Hilbert 第16問題 と その周辺

城崎スクール「幾何学の広がり」

2006.3.5 〜 3.8

http://www.math.sci.hokudai.ac.jp/~ishikawa/Hilbert16.html

- Hilbert 16問題とは何か
- どういうことが知られているか
  どのような 方法
- 関連する問題の紹介

4-dim. topology

transformation group
Smith theory

Hilbert's 16 th problem
Topology of real algebraic varieties

Toric geometry
Amoeba
Tropical geometry

Singularity theory

問題設定

実射影空間

$\mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times \hookrightarrow \mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$

$[x_0 : x_1 : \cdots x_n] = [x]$ homog. coord.

$V = V_F = \{[x] \in \mathbb{C}P^n \mid F(x_0, x_1, \cdots, x_n) = 0\}$ homog. $\deg F = d$

$V = \{[x] \in \mathbb{C}P^n \mid F_1(x_0, x_1, \cdots, x_n) = 0, \cdots, F_S(x_0, x_1, \cdots, x_n) = 0\}$

$F_1, \cdots, F_S$ 実係数 同次多項式 $\deg F_i = d_i$

$V_{\mathbb{R}} = V \cap \mathbb{R}P^n$ d次 実射影超曲面

conj : $\mathbb{C}P^n \to \mathbb{C}P^n$ $\overset{\mathrm{conj}[x]}{= [\bar{x}]}$ $\mathrm{conj}|V : V \to V$ $V_{\mathbb{R}} = \{[x] \in V \mid \mathrm{conj}[x] = [x]\}$

$n=2, S=1, d=d_1$ d次実平面曲線

$n=3, S=1, d=d_1$ d次実空間曲面

Hilbert 第16問題 ^{1900} $(n=2, d=6, n=3, d=4)$

非特異

6次実平面曲線 の位相的分類

4次実空間曲面 の位相的分類.

実代数多様体を位相的に分類せよ

実構造は思い出せ 複素化するが

代数構造は忘れて位相構造に注目せよ

(n, d) を固定して, isotopy 形 を分類せよ

（相対位相型）

$x \in \mathbb{C}^{n+1} \setminus \{0\}$

$x_0^2 + x_1^2 + \cdots + x_n^2 = 0$

Jacobi 行列

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_0}, & \frac{\partial F_1}{\partial x_1}, & \cdots, & \frac{\partial F_1}{\partial x_n} \\ & \cdots & & \\ \frac{\partial F_S}{\partial x_0}, & \frac{\partial F_S}{\partial x_1}, & \cdots, & \frac{\partial F_S}{\partial x_n} \end{pmatrix}$$

: const. rank r

whenever $F_1 = \cdots = F_S = 0$

$\dim V = n - r$

V is complex mfd of dim $n-r$

$x \in \mathbb{C}^{n+1} \setminus \{0\}$, $F(x) = 0$

$\Rightarrow \left(\frac{\partial F}{\partial x_0}(x), \frac{\partial F}{\partial x_1}(x), \cdots, \frac{\partial F}{\partial x_n}(x)\right) \neq 0$ if $F(x) = 0$

$(\mathbb{R}P^n, V_{\mathbb{R}}) \underset{\text{isotopic}}{\sim} (\mathbb{R}P^n, V'_{\mathbb{R}})$

$\exists \varphi_t$: family of homeo. of $\mathbb{R}P^n$ $0 \leq t \leq 1$

$\varphi_0 = id$, $\varphi_1(V_{\mathbb{R}}) = V'_{\mathbb{R}}$

$V_{\mathbb{R}}$ が $\varphi_t(V_{\mathbb{R}})$ を経由して $V'_{\mathbb{R}}$ に連続的に変形される.

$(n, S, d_1, \cdots, d_S)$ を固定して 実代数多様体 の isotopy 形 を分類せよ.

Charactor of the Problem

restriction　inequality, congruence

複素化 complexificat. encomplexicy

realization　patchwork

離散化, 組合せ化 combinatrics

endiscreting

実幾何 (real geometry)

signature　Sylvester の慣性法則

mapping degree, Descartes の定理
Sturm の定理

encomplexified invariant

$\mathbb{C}$の中での$\mathbb{R}$の定義は？

$$\begin{tikzcd}\mathbb{R} \arrow[r, "拡大"'] & \mathbb{C}\end{tikzcd}$$

拡大体であることを忘れる

complexification　思い出す

encomplexification

complex conjugation

$\mathrm{conj} : \mathbb{C} \longrightarrow \mathbb{C}$

fixed point set が $\mathbb{R}$

Suppose alg str. of $\mathbb{C} \ni 1$

$$\begin{tikzcd}\mathbb{Q} \arrow[r, hook] \arrow[dr, hook] & \mathbb{C} \\ & \mathbb{R} \arrow[u, dashed, "\exists ?"']\end{tikzcd}$$

12-w1

cf. The definition of top. str. on $\mathbb{C}$ needs reals ??

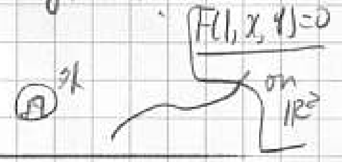
実平面曲線のトポロジー　$n=2$　$\mathbb{R}^2 \hookrightarrow \mathbb{C}P^2$ $x_0 \neq 0$ $\boxed{x=\frac{x_1}{x_0},\ y=\frac{x_2}{x_0}}$

非特異実平面代数曲線　$C=\{[x] \in \mathbb{C}P^2 \mid F(x_0,x_1,x_2)=0\}$

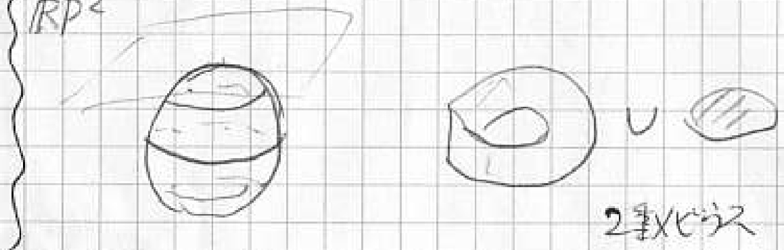
$C_{\mathbb{R}} = C \cap \mathbb{R}P^2$　$\deg F = d$

$C_{\mathbb{R}} \subset \mathbb{R}P^2$
連結成分は
oval か pseudo-line

$F(1,x,y)=0$



$\mathbb{R}P^2$



$H_1(\mathbb{R}P^2, \mathbb{Z}/2) \cong \mathbb{Z}/2$

Harnackの不等式

連結成分 $l \leqq \frac{1}{2}(d-1)(d-2)+1$　各 d に対して M-curve が存在する。

等号　M-curve.

