On signs of Whitney cusps on Gauss maps of cuspidal edges with bounded Gaussian curvature

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Cuspidal edge

- $f : \Sigma \to \mathbf{R}^3$; C^∞ map, $\Sigma \subset (\mathbf{R}^2; u, v)$; domain.
- f has a **cuspidal edge** at $p \in \Sigma$
  $\overset{\text{def.}}{\Longleftrightarrow}$ $\exists \phi : \mathbf{R}^2 \to \Sigma$; (local) diffeo., $\exists \Phi : \mathbf{R}^3 \to \mathbf{R}^3$; (local) diffeo. s.t.
  $\Phi \circ f \circ \phi^{-1}(u, v) = (u, v^2, v^3)(= f_C(u, v))$ (i.e., $f \sim_{\mathcal{A}} f_C$).

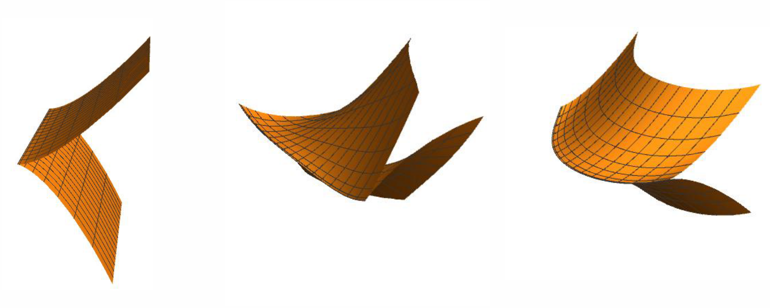


Figure: Surfaces with cuspidal edges

- f at p; cuspidal edge $\Longrightarrow$ rank $df_p = 1$ and
$\exists U \subset \Sigma$; nbd. of p, $\exists \gamma : (-\epsilon, \epsilon) \longrightarrow U$ regular curve ($p = \gamma(0)$).
s.t. $\gamma((-\epsilon, \epsilon)) = S(f) \cap U$ ($S(f)$; singular set of f).
γ; **singular curve** of f, $\hat{\gamma} = f \circ \gamma$; singular locus.
Moreover, $\exists \eta$; non-zero vector field on U s.t. $df(\eta) = 0$ on $S(f)$.
η; **null vector field**.

Fact (Kokubu-Rossman-Saji-Umehara-Yamada 2005)

f at p is a cuspidal edge $\Longrightarrow \det(\gamma'(t), \eta(t)) \neq 0$ ($' = d/dt$, $\eta(t) = \eta(\gamma(t))$).

- Further, $\exists \nu : U \longrightarrow S^2$ s.t. $\left\langle df_q(X), \nu(q) \right\rangle = 0$ ($\forall q \in U$, $\forall X \in T_q U$)
ν; the **Gauss map** of f, $\hat{\nu} = \nu \circ \gamma$.
Note: ν might have **singularities** at p (i.e., rank $d\nu_p < 2$),
but the pair $(f, \nu) : U \longrightarrow \mathbf{R}^3 \times S^2$; **immersion**.

For cuspidal edges, several geometrical invariants are known:
κ_s; **singular curvature**, κ_ν; **limiting normal curvature**,
$\kappa_c (\neq 0)$; **cuspidal curvature**, κ_t; **cusp-directional torsion**,
Note: κ_s is an **intrinsic invariant** (Saji-Umehara-Yamada 2009).
Note: ν has a singularity at $p \iff \kappa_\nu(p) = 0$ (Martins-Saji-Umehara-Yamada 2016, T.).

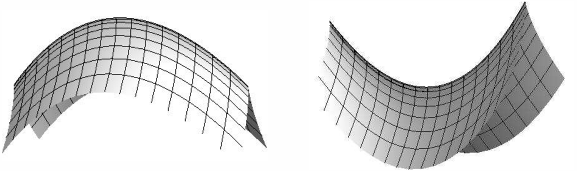


Figure: Cuspidal edges with positive κ_s (left) and negative κ_s (right).

In general, the Gaussian curvature K given by $K = \dfrac{\det(\nu_u, \nu_v, \nu)}{\det(f_u, f_v, \nu)}$ is **unbounded** near a cuspidal edge.
However, the following assertion is known.

Fact (Saji-Umehara-Yamada 2009, Martins-Saji-Umehara-Yamada 2016)

The Gaussian curvature K is **bounded** $(K \neq 0)$ near a cuspidal edge
$\Longleftrightarrow \det(\nu_u, \nu_v, \nu)(\gamma(t)) = 0 \Longleftrightarrow \kappa_\nu(\gamma(t)) = 0$.

This implies that the set of singular point of ν coincides with the set of singular point of f at least locally.

