Variational problems of anisotropic surface energy for hypersurfaces with singular points

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Today's objects: piecewise-C^2 hypersurfaces in $\mathbf{R}^{n+1}$

Today's subject: variational problem of anisotropic surface energy

Main references:

[1] M. Koiso. Uniqueness of stable closed non-smooth hypersurfaces with constant anisotropic mean curvature, preprint. arXiv:1903.03951 [math.DG]

[2] Y. Jikumaru and M. Koiso. Non-uniqueness of closed embedded non-smooth hypersurfaces with constant anisotropic mean curvature, preprint. arXiv:1903.03958 [math.DG]

2 Anisotropic surface energy

Formulation of piecewise-C^2 hypersurface

Let $M = \cup_{i=1}^{k} M_i$ be an n-dimensional oriented compact connected C^∞ manifold, where each M_i is an n-dimensional connected compact submanifold of M with piecewise-C^∞ boundary, and $M_i \cap M_j = \partial M_i \cap \partial M_j$, $(i, j \in \{1, \cdots, k\}$, $i \neq j)$. We call a map $X : M \to \mathbf{R}^{n+1}$ a piecewise-C^2 weak immersion (or a piecewise-C^2 hypersurface) if X satisfies the following conditions (A1) - (A3) for $i = 1, \cdots, k$.

(A1) X is continuous, and each $X_i := X|_{M_i} : M_i \to \mathbf{R}^{n+1}$ is of C^2.

(A2) The restriction $X|_{M_i^o}$ of X to the interior M_i^o of M_i is a C^2-immersion.

(A3) The unit normal vector field $\nu_i : M_i^o \to S^n$ along $X_i|_{M_i^o}$ can be extended to a C^1-mapping $\nu_i : M_i \to S^n$. Here the orientation of ν_i is determined so that, if $(u^1, \cdots, u^n)$ is a local coordinate system in M_i, $\{\nu_i, \partial/\partial u^1, \cdots, \partial/\partial u^n\}$ gives the canonical orientation in $\mathbf{R}^{n+1}$.

Let $S^n = \{\nu \in \mathbf{R}^{n+1} \; ; \; |\nu| = 1\}$ **be an** n**-dimensional round sphere,**

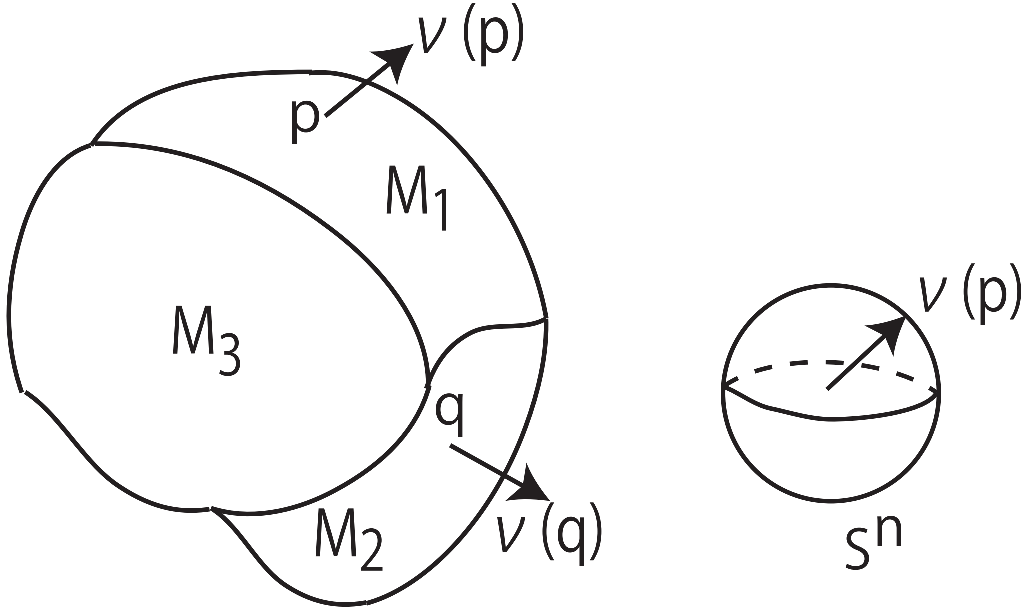
$\gamma : S^n \to \mathbf{R}_{\geq 0}$ **a continuous function, (energy density function),**

$M = \cup_{i=1}^k M_i$**,** n**-dimensional** C^∞ **manifold,**

$X : M \to \mathbf{R}^{n+1}$ **a piecewise-**C^2 **hypersurface,**

ν **: unit normal to each** M_i.