$g(C) = \frac{1}{2}(d-1)(d-2)$

$C \approx$

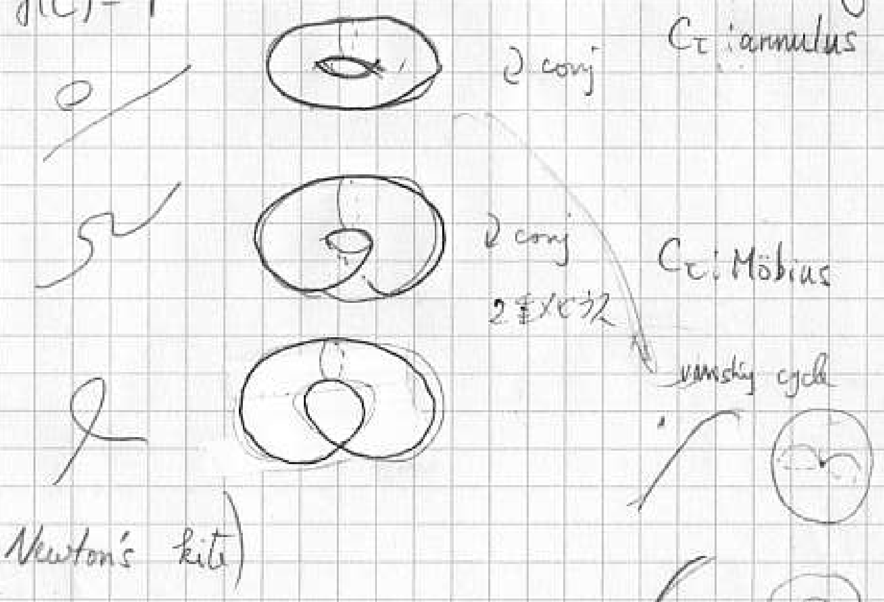
$\sum_i \dim H_i(C_{\mathbb{R}}, \mathbb{Z}/2) \leqq \sum_j \dim H_j(C, \mathbb{Z}/2)$

$l+l$　　$2+2 \cdot g$

Ex.　$C \subset \mathbb{C}P^2$　deg 3　$\frac{1}{2}(d-1)(d-2)$

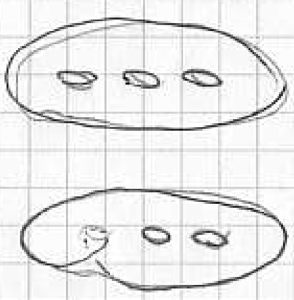
g

$g(C)=1$



(Newton's kite)

deg 4　$g=3$



$l \leqq 4$

$d=4$ $\langle 4\rangle$ $\langle 3\rangle$ $\langle 2\rangle$ $\langle 1\langle 1\rangle\rangle$ $\langle 1\rangle$ $\langle 0\rangle$

$l \leqq 7$

$d=5$ $\langle J \amalg 6\rangle$ $\langle J \amalg 5\rangle$ $\langle J \amalg 4\rangle$ $\langle J \amalg 3\rangle$ $\langle J \amalg 2\rangle$
$\langle J \amalg 1\langle 1\rangle\rangle$ $\langle J \amalg 1\rangle$ $\langle J\rangle$

$d=6$ M-curve $l = \frac{1}{2}(6-1)(6-2)+1 = \frac{5\cdot 4}{2}+1 = 11$

$\langle 1\langle 9\rangle \amalg 1\rangle$ $\langle 1\langle 5\rangle \amalg 5\rangle$ $\langle 1\langle 1\rangle \amalg 9\rangle$

(Gudkov)

$d \leqq 7$ 分類は完成している (Gudkov, Viro)

$d=8$ 未解決

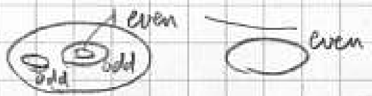
Orevkov (2002)

flexible M-curves of degree 8 の分類.

$C \subset \mathbb{C}P^2$ symplectic surface $conj(C)=C$

$[C] = 8[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2;\mathbb{Z})$

d: even $C_{\mathbb{R}}$: oval の和 p, n



$p = \#$even oval, $n = \#$ odd oval

$\chi(B_+) = p-n$, $\chi(B_-) = 1-(p-n)$

オイラー標数

$l = p+n$

Ex.
$\langle 1\langle 1\rangle \amalg 9\rangle$ $p=10$, $n=1$

Petrovskii の不等式

$$-\frac{3}{8}d(d-2) \leqq p-n \leqq \frac{3}{8}d(d-2)+1$$

$d=6$ $-\frac{3}{8}6\cdot 4 = -9 \leqq p-n \leqq 10$

$\langle 11\rangle$ 型は存在しない!

Rokhlin's congruence d: even $C_{\mathbb{R}}$ M-curve

$\Rightarrow \quad p-n \equiv \left(\frac{d}{2}\right)^2 \mod 8$

$d=6 \quad p+n=11 \quad p-n\equiv 1 \ (8)$

$(p,n) = (10,1),\ (6,5),\ (2,9)$

$d=8 \quad p+n=22 \quad p-n\equiv 0 \ (8)$

$(p,n)=(19,3),\ (15,7),\ (11,11),\ (7,15),\ (3,19)$

$\langle \alpha \amalg 1\langle\beta\rangle\rangle \qquad p=\alpha+1,\ n=\beta$

$\langle \alpha \amalg 1\langle\beta\rangle \amalg 1\langle\gamma\rangle\rangle \qquad p=\alpha+2,\ n=\beta+\gamma$

$\langle \alpha \amalg 1\langle\beta\rangle \amalg 1\langle\gamma\rangle \amalg 1\langle\delta\rangle\rangle \qquad p=\alpha+3,\ n=\beta+\gamma+\delta$

$\langle \alpha \amalg 1\langle \beta \amalg 1\langle\gamma\rangle\rangle\rangle \qquad p=\alpha+\gamma+1,\ n=\beta+1$

Viro's th

$\langle \alpha \amalg 1\langle\beta\rangle \amalg 1\langle\gamma\rangle \amalg 1\langle\delta\rangle\rangle \quad \beta\gamma\delta\neq 0 \Rightarrow \beta,\gamma,\delta$: odd

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~~Orevkov (2002)~~

~~Classification of flexible M-curves of degree 8~~

~~$C \subset \mathbb{CP}^2$ symplectic surface $\mathrm{conj}(C)=C$~~

~~$C \sim 8L \quad L$: line~~

Real complex manifold

X complex manifold

$R: X \to X$ anti-hol. involution (real str.) $\quad R\circ R = \mathrm{id}_X$

$R_*: T_xX \to T_{R(x)}X$

comp. vector sp. $\quad R_*(\sqrt{-1}v) = -\sqrt{-1}R_*(v)$

Ex. $X \subset \mathbb{CP}^n$ complex submfd

$\mathrm{conj}(X) = X$

$R = \mathrm{conj}|X$: anti-hol inv. on X

Ex $\mathbb{CP}^1 = \{[z_0 : z_1]\}$

$\mathrm{conj}([z_0 ; z_1]) = [\bar{z}_0 : \bar{z}_1]$

別の real str.

