Jacobian-squared function-germs

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REFERENCE

[N] T. N., *Jacobian-squared function-germs*, Pure and Applied Mathematics Quarterly, **13** (2017), 711-728.

MOTIVATION

The MOTIVATION of the reference is one fact found in the following Mather's prominent paper:

J. Mather, *Generic Projections*, Annals of Mathematics, **98** (1973), 226–245.

In order to explain motivation in detail, let me define several fundamental notions of this talk.

DEFINITION 1

(1)

$$\overset{\text{Projection}}{\Leftrightarrow}\quad \pi : \mathbb{R}^{n+1} \to \mathbb{R}^p \text{ linear surjective}$$

(2)

$$S^n = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} x_i^2 = 1 \right\}.$$

(3)

$$\overset{\text{Projection of } S^n}{\Leftrightarrow}\quad \pi|_{S^n} : S^n \to \mathbb{R}^p \text{ restriction}$$

FACT 1 (J. Mather) *Let n, p be positive integers such that $n+1 \geq p$. Then,*

(1) *Any two $\pi_1|_{S^n}, \pi_2|_{S^n}$ are $\mathcal{A}$-equivalent. More precisely, there exist a rotation $h : S^n \to S^n$ and a linear isomorphism $H : \mathbb{R}^p \to \mathbb{R}^p$ such that*

$$\boxed{\pi_1|_{S^n} = H \circ (\pi_2|_{S^n}) \circ h.}$$

(2) *Every $\pi|_{S^n}$ is stable. More precisely, the singular point set $\Sigma(\pi|_{S^n})$ is a $(p-1)$-dimensional sphere consisting of definite fold singular points and*

$$\pi|_{\Sigma(\pi|_{S^n})} : \Sigma(\pi|_{S^n}) \to \mathbb{R}^p$$

is an embedding.

This fact might be not so profound. But, I wanted to view a projected image of Whitney umbrella inside the unit sphere. So, I wanted to investigate what one can get by projecting a Whitney umbrella inside the unit sphere. In this talk, let me first recover my investigation. From now on, let's concentrate on the case $n = p = 3$ and the orthogonal projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