The anisotropic (surface) energy $\mathcal{F}_\gamma(X)$ **of** X **is defined as follows:**



$$\mathcal{F}_\gamma(X) := \sum_{i=1}^{k} \int_{M_i} \gamma(\nu(p)) \, dA.$$

Theorem A (J. E. Taylor, 1978) **Assume $\gamma : S^n \to \mathbf{R}_{>0}$ is positive and continuous. Let $V > 0$. Among all closed hypersurfaces in $\mathbf{R}^{n+1}$ enclosing the same $(n+1)$-dimensional volume V, there exists a unique absolute minimizer $W_\gamma(V)$ of $\mathcal{F}_\gamma = \int \gamma(\nu) dA$. And $W_\gamma(V)$ is a homothety of the Wulff shape W_γ.**

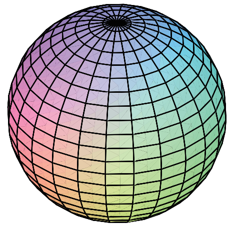
Here, $W_\gamma := \partial\left(\cap_{\nu \in S^n} \left\{ X \in \mathbf{R}^{n+1} \mid \langle X, \nu \rangle \leq \gamma(\nu) \right\} \right)$**.**

Hence, the Wulff shape is the solution (the minimizer) of the isoperimetric problem for the anisotropic surface energy $\mathcal{F}_\gamma = \int \gamma(\nu) dA$**.**

Remark 2.1 (i) $\gamma \equiv 1 \Longrightarrow W_\gamma$ **is the unit sphere. (i.e.** $W_1 = S^n(1)$**.)**

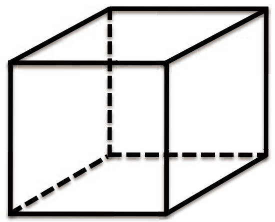
(ii) W_γ **is a closed convex hypersurface. It is not smooth in general.**

Example 2.1 **Recall the anisotropic surface energy is** $\mathcal{F}_\gamma = \int \gamma(\nu) dA$.

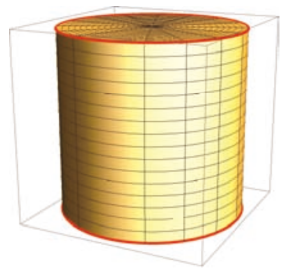


(i) For $\gamma \equiv 1$**, the Wulff shape** W_γ **is the unit sphere.**

(ii) If $\gamma(\nu) = \gamma(\nu_1, \cdots, \nu_{n+1}) = \sum_{i=1}^{n+1} |\nu_i|$**,** $(\nu \in S^n)$**,** W_γ **is the cube** $\{x = (x_1, \cdots, x_{n+1}) \in \mathbf{R}^{n+1} \mid \mathbf{max}\{|\mathbf{x_1}|, \cdots, |\mathbf{x_{n+1}}|\} = \mathbf{1}\}$**.**

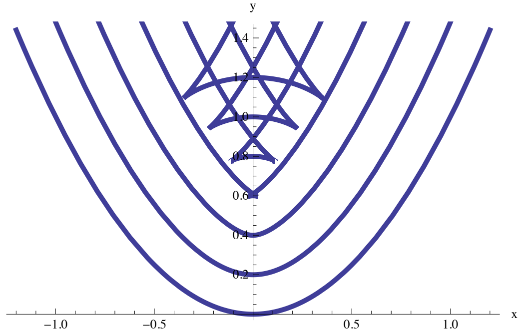


(iii) If $r > 0$**,** $h > 0$**, and** $\gamma(\nu) = r\sqrt{\nu_1^2 + \cdots + \nu_n^2} + h|\nu_{n+1}|$**, then**



W_γ **is the cylinder with radius** r **and height** $2h$**.**

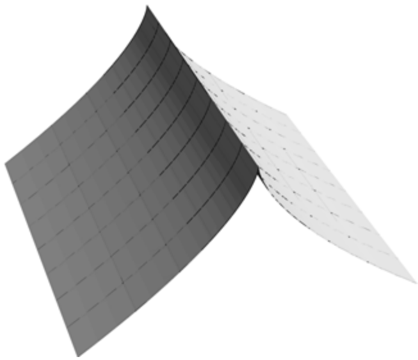
3 Parallel surfaces and (wave) fronts

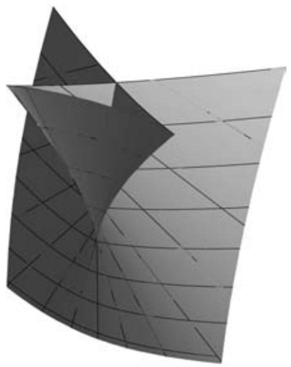


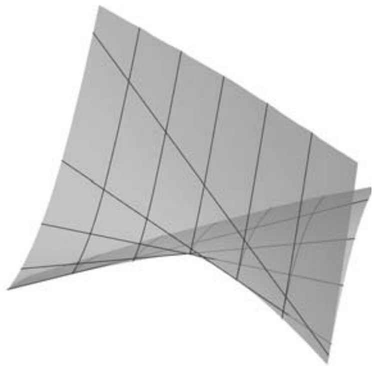
parallel curves $f^t(s) := f(s) + t\,N(s)$