$R([z_0 : z_1]) = [-\bar{z}_1 : \bar{z}_0]$

$\mathbb{CP}^1 \cong Q \subset \mathbb{CP}^3 \quad [w_0 : w_1 : w_2]$

$w_1^2 + w_1^2 + w_2^2 = 0$

復習

d: even　　non-sing. real plane curve of degree d

$$\ell \leqq \frac{1}{2}(d-1)(d-2)+1$$

$= M$-curve

$p = \#$ even oval　　$n = \#$ odd oval

Rokhlin　even d　M-curve

$$p - n \equiv \left(\frac{d}{2}\right)^2 \quad \underline{\mathrm{mod}\ 8}$$

Signature of involution

X (oriented) closed mfd　$\dim X = 4n$

$E = H_{2n}(X;\mathbb{R})$　交差形式 $E \times E \to \mathbb{R}$

$\sigma(X) = \#$正の固有値 $- \#$負の固有値.

$R: X \to X$　orientation preserving involution　$\tau \circ \tau = \mathrm{id}_X$

$E = E_+ \oplus E_-$　　$R_*: E \to E$ は固有値 ± 1

$\sigma_\pm$: 交差形式を $\underline{E_\pm}$ に制限したものの符号数

$\sigma_+ - \sigma_-$: R の signature

Th (Atiyah-Singer) X compact complex mfd $\dim_{\mathbb{C}} X = 2n$

$R: X \to X$ anti-hol inv　　$X^{\mathbb{R}}$ 不動点集合

$\Rightarrow (-1)^n \chi(X^{\mathbb{R}}) = \sigma_+ - \sigma_-$

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complex vector space

$R_*: T_xX \to T_{\tau(x)}X$

$R_*(\sqrt{-1}\,v) = -\sqrt{-1}\,\tau_*(v)$

Ex. of signature of involutions 2n,

$(-1)^n \chi(X^\tau) = \sigma_+ - \sigma_-$

$(\mathbb{CP}^2, \text{conj})$ $\sigma_+ = 0, \quad \sigma_- = 1$

$\chi(\mathbb{RP}^2) = 1$ $H_2(\mathbb{CP}^2; \mathbb{Z}) \cong \mathbb{Z}$ conj. acts on $\mathbb{Z}$ by -1

$(\mathbb{CP}^4, \text{conj})$ $\sigma_+ = 1, \quad \sigma_- = 0$

conj acts on $\mathbb{Z}$ by $+1$

$\chi(\mathbb{RP}^4) = 1$

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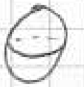
d=2 C ≈  $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$

$C_{\mathbb{R}}$  in $\mathbb{RP}^2$ oval

$F = x_0^2 + x_1^2 + x_2^2 = 0$

d=3  in $\mathbb{RP}^2$

Smith theory

X: finite polyhedron

$\tau : X \to X$ simplicial map, $\tau \circ \tau = id_X$

$X^\tau = \{x \in X \mid \tau(x) = x\}$

$X_\tau = X/\tau$ quotient space

Th 次の完全系列

$$\xrightarrow{\beta_{r+1}} H_{r+1}(X_\tau, X^\tau; \mathbb{Z}/2) \xrightarrow{\gamma_{r+1}} H_r(X_\tau, X^\tau; \mathbb{Z}/2) \oplus H_r(X^\tau; \mathbb{Z}/2)$$

$$\xrightarrow{\alpha_r} H_r(X; \mathbb{Z}/2) \xrightarrow{\beta_r} H_r(X_\tau, X^\tau; \mathbb{Z}/2) \xrightarrow{\gamma_r}$$

(Smith-Thom)

Th $\sum_i \dim_{\mathbb{Z}/2\mathbb{Z}} H_i(X^\tau; \mathbb{Z}/2\mathbb{Z}) \leq \sum_j \dim_{\mathbb{Z}/2\mathbb{Z}} H_j(X; \mathbb{Z}/2\mathbb{Z})$

等号のとき $\forall j$

$\tau_* : H_j(X; \mathbb{Z}/2\mathbb{Z}) \to H_j(X; \mathbb{Z}/2\mathbb{Z})$

" id

Rokhlin's th $\quad X$: complex mfd

(X, R) $\quad \dim_{\mathbb{C}} X = 2n$ $\quad$ M-mfd

$$\left(\sum_i \dim H_i(X^R; \mathbb{Z}/2) = \sum_j \dim H_j(X; \mathbb{Z}/2) \right)$$

$\Rightarrow \chi(X_R) \equiv \sigma(X) \mod 16$

Proof $\quad H_{2n}(X, \mathbb{Z})/_{Tor} = L_+ \oplus L_- \quad (\Leftarrow X: \text{M-mfd}$ Smith. Th$)$

$\sigma_+ \quad \sigma_-$

$(-1)^n \chi(X_R) = \sigma_+ - \sigma_-$ $\quad$ n: even $\chi(X_R) = \sigma_+ - \sigma_- = \sigma - 2\sigma_-$

$\chi(X_R) = \sigma_- - \sigma_+ = \sigma - 2\sigma_+$

n: even $\Rightarrow L_-$: even lattice σ_- is divisible by 8

n: odd $\Rightarrow L_+$: even lattice σ_+ " 8

Lemma $\quad$ n: even $\quad \exists y \in L_+ \quad$ ($y \bmod 2$ is PD of v_{2n})

$\forall x \in H_{2n}(X; \mathbb{Z})/Tor \quad x \cdot y \equiv x \cdot x \pmod 2$

$x \in L_- \quad \rho_2(x \cdot x) = \rho_2(x \cdot y) = 0$

$\boxed{E: \text{even unimodular} \Rightarrow \sigma(E) \text{ is divisible by } 8}$

たとえば Serre の本

Proof of Rokhlin's congruence

$\pi: Y \to \mathbb{C}P^2$ ramified double covering along $C \subset \mathbb{C}P^2$

deg d: even $= 2k$

M-curve

$\boxed{\cancel{\exists}\ C: \text{M-curve} \Rightarrow (Y, R_-)\ \underset{\sim}{\text{M-mfd}}}$

$Z = \{(z_0, z_1, z_2, \xi) \in \mathbb{C}^3 \times \mathbb{C} \mid z_0 z_1 z_2 \neq 0\} / \mathbb{C}^\times$

$\alpha(z_0, z_1, z_2, \xi) = (\alpha z_0, \alpha z_1, \alpha z_2, \alpha^k \xi)$

$Y = \{[z_0, z_1, z_2, \xi] \in Z \mid f(z_0, z_1, z_2) + \xi^2 = 0\}$

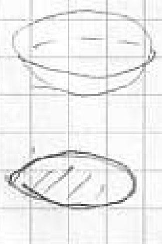
complex surface

$R([z_0, z_1, z_2, \xi]) = (\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{\xi})$

$Y^R = \{[z_0, z_1, z_2, \xi] \mid [z_0, z_1, z_2] \in \mathbb{R}P^2,\ f(z) \leqq 0,\ \xi = \pm\sqrt{-f(z)}\}$