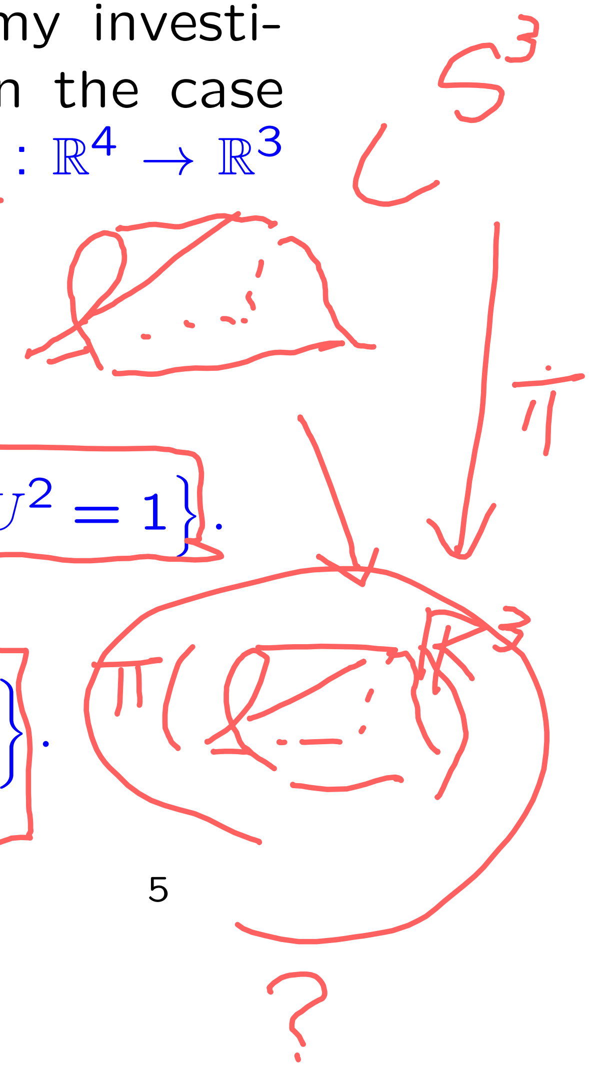
$$\boxed{\pi(X, Y, Z, U) = (X, Y, Z)}$$

and the restriction of π to

$$\boxed{S^3 = \left\{(X, Y, Z, U) \in \mathbb{R}^4 \;\middle|\; X^2 + Y^2 + Z^2 + U^2 = 1\right\}}.$$

Let $\mathcal{W} \in \mathbb{R}^3$ be the open set defined by

$$\boxed{\mathcal{W} = \left\{(\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3 \;\middle|\; -\frac{\pi}{2} < \theta_i < \frac{\pi}{2}\right\}}.$$



Let $\varphi : \mathcal{W} \to S^3$ be the parametrization defined by

$$
\begin{aligned}
X \circ \varphi(\theta_1, \theta_2, \theta_3) &= \cos\theta_1 \cos\theta_2 \cos\theta_3, \\
Y \circ \varphi(\theta_1, \theta_2, \theta_3) &= \cos\theta_1 \cos\theta_2 \sin\theta_3, \\
Z \circ \varphi(\theta_1, \theta_2, \theta_3) &= \cos\theta_1 \sin\theta_2, \\
U \circ \varphi(\theta_1, \theta_2, \theta_3) &= \sin\theta_1.
\end{aligned}
$$

So, θ_1 is the latitude and θ_2, θ_3 are longitudes.

S^3

Then $\pi \circ \varphi(\theta_1, \theta_2, \theta_3)$ is

$$(\cos\theta_1 \cos\theta_2 \cos\theta_3,\ \cos\theta_1 \cos\theta_2 \sin\theta_3,\ \cos\theta_1 \sin\theta_2)\,.$$

Set

$$\begin{aligned} \Psi(\theta_1, \theta_2, \theta_3) &= \left(1-\theta_1^2,\ \theta_2,\ \theta_3\right), \\ H(X, Y, Z) &= (\psi(X)\cos Y \cos Z,\ \psi(X)\cos Y \sin Z,\ \psi(X)\sin Y)\,, \end{aligned}$$

where $\psi(X) = 1 - \frac{1}{2!}(1-X) + \frac{1}{4!}(1-X)^2 - \frac{1}{6!}(1-X)^3 + \cdots$.

Then,

$$\psi(1-\theta_1^2) = 1 - \frac{1}{2!}\theta_1^2 + \frac{1}{4!}\theta_1^4 - \frac{1}{6!}\theta_1^6 + \cdots = \cos\theta_1.$$

Thus, we have

$$
\begin{aligned}
& H \circ \Psi(\theta_1, \theta_2, \theta_3) \\
= & (\cos\theta_1 \cos\theta_2 \cos\theta_3,\ \cos\theta_1 \cos\theta_2 \sin\theta_3,\ \cos\theta_1 \sin\theta_2) \\
= & \pi \circ \varphi(\theta_1, \theta_2, \theta_3).
\end{aligned}
$$

holds. Moreover, $H : (\mathbb{R}^3, (1,0,0)) \to (\mathbb{R}^3, (1,0,0))$ is clearly a germ of C^∞ diffeomorphism and Thus, $\pi \circ \varphi : (\mathcal{W}, \mathbf{0}) \to (\mathbb{R}^3, (1,0,0))$ is $\mathcal{L}$-equivalent to $\Psi : (\mathcal{W}, \mathbf{0}) \to (\mathbb{R}^3, (1,0,0))$.