For a cuspidal edge with bounded Gaussian curvature, the following holds.

Fact (Saji-Umehara-Yamada 2009)

K is bounded and $K > 0$ near a cuspidal edge $p \Longrightarrow \kappa_s(p) < 0$.

Remark: If $K < 0$, $\kappa_s > 0$ is **NOT** true in general.
$\because$ If $p = \gamma(0)$ is not a cuspidal edge satisfying $\det(\gamma', \eta)(0) = 0$, and $\gamma(t)$ is cuspidal edges $(t \neq 0)$, then the singular curvature κ_s behaves $\lim_{t\to 0} \kappa_s(\gamma(t)) = -\infty$.
Thus for a negative Gaussian curvature surface with a singular point $p = \gamma(0)$ satisfying $\det(\gamma', \eta)(0) = 0$, the singular curvature κ_s is **negative** near p.

Question 1

When does the following statement hold: if $K < 0$ near a cuspidal edge p, then $\kappa_s(p) > 0$?

On the other hand, the Gauss map $\nu : U \to S^2$ of a cuspidal edge with bounded Gaussian curvature has singularity.

Typical singularities of ν:

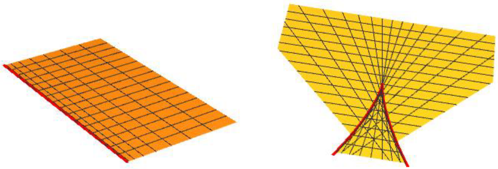
fold: $(u, v) \mapsto (u, v^2)$, **Whitney cusp**: $(u, v) \mapsto (u, \pm uv + v^3)$.

ν at p; fold (resp. **Whitney cusp**)

$\Rightarrow \exists \sigma : (-\delta, \delta) \to U$; regular curve ($\sigma(0) = p$)

s.t. $\sigma((-\delta, \delta)) = S(\nu) \cap U$ ($S(\nu)$; singular set of ν),

$\check{\nu} = \nu \circ \sigma$; spherical regular curve (resp. curve with **3/2-cusp** $(t \mapsto (t^2, t^3))$).



For cuspidal edge with bounded Gaussian curvature, the following assertion holds.

Fact (Saji-Umehara-Yamada 2012)

When the Gaussian curvature K is bounded and $K \neq 0$, and ν has fold along γ, then

$$\kappa_s(t)|\hat{\gamma}'(t)| = \mathrm{sgn}(K)\kappa_\#(t)|\hat{\nu}'(t)|,$$

where $\kappa_\#$; geodesic curvature of $\hat{\nu} = \nu \circ \gamma$.

On the other hand, when the Gauss map ν has a Whitney cusp at p, one can define the **cuspidal curvature** μ^{ν} for $\check{\nu}$ which measures a kind of wideness of cusps.

Definition

p; **positive** (resp. **negative**) cusp of $\nu \overset{\text{def.}}{\Longleftrightarrow} \mu^{\nu} > 0$ (resp. $\mu^{\nu} < 0$).

Note: $\mu^{\nu} > 0$ (resp. $\mu^{\nu} < 0$) $\Rightarrow \check{\nu}$; **right-turning** (resp. **left-turning**) cusp.

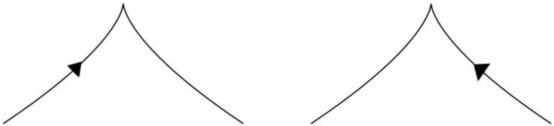


Figure: Positive cusp (left) and negative cusp (right).

Question 2

What relation between the signs of cusps of ν and geometric invariants of a cuspidal edge hold?

Characterization by geometric invariants

$f : (U; u, v) \longrightarrow \mathbf{R}^3$; C^∞ map with cuspidal edge $p \in U$.
$\nu : U \longrightarrow S^2$; Gauss map of f.
Set functions $\lambda, \Lambda : U \longrightarrow \mathbf{R}$ by

$$\lambda(u, v) = \det(f_u, f_v, \nu)(u, v), \quad \Lambda(u, v) = \det(\nu_u, \nu_v, \nu)(u, v).$$

Then $\lambda^{-1}(0) = S(f)$ and $\Lambda^{-1}(0) = S(\nu)$.
Assume that p is also a singular point of ν, i.e., $\Lambda(p) = 0$ ($\Longleftrightarrow$ $\kappa_\nu(p) = 0$).
Then p; **non-degenerate singular point** of ν $\overset{\text{def.}}{\Longleftrightarrow}$ $(\Lambda_u(p), \Lambda_v(p)) \neq (0, 0)$.
Note: folds and Whitney cusps are non-degenerate singular points of ν.