$\downarrow$

B_-



$\sum_{i=0}^{2} \dim H_i(Y^R; \mathbb{Z}/2\mathbb{Z}) = 2 + 2(p+n)$

$\sum_{i=0}^{2} \dim H_i(Y; \mathbb{Z}/2\mathbb{Z}) = 4 + 2g$

$p + n = 1 + g \quad 0, \underset{=}{2}, 4$

$$\begin{cases} \chi(Y^-) = 2\chi(B_-) = 2(1-(p-n)) \\ \sigma(Y) = 2\sigma(\mathbb{C}P^2) - (C \cdot C)_Y \\ \quad = 2 - \frac{1}{2}(C \cdot C)_{\mathbb{C}P^2} = 2 - \frac{1}{2}d^2 \\ 2 - 2(p-n) \equiv 2 - \frac{1}{2}d^2 \quad (16) \\ p - n \equiv \frac{1}{4}d^2 \quad (8) \end{cases}$$

//

$$\left(\begin{aligned} \chi(Y^R) &= \sigma_+(R) - \sigma_-(R) \\ &= \sigma(Y) - 2\sigma_-(R) \\ &= 2\sigma(\mathbb{C}P^2) - \frac{1}{2}(C \cdot C)_{\mathbb{C}P^2} - \underline{2\sigma_-(R_-)} \\ &\equiv \underline{2 - \frac{1}{2}d^2} \end{aligned} \right)$$

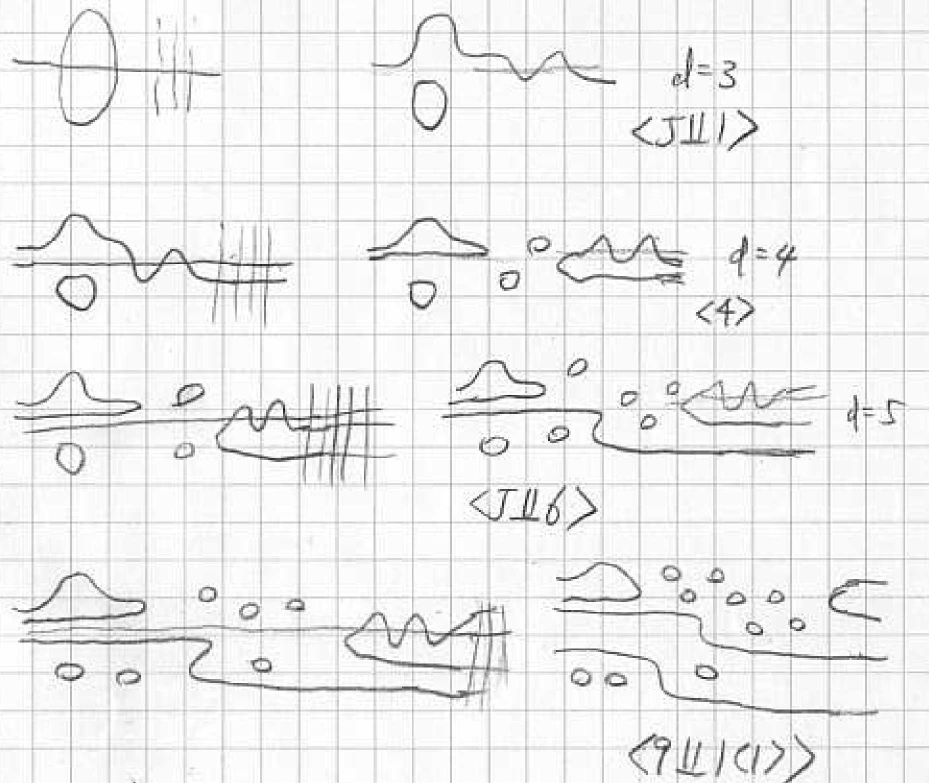
$$\chi(Y^\theta) = \sigma_+(\theta) - \sigma_-(\theta)$$

$$\overset{\shortparallel}{(C \cdot C)_Y} \quad = 2\sigma_+(\theta) - \sigma(Y)$$

$$= 2\sigma(\mathbb{C}P^2) - \sigma(Y)$$

Realization

Harnack の方法



Real algebraic curves and Amoebas

$f\in\mathbb{R}[z_1^{\pm1},\dots,z_n^{\pm1}]$ real Laurent polynomial

$V=V_f\subset(\mathbb{C}^*)^n$ real. $\boxed{\mathrm{Log}(V)=\mathcal{A}}$ Amoebas

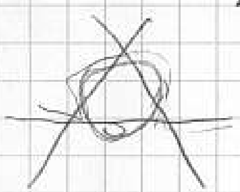
$\mathrm{Log}:(\mathbb{C}^*)^n\to\mathbb{R}^n$ $z\mapsto(\log|z_1|,\dots,\log|z_n|)$

$(\mathbb{R}V\subseteq\mathrm{Crit}(\mathrm{Log}|V))$

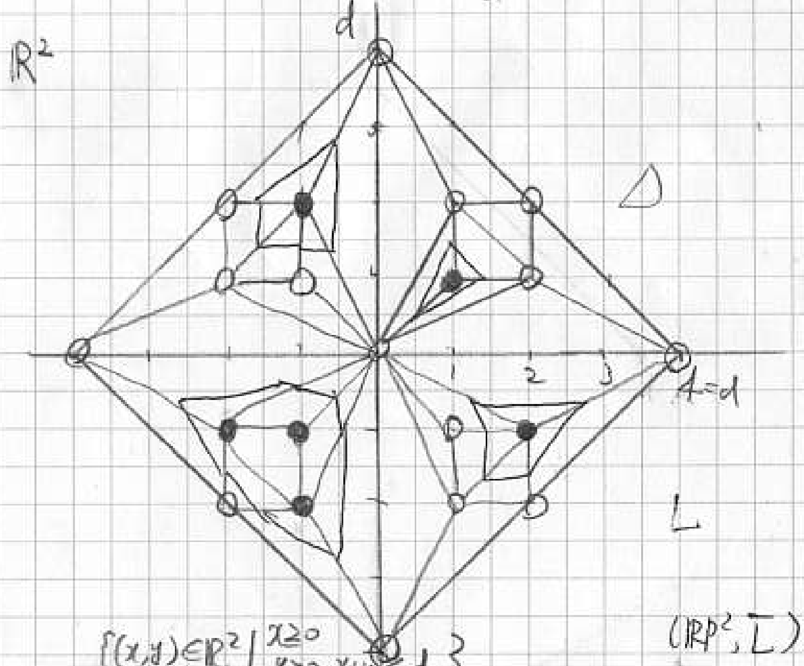
V: simple Harnack curve

$\Rightarrow \mathrm{Log}|\mathbb{R}V:\mathbb{R}V\to\mathbb{R}^n$ embedding

$\mathrm{Log}(\mathbb{R}V)=\partial\mathcal{A}$



Patchwork



$\{(x,y)\in\mathbb{R}^2 \mid x\ge0,\ y\ge0,\ x+y\le d\}$

- Δ を 3角形分割 （頂点は 整数点）
- 頂点に 符号 をつける ○ は ＋, ● は −
- ◇に 拡張する. $\boxed{\Delta_*}$
- 異符号 の辺を結ぶ $L\subset\Delta_*$
- $\mathbb{R}P^2\supset\bar{L}$ （辺をはり合わせて）