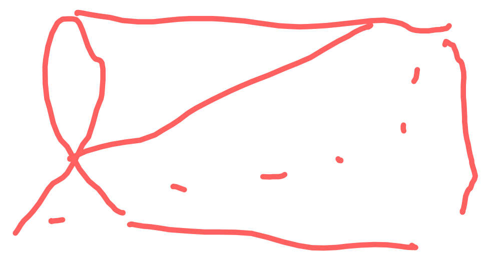
Next, let $\mathcal{V} \subset \mathbb{R}^2$ be a small open neighborhood of $\mathbf{0}$ $\in \mathbb{R}^2$
and let $f : \mathcal{V} \to \mathbb{R}^2$ be defined by

$$f(x, y) = \left(\frac{1}{3}x^3 + xy,\ y\right).$$

Any map-germ $g : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ $\mathcal{A}$-equivalent to f is called a *plane-to-plane cusp singularity*. Notice that the Jacobian determinant $|Jf|$ is $x^2 + y$ for our f.

Let $F : \mathcal{V} \to \mathbb{R} \times \mathbb{R}^2$ be defined by

$$\begin{aligned} F(x, y) &= (|Jf|(x, y),\ f(x, y)) \\ &= \left(x^2 + y,\ \frac{1}{3}x^3 + xy,\ y\right) \end{aligned}$$



Any map-germ $G : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ $\mathcal{A}$-equivalent to F is called a *Whitney umbrella*.

Assume $\mathcal{V}$ is sufficiently small so that
$F(\mathcal{V}) \subset \mathcal{W} = \left\{(\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3 \;\middle|\; -\frac{\pi}{2} < \theta_i < \frac{\pi}{2}\right\}$.
Let's calculate $\pi \circ \varphi \circ F : \mathcal{V} \to \mathbb{R}^3$.

$$\begin{aligned}
\pi \circ \varphi \circ F(x, y) &= H \circ \Psi\left(x^2 + y, \frac{1}{3}x^3 + xy, y\right) \\
&= H\left(1 - \left(x^2 + y\right)^2, \frac{1}{3}x^3 + xy, y\right)
\end{aligned}$$

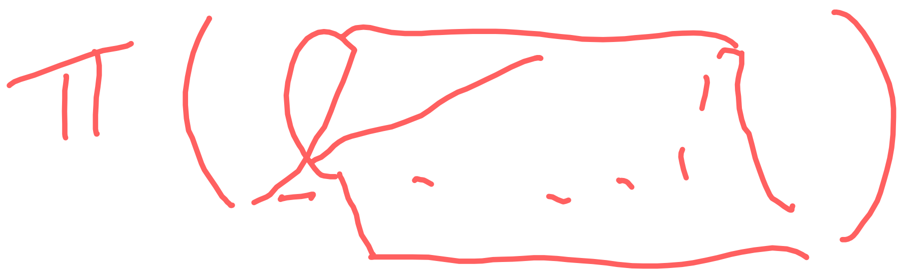
and H was a germ of C^∞ diffeomorphism.

Therefore, $\pi \circ \varphi \circ F$ is $\mathcal{L}$-equivalent to

$$\widetilde{F}(x,\ y) = \left((x^2+y)^2,\ \frac{1}{3}x^3 + xy,\ y \right) = (|Jf|^2(x,y),\ f(x,y)).$$

The function $|Jf|^2$ is called
the *Jacobian-**squared** function* of f.

Thus, in order to view the shape of projected image of Whitney umbrella inside S^3, it is sufficient to view the image of $\widetilde{F}$



$H_1 : \mathbb{R}^3 \to \mathbb{R}^3$

$$\widetilde{F}(x,\ y) = \left((x^2+y)^2,\ \frac{1}{3}x^3 + xy,\ y\right) = \left(x^4 + 2x^2y + y^2,\ \frac{1}{3}x^3 + xy,\ y\right).$$

Set $H_1(X,\ Y,\ Z) = (X - Z^2,\ Y,\ Z)$. Then, H_1 is a C^∞ diffeomorphism and we have

$$H_1 \circ \widetilde{F}(x,\ y) = \left(x^4 + 2x^2y,\ \frac{1}{3}x^3 + xy,\ y\right).$$

