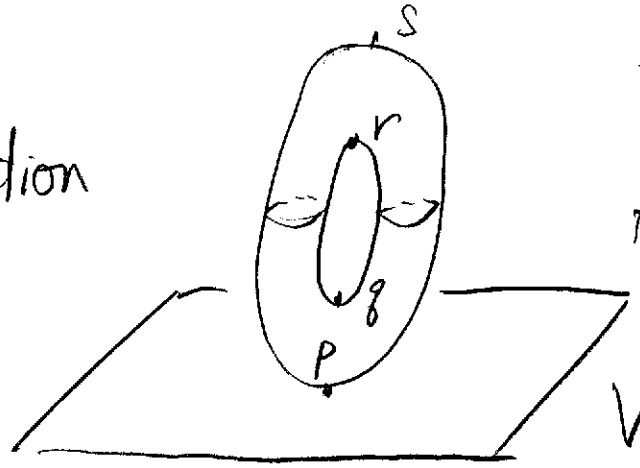
Part I non-degenerate smooth functions on a manifold

§1

M : torus

f ; height function

$f|V = 0$



$f = \text{const} - x^2 - y^2$

$f = \text{const} + x^2 - y^2$

$f = x^2 + y^2$

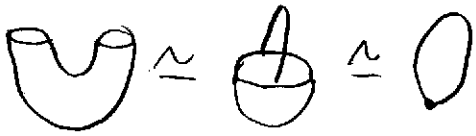
∃ local coordinates (x, y)

$f : M \to \mathbb{R}$ $a \in \mathbb{R}$

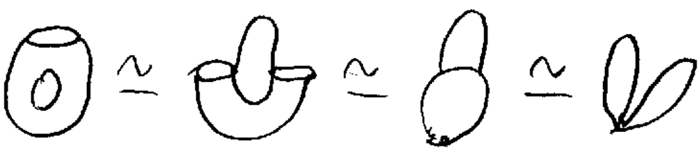
$M^a := \{x \in M \mid f(x) \leq a\}$

$a < 0 = f(p) \Rightarrow M^a = \emptyset$

$f(p) < a < f(q) \Rightarrow M^a \underset{\text{homeo.}}{\cong} D^2 \simeq \cdot$ 1pt

$f(q) < a < f(r) \Rightarrow M^a \cong S^1 \times I$ 

$(I = [0, 1])$

$f(r) < a < f(s) \Rightarrow M^a \cong$ 

$f(s) < a \Rightarrow M^a = M \simeq$  $\cup D^2$

Hopotopy theory

X, Y topological spaces

$X \simeq Y$ (homotopy equivalent)

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists f: X \to Y$, $\exists g: Y \to X$ continuous s.t.

$g \circ f \underset{\text{homotopic}}{\simeq} id_X : X \to X$

(i.e. $\exists H : X \times I \to X$ conti.
$H(x,0) = (g \circ f)(x)$, $H(x,1) = id_X(x) = x$.)

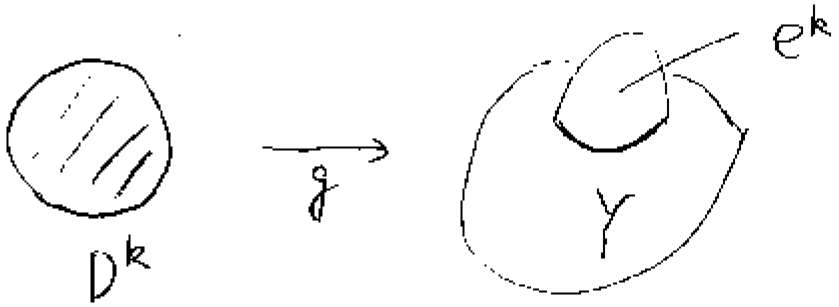
$f \circ g \simeq id_Y : Y \to Y$

$D^k := \{x \in \mathbb{R}^k \mid \|x\| \leq 1\}$ closed k-cell $\overset{\text{homeo.}}{\cong} I^k$

$S^{k-1} = \partial D^k := \{x \in \mathbb{R}^k \mid \|x\| = 1\} = S^{k-1}$ $(k-1)$-dimensional sphere

$g : \partial D^k \longrightarrow Y$ conti

$Y \cup e^k = Y \cup_g D^k := (Y \amalg D^k) / g(x) \sim x \in S^{k-1}$



X: topological space $e \subset X$ subset

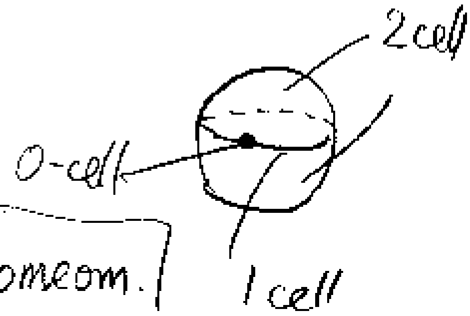
e: <u>n-cell</u> (n-胞体) n-dim. cell.

$\overset{def}{\Longleftrightarrow}$ $\exists \varphi : D^n \to X$ continuous

$\varphi(D^n) = \bar{e}$ (closure),

$\varphi(\partial D^n) \subseteq \bar{e} \setminus e$

$\varphi : D^n \setminus \partial D^n \xrightarrow{\cong} e$ homeom.



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top. space K : <u>cell complex</u> (胞複体)

$\overset{def}{\Longleftrightarrow}$ · K : Hausdorff.