$\nu:\Delta\to\mathbb{R}$ piecewise linear, convex

各3角形上で linear

2つの3角形の和集合上では linear でない

$\nu(\Delta\cap\mathbb{Z}^2)\subset\mathbb{Z}$ (*)

$$f_t(x,y) = \sum_{(i,j):頂点} \overset{\swarrow 符号}{\sigma_{i,j}} x^i y^j t^{\nu(i,j)}$$

d次多項式族

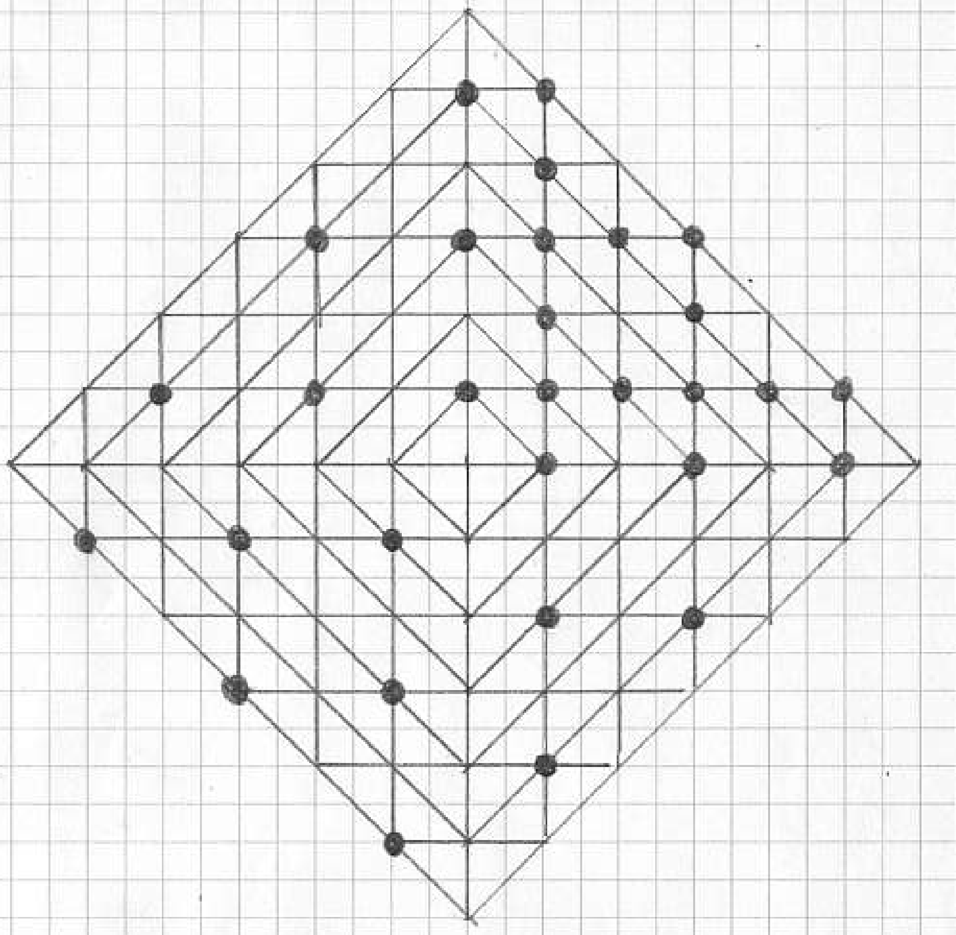
$$F_t(x_0,x_1,x_2) = x_0^d f_t\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right)$$ 同次化

$\boxed{Th}$ $\exists t_0 > 0$ $\forall t$ $0 < t < t_0$ $C_t = \{F_t = 0\} \subset \mathbb{R}P^2$

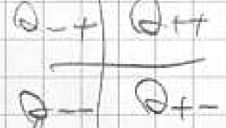
$(\mathbb{R}P^2, C_t)$ は $(\mathbb{R}P^2, \Gamma)$ と isotopic

(符号と係数を調節)

Patchwork construction of a curve
of type $\langle 1 \langle 1 \rangle \amalg 9 \rangle$ of degree 6



一般の Patchworking



$f(x,y) = \sum a_{ij} x^i y^j$ polygn

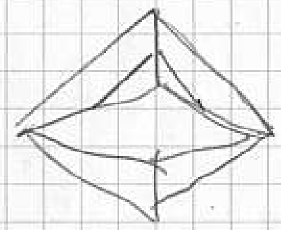
$\Delta(f) = \left(\{(i,j) \mid a_{ij} \neq 0\}\right.$ の 凸包$\left.\right)$ Newton polygon

Ex.

$f = x^2+y^2-y$

$\Delta = \Delta(f)$

$\Delta_* \supset \upsilon$ 部分集合 が $\underline{f の chart}$



$\overset{def}{\Longleftrightarrow}$ (1) $^\forall\Gamma$: Δ の辺

$\Gamma_{\pm1,\pm1} \cap \upsilon$ は

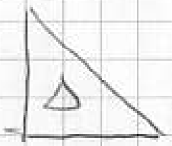
f_P の $Q_{\pm1,\pm1}$ での

零点の連結成分と

同じ個数

(2) $(\mathrm{Int}(\Delta_{\pm1,\pm1}), \mathrm{Int}(\Delta_{\pm1,\pm1} \cap \upsilon))$

$\sim (Q_{\pm1,\pm1}, Q_{\pm1,\pm1} \cap \{f=0\})$



$f_1, \cdots, f_r$ poly $\{x \geq 0, y \geq 0\ x+y \leq d\}$ //

$\Delta(f_1), \cdots, \Delta(f_r)$ が Δ の分割を与え

$f_i{}_{\Delta(f_i)\cap\Delta(f_j)} = f_j{}_{\Delta(f_i)\cap\Delta(f_j)}$ とすると

$\exists\ f = \sum a_{ij} x^i x^j$ $\Delta(f) = \Delta$ $f|_{\Delta(f_i)} = f_i$

$\nu: \Delta \to \mathbb{R}$ (*) と同様の性質

$f_t(x,y) = \sum a_{ij} x^i x^j t^{\nu(i,j)}$

Th

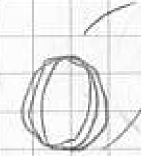
Th (Patchwork 定理) $^\exists t_0 > 0$ $^\forall t$ $0<t<t_0$

$f_1, \cdots, f_r$ の chart をつなぎ合わせたものが

$\{f_t = 0\}$ の isot. type を表す.