Set $H_2(X,\ Y,\ Z) = (3X,\ -12Y,\ 6Z)$. Then, H_2 is a C^∞ diffeomorphism and we have

$$H_2 \circ H_1 \circ \widetilde{F}(x,\ y) = \left(3x^4 + 6x^2y,\ -4x^3 - 12xy,\ 6y\right).$$

Finally, set $h_1(x, y) = \left(x, \frac{1}{6}y\right)$. Then,

$$H_2 \circ H_1 \circ \widetilde{F} \circ h_1(x, y) = \left(3x^4 + x^2y, -4x^3 - 2xy, y\right),$$

which is well-known as the normal form of *swallowtail*.

Thus, we confirmed that the image of Whitney umbrella inside S^3 by the canonical projection $\pi : S^3 \to \mathbb{R}^3$ is nothing but a swallowtail.

This is the motivation of my study on Jacobian-squared function germs.

What is the role of Jacobian-squared function-germs ?

Before stating my answer, let me explain several notions.

DEFINITION 2 A C^∞ map-germ $G : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^{n+\ell}, \mathbf{0})$ is called a *frontal* if there exist vector fields $\Phi_1, \ldots, \Phi_\ell$: $(\mathbb{R}^n, \mathbf{0}) \to T\mathbb{R}^{n+\ell}$ along G such that the three conditions in the next slide are satisfied.

(1) $\phi_i(x) \cdot tG(\xi)(x) = 0$ for any i $(1 \leq i \leq \ell)$ and any $\xi \in \theta(n)$, where $\Phi_i(x) = (G(x), \phi_i(x))$ and the dot in the center stands for the scalar product of two vectors in $T_{G(x)}\mathbb{R}^{n+\ell}$.

(2) $\phi_i(\mathbf{0}) \neq \mathbf{0}$ for any i $(1 \leq i \leq \ell)$.

(3) $\phi_1(\mathbf{0}), \ldots, \phi_\ell(\mathbf{0})$ are linearly independent.

$\Phi_i : \mathbb{R}^n \to \mathbb{R}^{n+\ell}$

$x \mapsto (G(x), \phi_i(x))$

DEFINITION 3 Let $f = (f_1, \ldots, f_n) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an equidimensional map-germ.

(1) Let Ω_n^1 denote the $\mathcal{E}_n$-module of 1-forms on $(\mathbb{R}^n, 0)$. Then, the $\mathcal{E}_n$-module generated by df_i $(i = 1, \ldots, n)$ in Ω_n^1 is called the *Jacobi module* of f and is denoted by $\mathcal{J}_f$, where dh for a function-germ $h : (\mathbb{R}^n, 0) \to \mathbb{R}$ stands for the exterior differential of h.

(2) The *ramification module* of f (denoted by $\mathcal{R}_f$) is defined as the $f^*(\mathcal{E}_n)$-module consisting of all function-germs φ such that $d\varphi$ belongs to $\mathcal{J}_f$.

$$\mathcal{R}_f = \{\varphi / d\varphi \in \mathcal{J}_f\}$$

$$\mathcal{J}_f = \langle df_1, \ldots, df_n \rangle_{\mathcal{E}_n}$$

THEOREM 1 *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an equidimensional map-germ. Then, the following inclusion holds:*

$$|Jf|\Omega_n^1 \subset \mathcal{J}_f.$$

$$\boxed{|Jf|\,\Omega_n^1} \subset \mathcal{J}_f$$

$|Jf| :$

Since $d(\mu|Jf|^2) = |Jf|(|Jf|d\mu + 2\mu\ d|Jf|) \in |Jf|\Omega^1_n$ for any $\mu \in \mathcal{E}_n$, the following corollary can be obtained from Theorem 1.