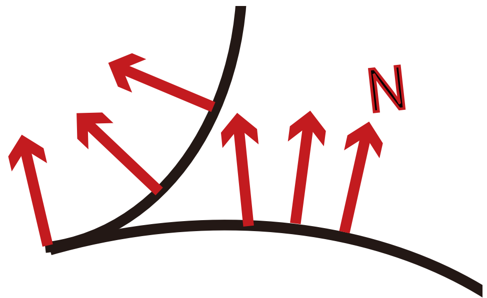
singular points appear at $\exists t$**.**

Similarly, for parallel surfaces $f^t(u,v) := f(u,v) + t\,N(u,v)$**, singular points appear at** $\exists t$**. These surfaces are called (wave) fronts.**







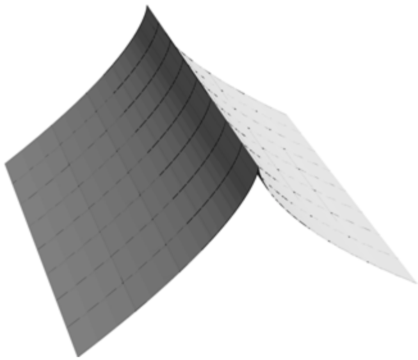
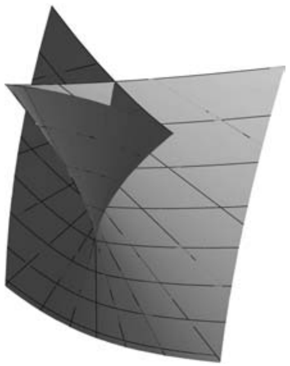
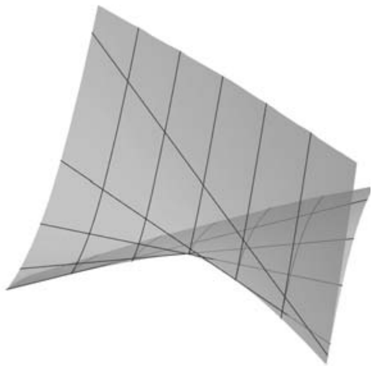
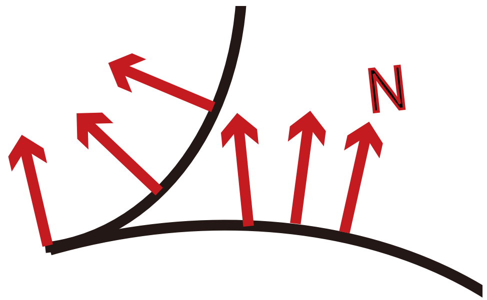


Cuspidal Edge, Swallow Tail, Cuspidal Butterfly. N **is smooth.**

Definition 3.1 **Let $r \geq 2$ be an integer. A mapping $X : M \to \mathbf{R}^{n+1}$ is called a C^r (wave) front if there exists a C^{r-1} mapping $\nu : M \to S^n$ such that the following (i) and (ii) hold.**

(i) $\langle \nu(p), X_*(u) \rangle = 0$ for all $p \in M$ and $u \in T_pM$.

(ii) The mapping $(X,\nu) : M \to \mathbf{R}^{n+1} \times S^n$ is an immersion.