Perturbation of sing. J_{10}-特異点

6次曲線を



の perturb で作る

$f(x,y)$ 原点に孤立特異点をもつ 解析関数

Newton diagram $f(x,y) = \sum_{\substack{i>0\\j>0}} a_{ij} x^i x^j$

$\Gamma_+(f) = \bigcup_{\substack{(i,j)\\a_{ij}\neq0}} \left((i,j) + \mathbb{R}^2_{\geq 0}\right)$ の凸包

$\Gamma(f)$

Δ を3角形分割 頂点に符号をおく

Local Patchwork th

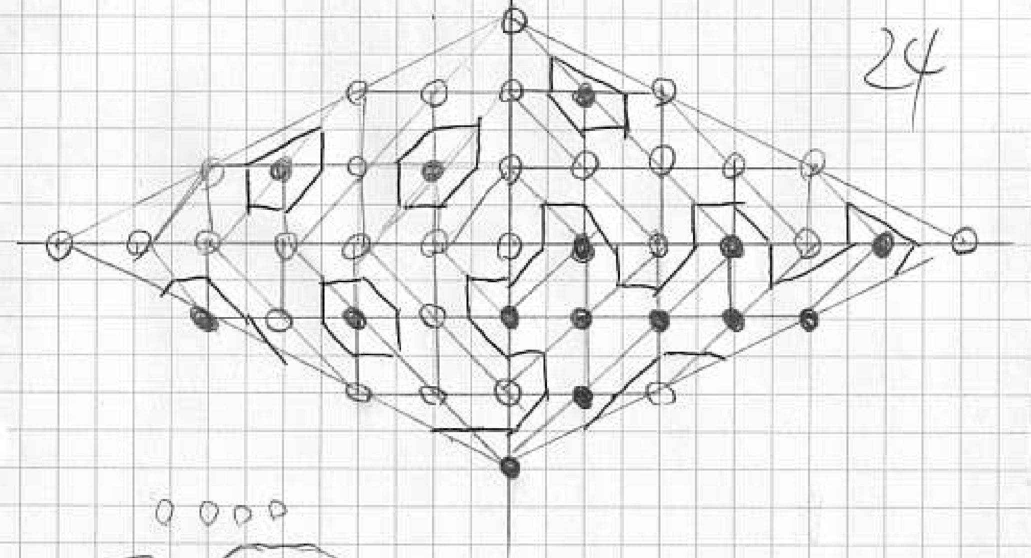
$f_t = f + \sum_{(i,j)} \sigma_{ij} x^i y^j t^{\nu(i,j)}$

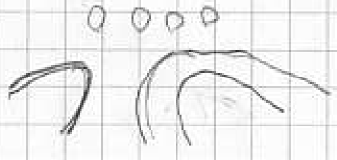
$\nu: \Delta \longrightarrow \mathbb{R}$ (*) $\nu|_\Gamma = 0$

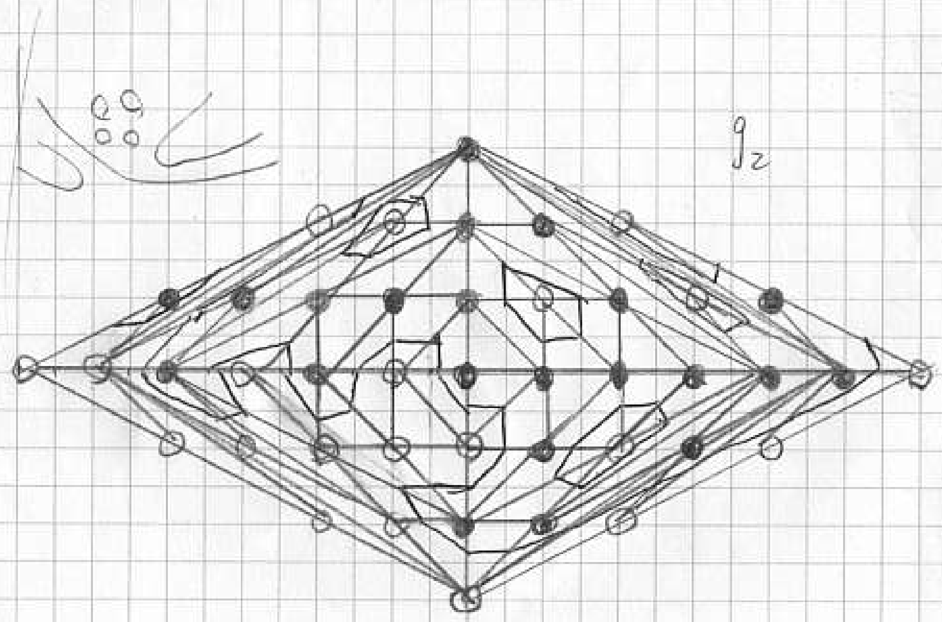
$0 < t_0 \ll \exists\varepsilon_0$ $0<t<\varepsilon_0$

$(D_\varepsilon, D_\varepsilon \cap f_t^{-1}(0))$ は patchwork で得られるものと isotopic

$\Delta_* \pm$

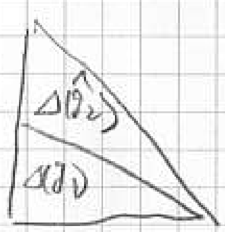


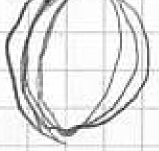




$g_1(x,y)$, $g_2(x,y)$

$\hat{g}_2(x,y) = y^6 g_2(\frac{x}{y}, \frac{1}{y})$ とおく





$\langle 1\langle 5\rangle \perp\!\!\!\perp 5\rangle$

$\langle 1\langle 9\rangle \perp\!\!\!\perp 1\rangle$

- surface case
- local case
- parametrized case

実空間4次曲面 $V_{\mathbb{R}} = V_{\mathbb{R}}(F) \subset \mathbb{RP}^3$

1886	Rohn	最大個数	$\leqq 12$
1911	Rohn	$\exists$ 10 comp.	
1967	Utkin		$\leqq 11$
1972	Kharlamov		$\leqq 10$

Petrovskii-Oleinik inequality

$\mathbb{R}V \subset \mathbb{R}P^n$

$$\begin{cases} |\chi(\mathbb{R}V)-1| \leqq N_{n+1}(d) & n:\text{odd} \\ |\chi(B_+)-\chi(B_-)| \leqq N_{n+1}(d) & n, d:\text{even} \end{cases}$$

$N_{n+1}(d)$

$$= \{ l \in \mathbb{N}^n \mid 0<l_i<d \ , (i=1,2,\dots,n) \quad \frac{n-1}{2}d < \sum_{i=1}^{n} l_i < \frac{n+1}{2}d \}$$

$F(x_0, x_1, \dots, x_n)$

$\deg\left(\left(\frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0)\right)$

の評価

Ex. $N_4(4) = 19$

$$N_4(4) = \{ l \in \mathbb{N}^3 \mid 0<l_i<4 \quad 4 < \Sigma l_i < 8 \}$$

Th. 実4次曲面の連結成分の最大個数は 10 (Kharlamov)