COROLLARY 1 *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an equidimensional map-germ. For any i $(1 \le i \le \ell)$, let $\mu_i : (\mathbb{R}^n, 0) \to \mathbb{R}$ be a function-germ. Then, the map-germ $F : (\mathbb{R}^n, 0) \to \mathbb{R}^{n+\ell}$ defined by*

$|Jf|^2$

$$F = \left(f, \mu_1|Jf|^2, \ldots, \mu_\ell|Jf|^2\right)$$

is always a frontal.

There are several advantages of Corollary 1.

(1) Construction of non-trivial frontals is very easy.

(2) Similarly as in the case of swallowtail, well-known frontals can be easily constructed by Theorem 1.

(3) (At least for me), normal forms of famous frontals (especially coefficients of them) are not easy to memorize. On the other hand, construction by using Jacobian-squared function-germs provides very simple forms.

$\Phi : \mathbb{R}^2 \to \mathbb{R}^\ell$

$3 \leq \ell$

EXAMPLE 1 (Open Swallowtail) ~~Normal form of Open~~ Swallowtail: $\Phi = \left(x^3 + \cancel{xy,\ y,\ x^4} + \frac{2}{3}x^2y,\ x^5 + \frac{5}{9}x^3y,\ 0, \dots, 0\right)$.

Corollary

By using ~~Theorem~~ 1, Φ is constructed as follows:

Materials: $f(x,y) = \boxed{\left(x^3 + xy,\ y\right)}$,

$\mu_1(x,y) = 1$, $\mu_2(x,y) = x$,

$\mu_i(x,y) = 0 \ (3 \leq i \leq \ell)$.

In this case, our frontal F has the form

$$\begin{aligned}
F(x,y) &= \left(f(x,y), \mu_1(x,y)|Jf|^2(x,y), \dots, \mu_\ell(x,y)|Jf|^2(x,y)\right) \\
&= \left(x^3 + xy,\ y,\ (3x^2 + y)^2,\ x(3x^2 + y)^2, 0, \dots, 0\right) \\
&= \left(x^3 + xy,\ y,\ 9x^4 + 6x^2y + \cancel{y^2},\right. \\
&\qquad \left. 9x^5 + 6x^3y + \cancel{xy^2}, 0, \dots, 0\right).
\end{aligned}$$

Set

$$H_1(X, Y, U_1, U_2, U_3, \ldots, U_\ell) = (X,\ Y,\ U_1 - Y^2,\ U_2 - XY,\ U_3, \ldots, U_\ell).$$

$$H_2(X, Y, U_1, U_2, U_3, \ldots, U_\ell) = \left(X,\ Y,\ \frac{1}{9}U_1,\ \frac{1}{9}U_2, U_3, \ldots, U_\ell\right).$$

Then,

$$H_1 \circ F(x, y) = \left(x^3 + xy,\ y,\ 9x^4 + 6x^2y,\ 9x^5 + 5x^3y, 0, \ldots, 0\right).$$

$$\begin{aligned} &H_2 \circ H_1 \circ F(x, y) \\ &= \left(x^2 + xy,\ y,\ x^4 + \frac{2}{3}x^2y, x^5 + \frac{5}{9}x^3y, 0, \ldots, 0\right) \\ &= \Phi(x, y). \end{aligned}$$

Since H_1, H_2 are C^∞ diffeomorphisms, F and Φ are $\mathcal{L}$-equivalent.

Proof of Theorem 1

Let $\widetilde{Jf}$ be the **cofactor matrix of the Jacobian matrix** $\widetilde{Jf}$. Then, notice that $\widetilde{Jf}Jf = |Jf|E_n$ where E_n is the $n \times n$ unit matrix. For any 1-form $\alpha = \sum_{i=1}^{n} a_i dx_i$, we have the following:

$$\begin{aligned} |Jf|\alpha &= (a_1, \ldots, a_n)\,\widetilde{Jf}\,Jf \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix} \\ &= (a_1, \ldots, a_n)\,\widetilde{Jf} \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} \in \mathcal{J}_f. \end{aligned}$$

This completes the proof. $\square$

$$\alpha = (a_1, \ldots, a_n)\begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix} = (a_1, \ldots, a_n) E_n \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}$$

Is any frontal germ constructed in this way ?