Cuspidal Edge, Swallow Tail, Cuspidal Butterfly. **N is smooth.**

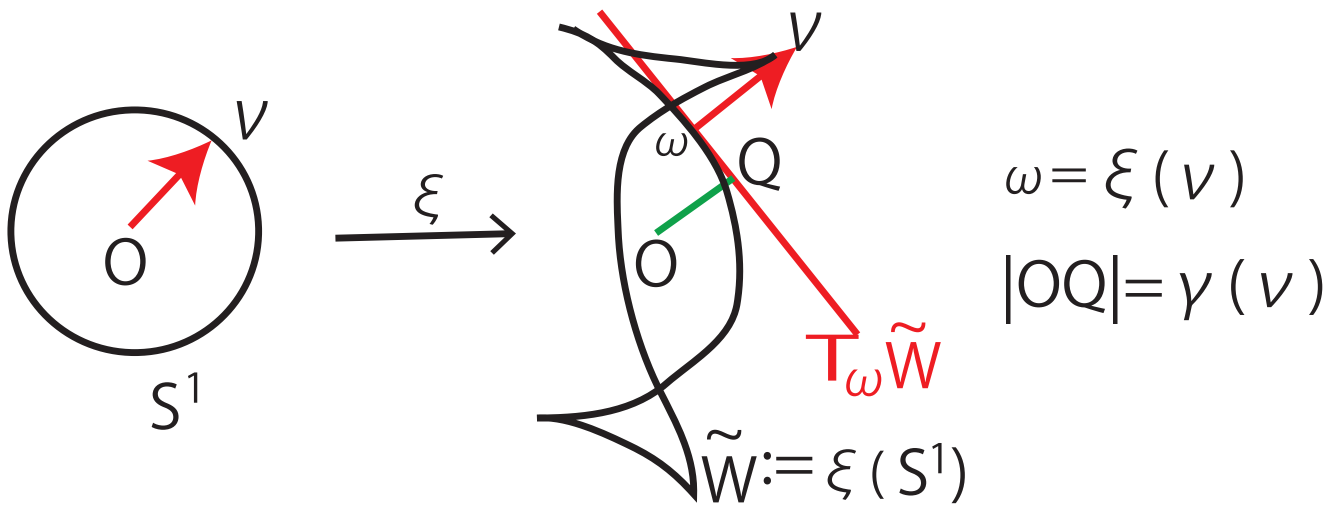
4 Cahn-Hoffman map and the Wulff shape

How to represent the Wulff shape? **Let** $\gamma : S^n \to \mathbf{R}_{\geq 0}$ **be of** C^2**.**

Definition 4.1 **The Cahn-Hoffman map** ξ_γ **is defined as**

$$\xi := \xi_\gamma : S^n \to \mathbf{R}^{n+1}, \quad \xi_\gamma(\nu) := D\gamma(\nu) + \gamma(\nu)\nu, \quad \forall \nu \in S^n.$$

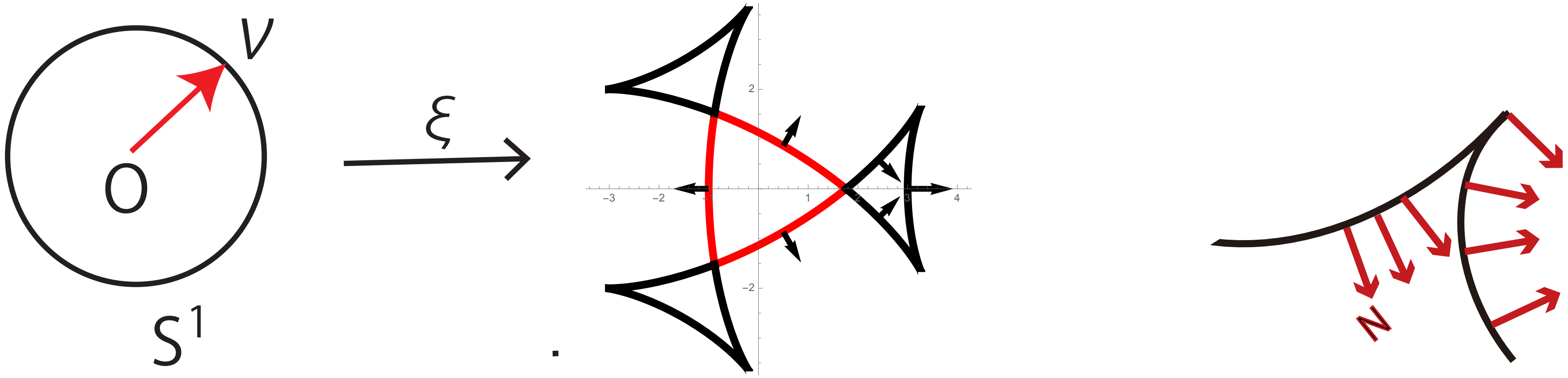
Remark 4.1 **(i)** ξ **is a front (§3).** **(ii)** γ **is the support function of** $\xi(S^n)$**. That is, in the picture below,** $\omega := \xi(\nu)$**, and** $|OQ| = \gamma(\nu)$**.**
(ii) ν **is the unit normal to** $\xi(S^n)$ **at** $\omega := \xi(\nu)$**.** **(iii)** $W_\gamma \subset \xi(S^n)$**.**



Example 4.1 **Let $n = 1$. For $\nu = (\nu_1, \nu_2) \in S^1 \subset \mathbf{R}^2$, define $\gamma(\nu) := 4\nu_1^3 - 3\nu_1 + 2$. The image of the Cahn-Hoffman map**

$$\xi : S^n \to \mathbf{R}^{n+1}, \quad \xi(\nu) := D\gamma(\nu) + \gamma(\nu)\nu, \quad \forall \nu \in S^n.$$

is given by the central picture below.



Fact 4.1 **The Wulff shape W_γ is the unique convex subset of $\xi_\gamma(S^n)$ s.t. the origin $0 \in \mathbf{R}^{n+1}$ is inside of the domain bounded by W_γ.**

5 Convex integrand

The homogeneous extension of $\gamma : S^n \to \mathbf{R}_{>0}$ is defined as follows.