Proof $X = X_{\mathbb{C}}$ deg 4 non-singular (K-3 曲面)

$\dim H_*(X, \mathbb{Z}/2) = 24$

$\dim H_0 = 1, \ \dim H_2 = 22 \quad \dim H_4 = 1$ $\boxed{\sigma(X) = 3-19 = -16}$

$\dim H_*(X_{\mathbb{R}}) \leqq 24$ Harnack-Thom-Smith

$|\chi(X_{\mathbb{R}}) - 1| \leqq N_{n+1}(d) = N_4(4) = 19$ $n=3$ $d=4$

$N_4(d) = \# \{ l \in \mathbb{N}^3 \mid 0 < l_i < d, \ d < l_1 + l_2 + l_3 < 2d \}$ 。

$4 < l_1 + l_2 + l_3 < 8$ $\underline{\underline{5, 6, 7}}$

5 (1, 1, 3) (1, 2, 2) (1, 3, 1)
(2, 1, 2) (2, 2, 1) (3, 1, 1) $\underline{\underline{6}}$

6 ~~(1, 1, 4)~~ (1, 2, 3) (1, 3, 2) ~~(1, 4, 1)~~
(2, 1, 3) (2, 2, 2) (2, 3, 1)
(3, 1, 2) (3, 2, 1) 7
~~(4, 1, 1)~~

7 ~~(1, 1, 5)~~ ~~(1, 2, 4)~~ (1, 3, 3) ~~(1, 4, 2)~~ ~~(1, 5, 1)~~
~~(2, 1, 4)~~ (2, 2, 3) (2, 3, 2) ~~(2, 4, 1)~~
(3, 1, 3) (3, 2, 2) (3, 3, 1) $\underline{6}$
~~(4, 1, 2)~~ ~~(4, 2, 1)~~
~~(5, 1, 1)~~

$|\chi(X_{\mathbb{R}}) - 1| \leqq 19$

$-19 \leqq \chi(X_{\mathbb{R}}) - 1 \leqq 19$

$-18 \leqq \chi(X_{\mathbb{R}}) \leqq 20$

$b_i = \dim H_i(X_{\mathbb{R}}; \mathbb{Z}/2)$

$b_0 + b_1 + b_2 \leqq 24$
$b_0 - b_1 + b_2 \leqq 20$

$X_{\mathbb{R}}$: orientable $\therefore b_2 = b_0$

$2b_0 + b_1 \leqq 24$, $2b_0 - b_1 \leqq 20$ $\quad \boxed{b_0 \leqq 11}$

If $b_0 = 11 \Rightarrow \begin{cases} b_0 + b_1 + b_2 = 24 \\ b_0 - b_1 + b_2 = 20 \end{cases}$ $\quad b_0 = 11,\ b_2 = 11$, $b_1 = 2$

$$-\chi(X_{\mathbb{R}}) \equiv \sigma(X) \ (16)$$
$$-20 \quad = -16$$

(Rokhlin の定理)

contradict

(3, 19)

Ex $\exists \Sigma_2 \amalg 9\Sigma_0$ $\quad b_0 = 10,\ b_2 = 10,\ b_1 = 4$
$\chi = 16 \ (= \sigma = -16)$

Topological Classification of surfaces of deg 4

$\Sigma_p \amalg a\Sigma_0 \quad a \geqq 0,\ p \geqq 0 \quad a + p \leqq 9$ (Kharlamov)

$\Sigma_p \amalg a\Sigma_0 \quad a \geqq 0,\ p > 0,\ a + p = 10$
$\qquad a - p \equiv 0, -2 \ (8)$

$\Sigma_p \amalg a\Sigma_0 \quad a \geqq 0,\ p > 0,\ a + p = 11$
$\qquad a - p \equiv -1 \ (8)$

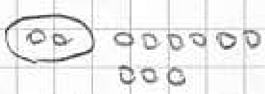
$\Sigma_1 \amalg \Sigma_1$, ϕ

(Kharlamov)

$2 + 2p + 2a = 2(a + p) + 2 = 24$
$\underline{a + p = 11}$

$\underline{9\text{-}2}$ $\qquad \underline{5\text{-}6}$ $\qquad$ 1-9
2-8
2-4
1-10

$\Sigma_2 \amalg 9\Sigma_0$, $\Sigma_6 \amalg 5\Sigma_1$



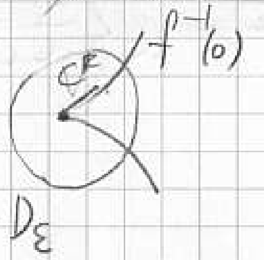
1-10

$\Sigma_{10} \amalg \Sigma_0$

local version

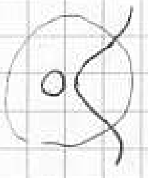
$f(x,y)$　$\mathbb{R}^2$の原点の近傍での real analytic

$\mu < \infty$　$\mu = \dim_{\mathbb{R}} \mathbb{R}[[x,y]] \big/ \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ ↑ideal



$F(x,y,t)$　fの変形　$f_t(x,y) = F(x,y,t)$

$t \neq 0$ 小



$C^R_t = f_t^{-1}(0) \cap D_\varepsilon$

：非特異

D_εの 1次元部分多様体

連結成分は　S^1　または $I = [0,1]$ と diffeo

oval　　interval

$f = 0$ の定める 複素解析的曲線 C

既約成分の個数 k　(link の comp 数)

k_1：実既約成分の個数

k_0：非 ——　　k_0：偶数

$k = k_1 + k_0$

Ⅱa (local Harnack inequality)