Proposition

p; a non-degenerate singular point of ν
$\Longleftrightarrow$ $\kappa_\nu'(p) \neq 0$ or $4\kappa_t(p)^2 + \kappa_s(p)\kappa_c(p)^2 \neq 0$.

When $K = \Lambda/\lambda$ is bounded near p, $\kappa_\nu(\gamma(t)) = 0$, and hence non-degeneracy is equivalent to $4\kappa_t(p)^2 + \kappa_s(p)\kappa_c(p)^2 \neq 0$.

We give characterizations of folds and cusps appearing on ν.

Proposition

$f : U \longrightarrow \mathbf{R}^3$; C^∞ map with cuspidal edge p. $\nu : U \longrightarrow S^2$; Gauss map of f. Assume that the Gaussian curvature K of f is bounded near p. Then

1. ν at p; **fold** $\iff$ $\kappa_t(p)(4\kappa_t(p)^2 + \kappa_s(p)\kappa_c(p)^2) \neq 0$,
2. ν at p; **Whitney cusp** $\iff$ $\kappa_t(p) = 0$, $\kappa_t'(p) \neq 0$ and $\kappa_s(p) \neq 0$

By this proposition, we have the following.

Corollary

Under the same assumption as in the above proposition,
p is a non-degenerate singular point of ν which is **NOT** a fold
$\iff$ $\kappa_t(p) = 0$ and $\kappa_s(p) \neq 0$.

Signs of Gaussian curvature

We consider the sign of the Gaussian curvature K which is bounded near a cuspidal edge p.
It is known that K at p can be written as

$$4K(p) = -(4\kappa_t(p)^2 + \kappa_s(p)\kappa_c(p)^2) \cdot (\text{positive constant}).$$

Thus we have the following.

Theorem 1

$f\colon \Sigma \longrightarrow \mathbf{R}^3$; C^∞ map, p; cuspidal edge of f, $\nu\colon \Sigma \longrightarrow S^2$; its Gauss map.
Suppose that the Gaussian curvature K of f is bounded on a sufficiently small neighborhood U of p.
When p; a non-degenerate singular point of ν **other than a fold**,
$K > 0$ (resp. $K < 0$) on $U \iff \kappa_s(p) < 0$ (resp. $\kappa_s(p) > 0$).

In particular, when ν at p is a Whitney cusp, the assertion holds.
This theorem gives an answer of the first question.

Example

Let us consider the surface parametrized by

$$f(u,v) = \left(\sin u \cos v, \sin u \sin v, \cos u + \log\left(\tan\left(\frac{u}{2}\right)\right)\right),$$

where $(u,v) \in (0,\pi) \times [0,2\pi)$.
This is a **pseudo-sphere**, and f has cuspidal edges on $S(f) = \{u = \pi/2\}$.
Thus the singular curve γ is $\gamma(v) = (\pi/2, v)$.
It is well known that the Gaussian curvature K is constant $K = -1$.
The Gauss map ν of f is given by

$$\nu(u,v) = (-\cos u \cos v, -\cos u \sin v, \sin u).$$

Since $\Lambda(u,v) = \det(\nu_u, \nu_v, \nu)(u,v) = -\cos u$, $S(f) = S(\nu)$ holds.
By direct calculations, we have

$$\kappa_s = 1 > 0, \quad \kappa_\nu = \kappa_t = 0$$

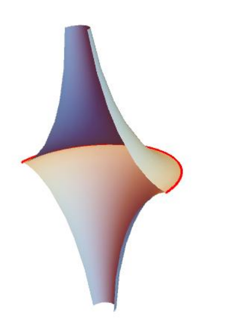
along γ.
In this case, γ is a curvature line of f (cf. Izumiya-Saji-Takeuchi 2017).

Example (continue)

Moreover, the singular locus of ν degenerates to a point $(0, 0, 1)$.

We call such a singular point a **cone like singular point**.

We note that similar phenomena occur in the case of flat fronts in H^3 and their Δ_1-dual fronts in S^3_1 (cf. Saji-T.)



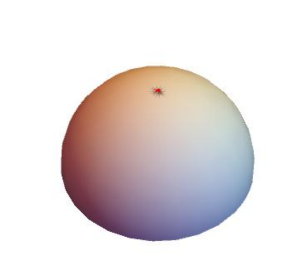


Figure: Pseudo-sphere (left) and image of its Gauss map (right).

In general, we have the following assertion.

Proposition

$f\colon U \longrightarrow \mathbf{R}^3$; C^∞ map with cuspidal edge $p \in U$.
$\nu\colon U \longrightarrow S^2$; the Gauss map of f.
$\gamma(t)$; singular curve of f through p.
Suppose that K; bounded near p,
p; non-degenerate singular point of ν and $\gamma(t)$; line of curvature
$\Rightarrow p$; **cone like singular point** of ν, i.e., $\nu(\gamma(t))$ degenerates to a point.