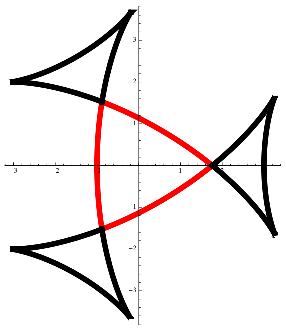
$$\overline{\gamma} : \mathbf{R}^{n+1} \to \mathbf{R}_{\geq 0}, \qquad \overline{\gamma}(rX) := r\gamma(X), \ \forall X \in S^n, \ \forall r \geq 0.$$

♣ **γ is called a convex integrand if and only $\overline{\gamma}$ is a convex function (that is, $\overline{\gamma}(X+Y) \leq \overline{\gamma}(X) + \overline{\gamma}(Y)$).**

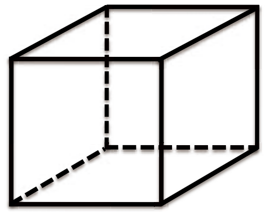
♡ **Assume γ is of C^1. Then, γ is a convex integrand.** $\Longleftrightarrow \xi(S^n) = W_\gamma$.

Example 5.1 (i) (convex) For $\gamma(\nu) = \sum_{i=1}^{n+1} |\nu_i|$, W_γ is a cube.

(ii) (non-convex) Let $n = 1$. For $\nu = (\nu_1, \nu_2) \in S^1$, define $\gamma(\nu) := 4\nu_1^3 - 3\nu_1 + 2$. $\xi(S^1)$ is given by the picture below. W_γ is the convex red curve.



$\xi(S^n) = W_\gamma$ **for (i)**

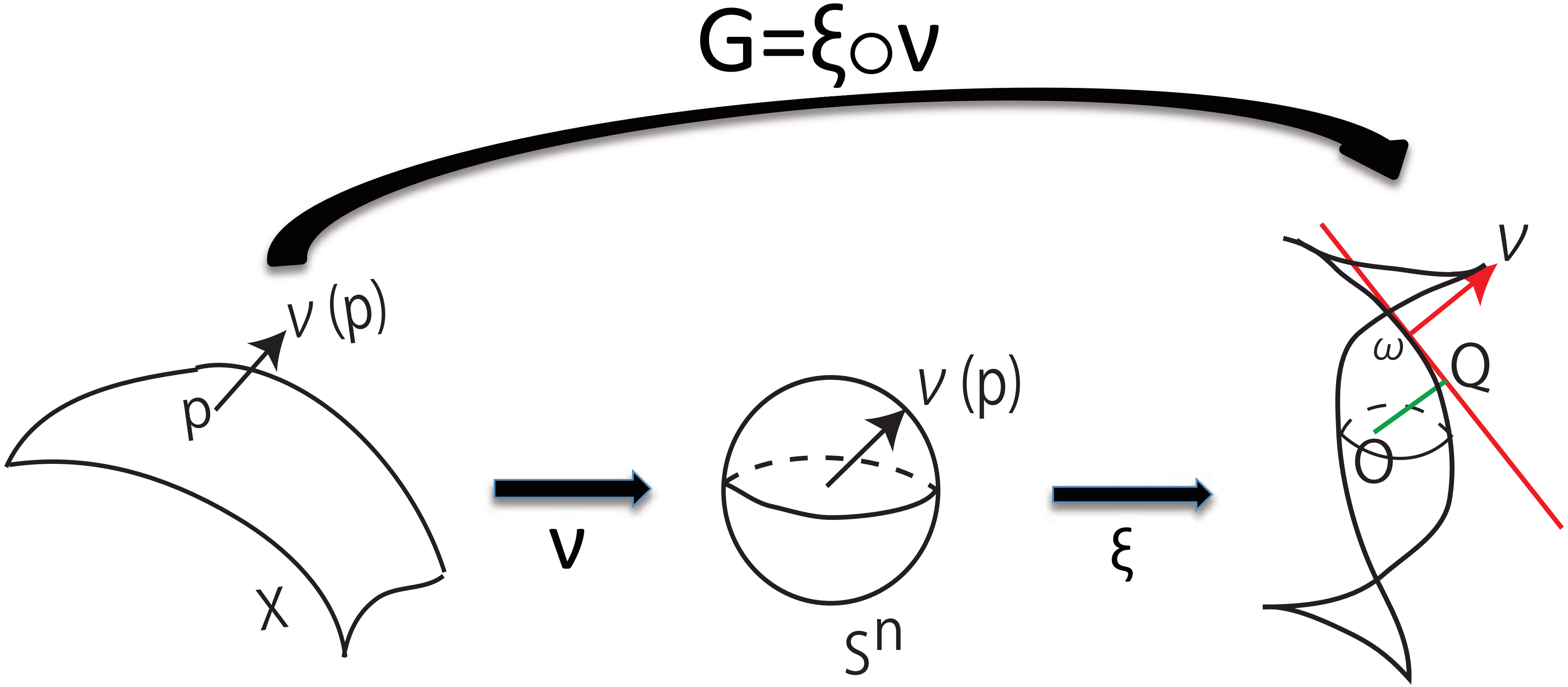


6 Cahn-Hoffman field and Euler-Lagrange equations

Assume $\gamma \in C^2(S^n)$. We will compare the shape of an arbitrary piecewise-C^2 hypersurface X with the Cahn-Hoffman map.

