Topology of complements to real affine space line arrangements

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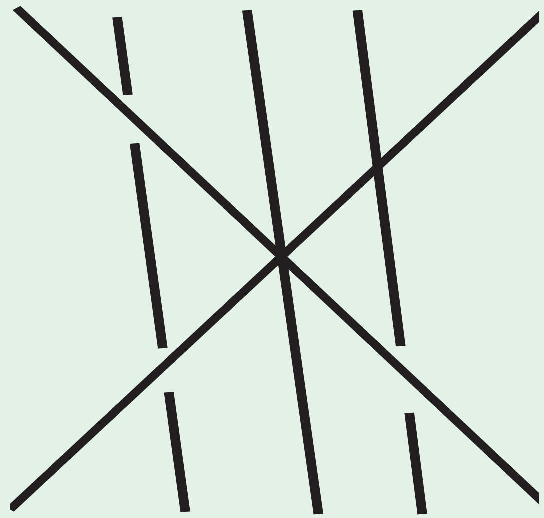
【 Space line arrangements 】

Let $\mathcal{A} = \{\ell_1, \ell_2, \dots, \ell_d\}$ be a real space line arrangement, or a configuration, consisting of affine d-lines in $\mathbf{R}^n$ $(n \geq 2)$.



In the above example, $d = 5$ and there are one double point and one triple point.

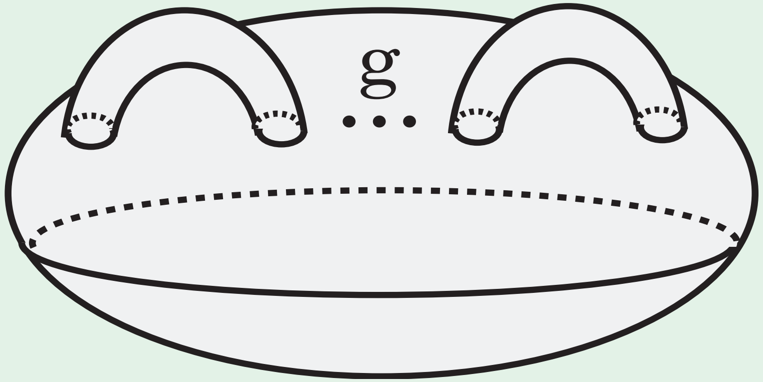
【 Theorem 】

Let $t_i = t_i(\mathcal{A})$ denote the number of multiple points with multiplicity i, $i = 2, \dots, d$. The vector $(t_d, t_{d-1}, \dots, t_2)$ provides a degree of degeneration of the line arrangement $\mathcal{A}$ combinatorially. Set $g := d + \sum_{i=2}^{d}(i-1)t_i$. Then we have:

Theorem. The complement $M(\mathcal{A}) := \mathbf{R}^n \backslash (\cup_{i=1}^{d} \ell_i)$ of the real line arrangement $\mathcal{A}$ is diffeomorphic to the interior of n-ball B_g with trivially attached g-handles of index $n-2$.

Corollary. $M(\mathcal{A})$ is homotopy equivalent to the bouquet $\bigvee_{k=1}^{g} S^{n-2}$.

$n = 3$



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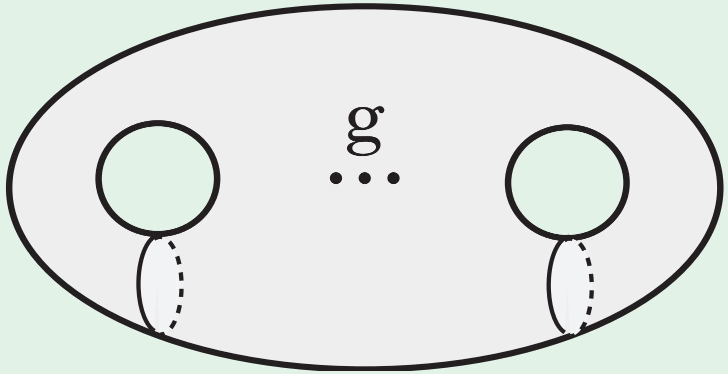


The topology of complements of real space line arrangements is completely determined by the combinational data, $g := d + \sum_{i=2}^{d}(i-1)t_i$, which is given by the number of lines d and the numbers of multiple points $t_d, t_{d-1}, \dots, t_2$.