· K is a union of cells

· $\forall e \subset K$ n-cell

$\bar{e} \setminus e$ is a union of cells of dim. $< n$

$L \subset K$ <u>sub cell complex</u> (部分胞複体)

$\overset{def}{\Longleftrightarrow}$ $\forall$cell $e \subset K$, $e \cap L \neq \phi \Rightarrow \bar{e} \subseteq L$

K^p : union of cells of dim $\leqq p$: <u>p-skeleton</u> (p-骨格)

K^p is a sub cell complex of K

K : <u>CW complex</u> (CW 複体)

$\overset{def}{\Longleftrightarrow}$ (i) closure finite ie. $\forall e$ cell $\subset {}^{\exists}$finite sub cell complex

(ii) weak topology, i.e. $S \subset K$ closed set

$\Leftrightarrow \forall$cell e, $S \cap \bar{e}$ closed

§2

M, N smooth ($= C^\infty$) manifold
$g : M \to N$ smooth ($= C^\infty$) map
$p \in M$ $g(p) = q$
$g_*(= dg_p) : T_pM \to T_qM$ differential map of g at p

$f : M \to \mathbb{R}$ smooth function

$p \in M$ <u>critical point</u> (臨界点) of f
$\overset{\text{def}}{\Longleftrightarrow}$ $f_* : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$ is 0-map
$\Longleftrightarrow$ $\forall$ local coord. $(x^1, \cdots, x^n)$ around p on M,
$$\frac{\partial f}{\partial x^1}(p) = 0, \ \cdots, \ \frac{\partial f}{\partial x^n}(p) = 0$$

($f(p)$ is called a <u>critical value</u> of f)

$p \in M$ <u>regular point</u> (正則点) of f
$\overset{\text{def}}{\Longleftrightarrow}$ p is <u>not</u> a critical point of f.

$a \in \mathbb{R}$ <u>regular value</u> (正則値) of f
$\overset{\text{def}}{\Longleftrightarrow}$ $\forall p \in f^{-1}(a)$, p is a regular point of f
($f(p) = a \Rightarrow$)

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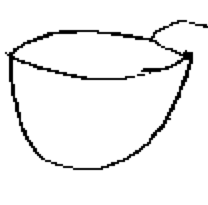
a is regular value of f

$\Rightarrow$ $M^a := \{x \in M \mid f(x) \leq a\}$
is a manifold (possibly) with boundary



$f^{-1}(a)$

M^a

A critical point p of f is called non-degenerate (非退化)
$\overset{\text{def}}{\Longleftrightarrow}$ Hessian (determinant)

$$\det\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right)_{1\le i,j\le n} \neq 0$$

$f_{**} : T_pM \times T_pM \to \mathbb{R}$

$v, w \in T_pM$, $\tilde{v}, \tilde{w}$ extension vector field

$$f_{**}(v, w) := v(\tilde{w}(f)) = \tilde{v}(w(f))$$

f_{**} : bilinear, symmetric

The <u>index</u> of f at p $= \lambda$

$\overset{\text{def}}{\Longleftrightarrow}$ $\lambda = \max\{\dim W \mid W \subset T_pM$ subspace, $f_{**}|_{W\times W}$ negative definite$\}$

$$\left(\overset{w\in W}{f_{**}(w,w) \le 0}, \quad = 0 \Leftrightarrow w = 0\right)$$

The <u>nullity</u> of f at p $= \nu$

$\overset{\text{def}}{\Longleftrightarrow}$ $\nu = \dim\{w \in V \mid f_{**}(v, w) = 0 \text{ for } \forall v \in V\}$

Lemma 2.2 (Morse's lemma) $\nu = 0$ (of index λ)
$f : M \to \mathbb{R}$ smooth , p ; non-degenerate critical point of f
$\exists$ local coord. $(y^1, \cdots, y^n)$ around p on M , $y^i(p) = 0$
s.t. $f = f(p) - (y^1)^2 - \cdots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \cdots + (y^n)^2$

Lemma 2.3 non-degenerate critical point is isolated.

(!) $f = f(p) - (y^1)^2 - \cdots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \cdots + (y^n)^2$

$$\frac{\partial f}{\partial y^1} = -2y^1, \cdots, \frac{\partial f}{\partial y^\lambda} = -2y^\lambda, \frac{\partial f}{\partial y^{\lambda+1}} = 2y^{\lambda+1}, \cdots, \frac{\partial f}{\partial y^n} = 2y^n$$

$y = (y^1, \cdots, y^n)$ crit. pt. of $f \iff y^1 = 0, \cdots, y^n = 0$

M : smooth manifold

smooth map $\varphi : \mathbb{R} \times M \to M$

1-parameter group of diffeomorphisms of M

$\overset{\text{def}}{\iff}$ (1) $\forall t \in \mathbb{R}$ $\varphi_t : M \to M$ $\varphi_t(q) = \varphi(t, q)$

is a diffeomorphism of M

(2) $\varphi_{t+s} = \varphi_t \circ \varphi_s : M \to M$ for $\forall t, s \in \mathbb{R}$.

X : vector field over M

X generates φ

$\overset{\text{def}}{\iff}$ $\forall q \in M$, $\forall f : M \to \mathbb{R}$ smooth

$$X_q f = \lim_{h \to 0} \frac{f(\varphi_h(q)) - f(q)}{h}$$

Lemma 2.4 X : smooth vector field over M

$\exists K \subset M$ compact, $X|_{M \setminus K} = 0$

$\Rightarrow \exists 1\, \varphi$: 1-para. group of diffeom. of M

s.t. X generates φ.