In order to do it, we introduce the "Cahn-Hoffman field" (or the anisotropic Gauss map) of X. The Cahn-Hoffman field (ξ-vector) was developed by Hoffman and Cahn (1972, 1974) in order to describe surface energy anisotropy in a first order phase transition represented by a sharp interface.

Definition 6.1 Let $X : M \to \mathbf{R}^{n+1}$ be a piecewise-C^2 hypersurface with unit normal ν. The map $G := \xi \circ \nu = D\gamma + \gamma(\nu)\nu : M \to \mathbf{R}^{n+1}$ is called the anisotropic Gauss map (or the Cahn-Hoffman field) of X. (Remark: In the picture below, $\omega := \xi(\nu) = D\gamma(\nu) + \gamma(\nu)\nu, \nu \in S^n$.)



Remark 6.1 **(i) Denote by** $\mu_1, \cdots, \mu_n$ **the principal curvatures of** $\xi_\gamma(\nu(p))$**. Let** w_{ii} **be the normal curvature of** X **at** p **for the principal direction of** μ_i**. Then the anisotropic mean curvature** Λ **of** X **is**

$$\Lambda = \frac{1}{n}\Big(\frac{w_{11}}{\mu_1} + \cdots + \frac{w_{nn}}{\mu_n}\Big).$$

(ii) When $\gamma \equiv 1$**,** $\Lambda = H$ **: the mean curvature of** X**.**

(iii) The anisotropic mean curvature of the Cahn-Hoffman map $\xi_\gamma : S^n \to \mathbf{R}^{n+1}$**, (**$\xi_\gamma(\nu) = D\gamma(\nu) + \gamma(\nu)\nu, \nu \in S^n$**) is** -1 **for the normal** ν**.**

Proposition 6.1 **[Euler-Lagrange equations, K2018]** **A piecewise-**C^2 **hypersurface** X**:** $M = \cup M_i \to \mathbf{R}^{n+1}$ **is a critical point of the anisotropic energy** $\mathcal{F}_\gamma(X) = \int_M \gamma(\nu)\, dA$ **for** $(n+1)$**-dimensional volume-preserving variations if and only if**

(i) $\Lambda \equiv$ **constant on** M**, and**

(ii) $(G|_{M_i} - G|_{M_j})(\zeta) \in T_\zeta(\partial M_i \cap \partial M_j)$ **at** $\forall \zeta \in \partial M_i \cap \partial M_j$**.** **(**$G = \xi \circ \nu$**.)**

Example 6.1 **Set** $\gamma : S^2 \to$ **R as** $\gamma(\nu) := \sqrt{|\nu_3^2 - \nu_1^2 - \nu_2^2|}$. **A CAMC surface is a CMC surface in** $\mathbf{R}_1^3 = (\mathbf{R}^3, dx^2 + dy^2 - dt^2)$. **For a graph** $t = f(x, y)$, Λ **and the mean curvature** H_L **as a surface in** $\mathbf{R}_1^3$ **satisfy**

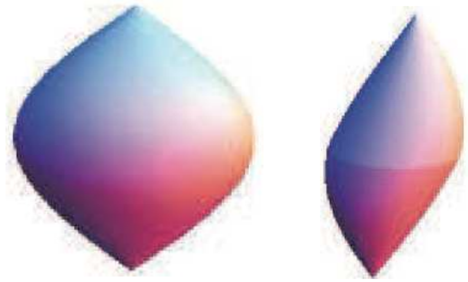
$$\Lambda|1 - f_x^2 - f_y^2|^{3/2} = (1 - f_y^2)f_{xx} + 2f_x f_y f_{xy} + (1 - f_x^2)f_{yy}, \qquad \Lambda = H_L.$$

Hence, the equation $\Lambda \equiv$ **constant is**

(a) **elliptic on "space-like parts", i.e.** $1 - f_x^2 - f_y^2 > 0$.

(b) **hyperbolic on "time-like parts", i.e.** $1 - f_x^2 - f_y^2 < 0$.

(a) $\Lambda = 0$



(b) $\Lambda = 0$

The image of the Cahn-Hoffman map is the elliptic hyperboloid of two sheets (one sheet) for the space-like (time-like) part (Honda-K-Tanaka '13).

7 Uniqueness and non-uniqueness for closed equilibria

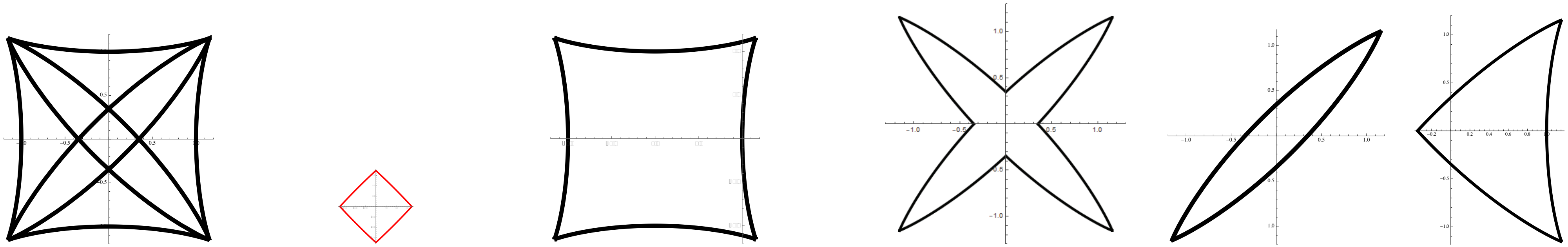
Is a closed CAMC hypersurface only the Wulff shape?