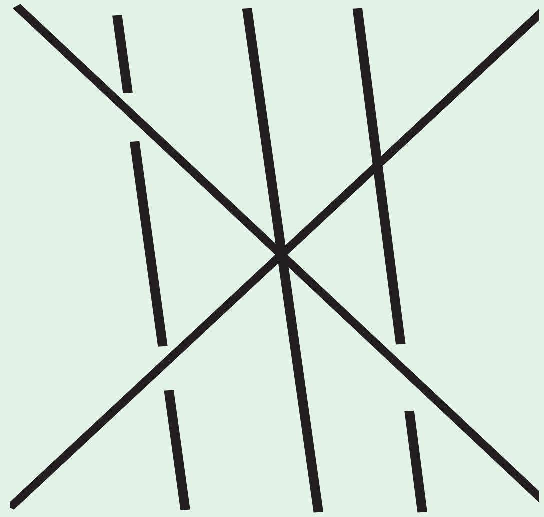
【 Remarks 】

The relative classification problem of line arrangements $(\mathbf{R}^n, \cup_{i=1}^d \ell_i)$ is classical and very difficult. Moreover it has much difference in differentiable category and topological category. In fact even the local classification near multiple points of high multiplicity i, $i \geq n+2$ has moduli in differentiable category while it has no moduli in topological category.

The classification of complements turns out to be easier and simpler.

【 Example 】

For the line arrangement



we have $d = 5, t_2 = 1, t_3 = 1$ and $g = 5+1+2 = 8$. Therefore the complement is diffeomorphic to interior of 3-ball B_8 with trivially attached 8 number of handles of index 1.

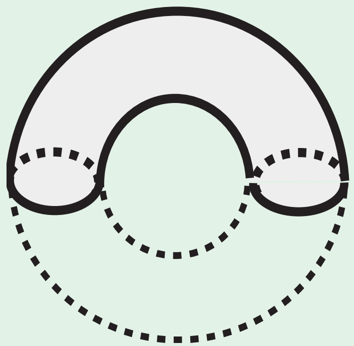
【 Trivial handle attachments 】

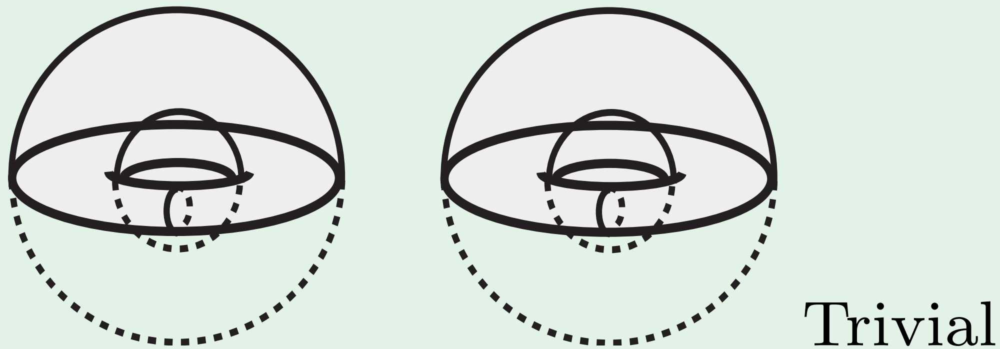
The pair $(D^i \times D^j, D^i \times \partial(D^j))$ with $i+j=n, 0 \leq i, 0 \leq j$, is called an n-dimensional handle of index j

Let M be a differentiable n-manifold with a connected boundary ∂M.

Let $p \in \partial M$. A coordinate neighbourhood (U, ψ), $\psi : U \to \psi(U) \subset \mathbf{R}^{n-1} \times \mathbf{R}$ around p in M is called adapted if $\psi : U \to \mathbf{R}^n$ is a homeomorphism of U and $\psi(U) \cap \{x_n \leq 0\}$ which maps $U \cap \partial M$ to $\mathbf{R}^{n-1} = \{x_n = 0\}$.

A handle attaching map $\varphi : \bigsqcup_{k=1}^{\ell}(D_k^i \times \partial(D_k^j)) \to \partial M$ is called trivial if there exist disjoint adapted coordinate neighbourhoods $(U_1, \psi_1), \dots, (U_\ell, \psi_\ell)$ on M such that $\varphi(D_k^i \times \partial(D_k^j)) \subset U_k$ and each $\psi_k \circ \varphi : D_k^i \times \partial(D_k^j) \to \mathbf{R}^{n-1} \times \mathbf{R}$ is the "standard" attachment for $k = 1, \dots, \ell$.





handle attachments: the cases $n = 3, j = 1, \ell = 1$ and $n = 4, j = 2, \ell = 2$

Let $n \geq 2$.

We consider line arrangements in $\mathbf{R}^n$ or more generally consider a subset X in $\mathbf{R}^n$ which is a union of finite number of closed line segments and half lines. Then X may be regarded as a finite graph (with non-compact edges) embedded as a closed set in $\mathbf{R}^n$.