$\#$intervals $= k_1$

$k_1 \geqq 1 \Rightarrow l_0 \leqq \frac{1}{2}(\mu - k + 1)$

$k_1 = 0 \Rightarrow l_0 \leqq 1 + \frac{1}{2}(\mu - k_0 + 1)$

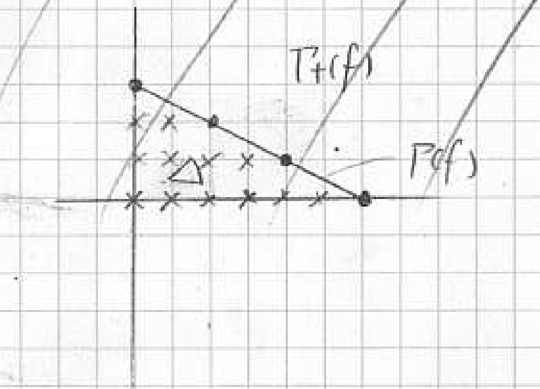
等号　M-perturbation　とよぶ

Ex　J_{10}　$\mu = 10$,　$k_1 = 3$, $k_0 = 0$　$k = 3$

$l_0 \leqq \frac{1}{2}(10 - 3 + 1) = 4$

J_{10}　$f = (y - x^2)(y + x^2)(y - 2x^2)$

$= 2x^6 - x^4 y - 2x^2 y^2 + y^3$



∘ Prob　M-perturbation は存在するか？

in $\mathbb{C}^2$ $f_t(x,y)=0$ $(t\neq 0)$

$\tau : \Sigma \to \Sigma$

$\chi(M) = 1-\mu$

$\chi(\Sigma) = 1-\mu+k$

Σの genus を g とおくと $1-\mu+k = 2-2g$

$$g = \frac{1}{2}(\mu-k+1)$$

$k_1 \geqq 1$

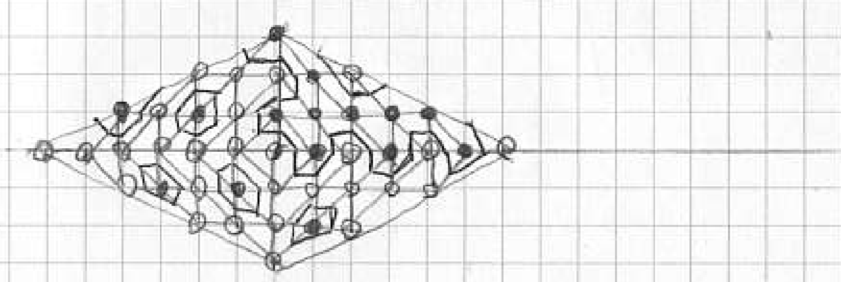
$1+l_0 \leqq \frac{1}{2}(\mu-k+1)+1$ $\quad l_0 \leqq \frac{1}{2}(\mu-k+1)$

$k_1=0$

$l_0 \leqq \frac{1}{2}(\mu-k_0+1)+1$

特異点によっては M-perturb が存在しない場合がある

$\boxed{J_{10}}$



$f(x,y) = (y-x^2)(y-2x^2)(y-3x^2)$

$= (y^2-2x^2y-x^2y+2x^4)(y-3x^2)$

$= (y^2-3x^2y+2x^4)(y-3x^2)$

$= y^3-3x^2y^2+2x^4y$

$\quad -3x^2y^2+9x^4y-6x^6$

$y^3-6x^2y^2+11x^4y-6x^6$

$f(x,y) = (x^2-y)(2x^2-y)(3x^2-y)$

Topology of real rational curves

$\left\| \begin{array}{l} f(x,y)=0 \\ (x(t),\ y(t)) \end{array} \right.$ 曲線とは

$f: \mathbb{R}P^1 \longrightarrow \mathbb{R}P^2$

$f(x_0 : x_1) = (f_0(x_0,x_1) : f_1(x_0,x_1) : f_2(x_0,x_1))$

f_i : homog polyn of degree d.

共通零点が $(x_0, x_1)=(0,0)$

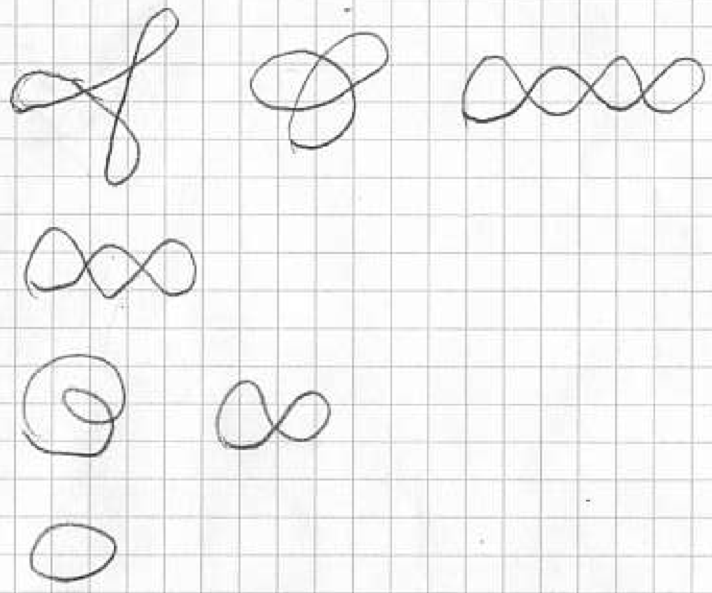
generic

Lemma : #selfint pt $l \leqq \frac{1}{2}(d-1)(d-2)$

(Harnack型不等式)

等号 M-curve.

$d=4$



Fourier curve

$f: \underset{\underset{\mathbb{C}}{\cap}}{S^1} \to \mathbb{C}$ n次 plane Fourier curve

$\Leftrightarrow f(\theta) = (f_1(\theta), f_2(\theta))$

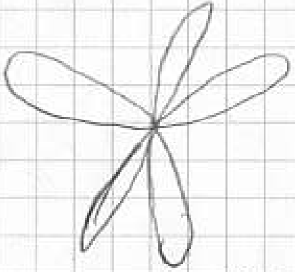
$f_i(\theta)$: $\sin\theta$, $\cos\theta$ の n次多項式

Ex. $z^3 - z^{-2}$

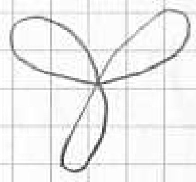
$(\cos 3\theta + i\sin 3\theta) - (\cos(-2\theta) + i\sin(-2\theta))$

$= \cos 3\theta - \cos 2\theta + i(\sin 3\theta + \sin 2\theta)$

$\begin{cases} f_1 = \cos 3\theta - \cos 2\theta \\ f_2 = \sin 3\theta + \sin 2\theta \end{cases}$ $\left| \begin{array}{l} f_1 = \cos 2\theta - \cos\theta \\ f_2 = \sin 2\theta + \sin\theta \end{array} \right.$



サクラ



エンレイソウ

○ n次 Fourier curve は $2n$次 real rational curve.

$\underline{\mathrm{Th}}$ $f: S^1 \to \mathbb{C} = \mathbb{R}^2$ n次 generic Fourier curve

f : M-curve $\Rightarrow$ fは <u>変曲点をもたない</u>.

Cor  型の 2次Fourier curve は 存在しない

$C \subset \mathbb{C}P^2$ degree m

$C, C^\vee$: generic front

$$k = m(m-1) - 2d - 3r$$
$$m = k(k-1) - 2t - 3w$$
$$w = 3m(m-2) - 6d - 8r$$
$$r = 3k(k-2) - 6t - 8w$$

$$m + w' + 2t'' = k + r' + 2d''$$

$m = \deg C$ $\quad$ d : Cの double pt の個数

$k = \deg C^\vee$ $\quad$ r: — cusp —

t: Cの double tangent

w Cの inflexion point

w' real inf $\quad$ r': real cusp $\quad$ <u>t'' 孤立2重接線</u> $\quad$ d'' 孤立2重点.

"encomplexed invariant"

If C is Fourier M-curve

$\Rightarrow$ $m = 2n$, $\quad d = \ell = (2n-1)(n-1)$

$r = 0$

$k = 2(2n-1)$

$r' = 0$, $d'' = 0$

$2n + w' + 2t'' = 2(2n-1)$

$w' + 2t'' = 2n-2$

$t'' \geq n-1$ $\quad$ $\boxed{w' = 0}$

おしまい