PROPOSITION 1 (Ishikawa) *For any frontal germ* $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n+\ell}, 0)$, *there exist germs of diffeomorphism* $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *and* $H : (\mathbb{R}^{n+\ell}, 0) \to (\mathbb{R}^{n+\ell}, 0)$, *an equidimensional map-germ* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *and elements* $\psi_1, \ldots, \psi_\ell$ *of* $\mathcal{R}_f$ *such that the following equality holds:*

$$H \circ F \circ h = (f, \psi_1, \ldots, \psi_\ell).$$

Based on Proposition 1, it is natural to ask the converse of Corollary 1. However, if $\dim_{\mathbb{R}} Q(f) > 3$, then there exist counterexamples against the converse of Corollary 1. Thus, we ask the converse of Corollary 1 in the case $\dim_{\mathbb{R}} Q(f) \leq 3$.

THEOREM 2 *Let $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n+\ell}, 0)$ be a frontal germ. Suppose that there exist germs of diffeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $H : (\mathbb{R}^{n+\ell}, 0) \to (\mathbb{R}^{n+\ell}, 0)$, an equidimensional map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ with $\dim_{\mathbb{R}} Q(f) \leq 3$ and elements $\psi_1, \ldots, \psi_\ell$ of $\mathcal{R}_f$ such that the following equality holds:*

$$H \circ F \circ h = (f, \psi_1, \ldots, \psi_\ell).$$

Then, the following holds:

$$\left\langle |Jf|^2 \right\rangle_{\mathcal{E}_n} + f^*(\mathcal{E}_n) = \mathcal{R}_f.$$

COROLLARY 2 *Let $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n+\ell}, 0)$ be a frontal germ. Suppose that there exist germs of diffeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $H : (\mathbb{R}^{n+\ell}, 0) \to (\mathbb{R}^{n+\ell}, 0)$, an equidimensional map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ with* $\dim_{\mathbb{R}} Q(f) \leq 3$ *and elements $\psi_1, \ldots, \psi_\ell$ of $\mathcal{R}_f$ such that the following equality holds:*

$$\boxed{H \circ F \circ h} = \boxed{(f, \psi_1, \ldots, \psi_\ell).}$$

Then, there exist a germ of diffeomorphism $\widetilde{H} : (\mathbb{R}^{n+\ell}, 0) \to (\mathbb{R}^{n+\ell}, 0)$ and function-germs $\mu_i : (\mathbb{R}^n, 0) \to \mathbb{R}$ $(1 \leq i \leq \ell)$ such that

$$\widetilde{H} \circ H \circ F \circ h = \boxed{(f, \mu_1 |Jf|^2, \ldots, \mu_\ell |Jf|^2).}$$

QUESTION 1 *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an equidimensional map-germ. Then, does there exist a finitely generated $\mathcal{E}_n$-module A such that the following holds ?*

$$\boxed{A + f^*(\mathcal{E}_n) = \mathcal{R}_f.}$$

Notice that by Ishikawa, it is known if "f is finite and of corank one" or "it is $\mathcal{A}$-equivalent to a finite analytic map-germ", then there exists a finitely generated $f^*(\mathcal{E}_n)$-module B satisfying the equality:

$$\boxed{B + f^*(\mathcal{E}_n) = \mathcal{R}_f.}$$

Notice also that in the case of Mather's $\mathcal{A}_e$ tangent space for a map-germ $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, the corresponding $\mathcal{E}_n$-module is nothing but $tg(\theta(n))$. Thus, Question 1 asks whether or not the ramification module $\mathcal{R}_f$ has a similar structure as $TA_e(g)$.

$p = n$

$$TA_e(g) = \boxed{tg(\theta(n))} + \boxed{\omega g(\theta(p))}$$

$\mathcal{E}_n$

$g^*\mathcal{E}_n$

Thank you
for your kind attention!