Signs of cuspidal curvature μ^ν

We consider signs of cuspidal curvature μ^ν for ν.
$f : U \longrightarrow \mathbf{R}^3$; C^∞ map, $p \in U$; cuspidal edge of f.
$\nu : U \longrightarrow S^2$; the Gauss map of f.
Suppose that K is bounded near p, $K \neq 0$, and ν at p is a **Whitney cusp**.
Note: $\hat{\nu}'' \neq 0$ at p.
Then μ^ν is given by

$$\mu^\nu = \left.\frac{\det(D_t\hat{\nu}', D_tD_t\hat{\nu}', \hat{\nu})}{|D_t\hat{\nu}'|^{5/2}}\right|_{t=0},$$

where $D_t\hat{\nu}' = \hat{\nu}'' - \langle\hat{\nu}'', \hat{\nu}\rangle\,\hat{\nu}$, $\hat{\nu}(t) = \nu \circ \gamma(t)$ and $p = \gamma(0)$.

Lemma

$$\mu^\nu = \frac{2\kappa_s(p)}{\sqrt{|\kappa_t'(p)|}} \cdot (\text{positive constant}).$$

By the previous lemma, we have the following.

Theorem 2

$f : U \longrightarrow \mathbf{R}^3$; C^∞ map, p; cuspidal edge of f, ν; Gauss map of f.
Suppose that K is bounded near p, $K \neq 0$, and ν at p is a cusp.
Then p; **positive cusp** (resp. **negative cusp**) of ν
$\iff$ $\kappa_s(p) > 0$ (resp. $\kappa_s(p) < 0$).

By this theorem, we have the following.

Corollary

Under the same assumptions as in Theorem 2,
if $K > 0$ (resp. $K < 0$), then p; **negative cusp** (resp. **positive cusp**) of ν.

Example

Let $f\colon \mathbf{R}^2 \to \mathbf{R}^3$ be a C^∞ map defined by

$$f(u,v) = \left(u, 3u^2 + \frac{v^2}{2}, \frac{v^3}{3} + u^4 + u^2v^2\right).$$

This map has a cuspidal edge at the origin and $S(f) = \{v = 0\}$.
The Gauss map ν of f is given by

$$\nu(u,v) = \frac{\left(8u^3 - 2uv(v-3), -2u^2 - v, 1\right)}{\sqrt{1 + (v + 2u^2)^2 + (8u^3 - 2uv(v-3))^2}}.$$

By direct calculations, we have

$$\kappa_\nu(u) \equiv 0,$$

$$\kappa_s(u) = \frac{6\left(1 + 24u^4 + 64u^6\right)}{\sqrt{1 + 4u^2 + 64u^6}\,(1 + 36u^2 + 16u^6)^{3/2}}, \quad \kappa_t(u) = \frac{4u}{1 + 4u^2 + 64u^4}.$$

Example(continue)

Thus we have $\kappa_s(0) = 1 > 0$, $\kappa_t(0) = 0$ and $\kappa_t'(0) = 4 \neq 0$.
$\therefore \nu$ at the origin is a **cusp**.
Moreover, the Gaussian curvature K is bounded near the origin and $K < 0$.
In fact, K is written as

$$K = \frac{2\left(-3 - 8u^2 + v\right)}{\left(1 + 64u^6 + v^2 + u^4\left(4 + 96v - 32v^2\right) + 4u^2v\left(1 + 9v - 6v^2 + v^3\right)\right)^2}$$

and especially $K < 0$ on sufficiently small neighborhood of the origin.

Example(continue)

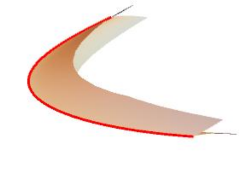
On the other hand, the singular locus $\check{\nu}(u) = \nu(u, 0)$ is

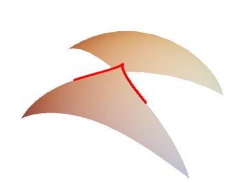
$$\check{\nu}(u) = \frac{\left(8u^3, -2u^2, 1\right)}{\sqrt{1 + 4u^4 + 64u^6}}.$$

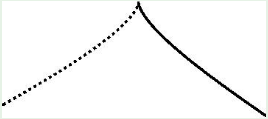
This has an ordinary cusp at $u = 0$, and the cuspidal curvature μ^{ν} at $u = 0$ is

$$\mu^{\nu} = 6 = \frac{2\kappa_s(0)}{\sqrt{|\kappa_t'(0)|}} > 0.$$

Thus $(0, 0)$ is a positive cusp of ν.







Thank you for your attention!