The answer is "No!" in general. However, if a $\gamma : S^n \to \mathrm{R}_{>0}$ is of C^∞ and strictly convex, then any closed CAMC hypersurface in R^{n+1} without self-intersection is a homothety of the Wulff shape! (For $\gamma \equiv 1$: Alexandrov 1962. For general γ: He-Li-Ma-Ge 2009.)

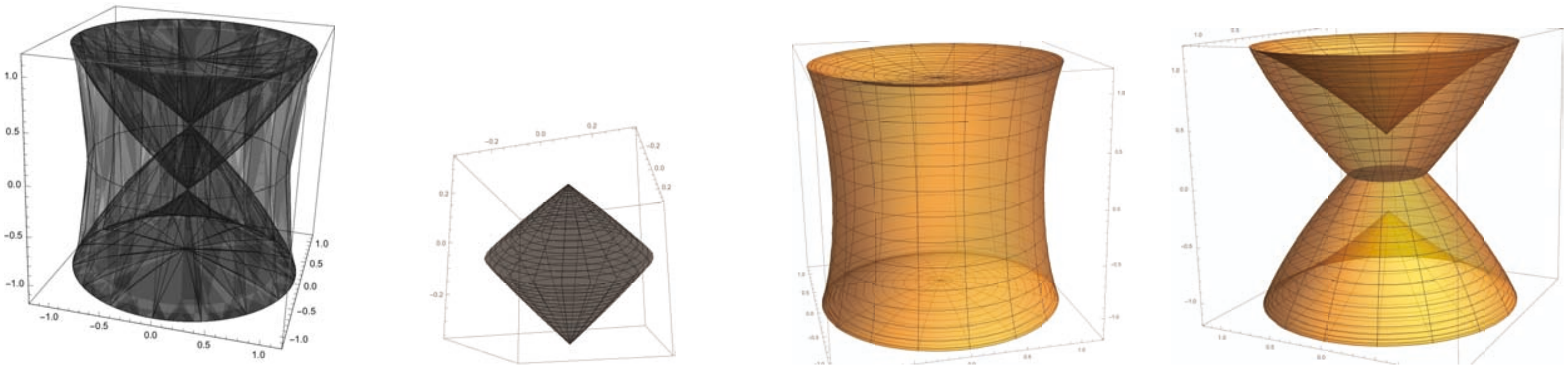
How about the case of weaker convexity of γ and lower regularity of hypersurfaces?

Theorem 7.1 (Jikumaru-K) There exists C^∞ functions $\gamma : S^n \to \mathrm{R}_{>0}$ s.t. there exist closed embedded CAMC hypersurfaces for γ which are not (any homotheties or translations of) the Wulff shape.

Example 7.1 **Define $\gamma : S^1 \to \mathbf{R}_{>0}$ as $\gamma(\nu_1, \nu_2) := \nu_1^6 + \nu_2^6$. The image of the Cahn-Hoffman map is the picture below left. Simple closed red curve and black curves are CAMC!**



We can get higher dimensional examples by rotation. For example, define $\gamma_1 : S^2 \to \mathbf{R}_{>0}$ by $\gamma_1(\nu_1, \nu_2, \nu_3) = (\nu_1^2 + \nu_2^2)^3 + \nu_3^6$. The image of the Cahn-Hoffman map is the picture below left. The black surface is the Wulff shape. Yellow surfaces are CAMC!

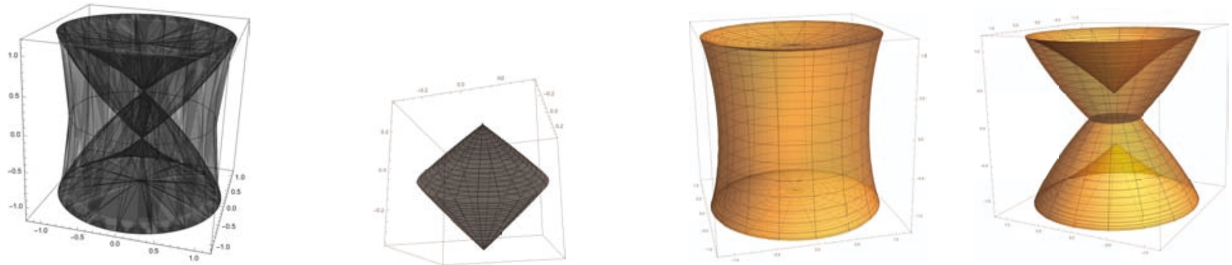


8 Application to the anisotropic mean curvature flow

Let $\gamma : S^n \to \mathbf{R}_{>0}$ be of C^2 with Cahn-Hoffman map $\xi_\gamma : S^n \to \mathbf{R}^{n+1}$. One-parameter family of hypersurfaces $X_t : M \to \mathbf{R}^{n+1}$ that satisfies $\frac{\partial}{\partial t} X_t = \Lambda_t \xi_\gamma \circ \nu_t$. is called the anisotropic mean curvature flow.
Anisotropic mean curvature flow diminishes the energy if $\Lambda_t \neq 0$:

$$\frac{d\mathcal{F}_\gamma(X_t)}{dt} - \int_M n\Lambda_t^2 \gamma(\nu_t)\, dA_t \leq 0.$$

Closed surfaces in the picture below are self-similar solutions $X_t := \sqrt{2(c-t)}\, \xi_\gamma$ of the aniso. mean curvature flow for $n{=}2$, $\gamma(\nu_1, \nu_2, \nu_3) = (\nu_1^2 + \nu_2^2)^3 + \nu_3^6$.



Remark! S^2 is the only closed embedded self-similar shrinking solution of mean curvature flow in $\mathbf{R}^3$ with genus zero (Brendle 2016).

9 Future works

(I) Weaken the assumption "$\gamma \in C^2(S^n)$".

If γ is of C^2, the Wulff shape W_γ has no straight line segments and hence no flat faces because of the following results.

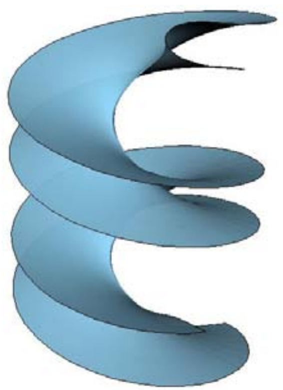
Assume $\gamma : S^n \to \mathbf{R}_{>0}$ is convex. Then,

F. Morgan 1991: $\gamma \in C^{1,1}$ if and only if W_γ is uniformly convex.

H. Han-T. Nishimura 2017: $\gamma \in C^1$ if and only if W_γ is strictly convex.

(II) Classify the singular points of CAMC hypersurfaces.

For example, we observed the following: For $\gamma : S^2 \to \mathbf{R}$ defined by $\gamma(\nu) := \sqrt{|\nu_3^2 - \nu_1^2 - \nu_2^2|}$, CAMC surfaces in $\mathbf{R}^3$ are CMC surfaces in $\mathbf{R}^3_1 = (\mathbf{R}^3, dx^2 + dy^2 - dt^2)$. [Honda-K-Saji, 2018] proved: Space-like piecewise-smooth CMC surfaces can have $(2, 5)$-cuspidal edges. (locally, $X_{(2,5)}(u, v) = (u,\ v^2,\ v^5)$.) On the other hand, any space-like zero mean curvature surface (figure below) does not have $(2, 5)$-cuspidal



edges. **(Figure from "Fujimori-Rossman-Umehara-Yamada-Yang, Spacelike mean curvature one surfaces in de Sitter 3-space, Comm. Anal. Geom. 17 (2009), 383-427. ")**

Appendix. Is a closed CAMC hypersurface only the Wulff shape?

	regularity	$n=2$, genus 0	$\forall n$, stable	$\forall n$, embedded
$\gamma \equiv 1$ ($W = S^n$)	smooth	○ Hopf '51	○ Barbosa-do Carmo'84	○ Alexandrov '62
$\gamma, W_\gamma \in C^\infty$, $D^2\gamma + \gamma 1 > 0$	smooth	○'10 K-Palmer 2012 Ando	○ 1998 Palmer	○ He-Li-Ma-Ge, 2009
$\gamma \in C^2$ γ:convex	piecewise C^2, non-flat	?	○ 2018 Koiso	?
$\gamma \in C^0$ Lip γ:convex	can have flat faces	?	? $n = 1$ ○'05 Morgan	?
$\gamma \in C^\infty$ non-convex	piecewise smooth, non-flat	× Jikumaru-K 2018	?	× Jikumaru-K 2018

References

[1] **A. Honda, M. Koiso and Y. Tanaka. Non-convex anisotropic surface energy and zero mean curvature surfaces in the Lorentz-Minkowski space. *Journal of Math-for-Industry* 5 (2013), 73–82.**

[2] **Y. Jikumaru and M. Koiso. Non-uniqueness of closed embedded non-smooth hypersurfaces with constant anisotropic mean curvature, preprint. arXiv:1903.03958 [math.DG]**

[3] **M. Koiso. Uniqueness of stable closed non-smooth hypersurfaces with constant anisotropic mean curvature, preprint. arXiv:1903.03951 [math.DG]**