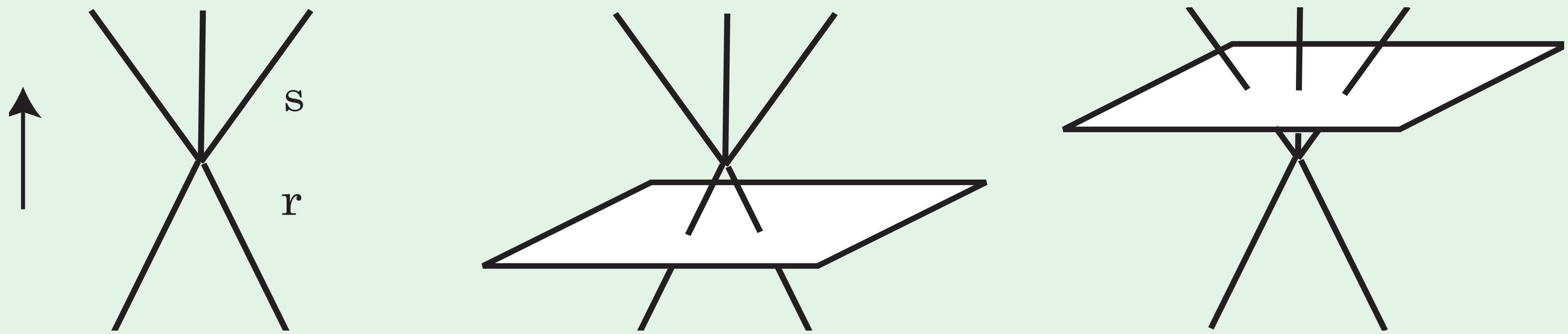
Take a generic hight function $h : \mathbf{R}^n \to \mathbf{R}$. After a rotation of $\mathbf{R}^n$, we may suppose $h(x) = x_n$.

Set $M = \mathbf{R}^n \setminus X$ and, for any $c \in \mathbf{R}$,

$$M_{\leq c} := \{x \in M \mid x_n \leq c\}, \quad M_{<c} := \{x \in M \mid x_n < c\}.$$

Let $V \subset X$ be the totality of vertices of X.

Set $V = \{u_1, u_2, \ldots, u_r\}, c_i = h(u_i)$ and
$C = h(V) = \{c_1, c_2, \ldots, c_r\}$ with $c_1 < c_2 < \cdots < c_r$.


s
r

Lemma (Topological bifurcation). Let u be a vertex of X and let $c = h(u)$. Let $s = s(u)$ denote the number of edges of X which are adjacent to u from above with respect to h.

Then, for a sufficiently small $\varepsilon > 0$, the open set $M_{<c+\varepsilon}$ is diffeomorphic to the interior of $M_{\leq c-\varepsilon} \bigcup_{\varphi} (\bigsqcup_{i=1}^{s-1} (D_i^2 \times D_i^{n-2}))$, obtained by an attaching map

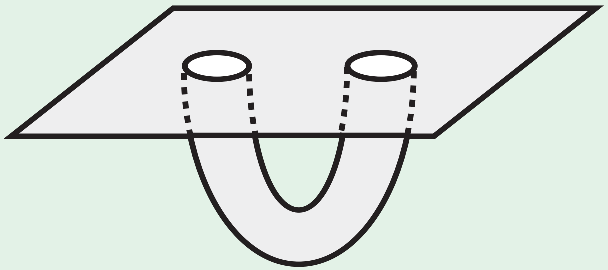
$$\varphi : \bigsqcup_{i=1}^{s-1} D^2 \times \partial(D^{n-2}) \longrightarrow h^{-1}(c-\varepsilon) \setminus X \; = \partial(M_{\leq c-\varepsilon}) \subset M_{\leq c-\varepsilon},$$

of $(s-1)$ number of trivial handles of index $n-2$, provided $s \geq 1$.

In particular $M_{<c+\varepsilon}$ is diffeomorphic to $M_{<c-\varepsilon}$ if $s = 1$.

If $s = 0$ then $M_{<c+\varepsilon}$ is diffeomorphic to the interior of $M_{\leq c-\varepsilon} \bigcup_{\varphi} (D^1 \times D^{n-1})$ obtained by an attaching map $\varphi : D^1 \times \partial(D^{n-1}) \to h^{-1}(c-\varepsilon) \setminus X$ of a (not necessarily trivial) handle of index $n-1$.

$s = 2, r = 0$



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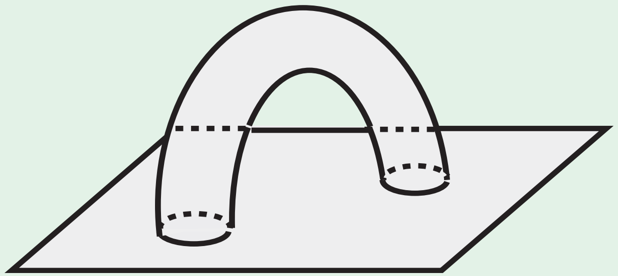
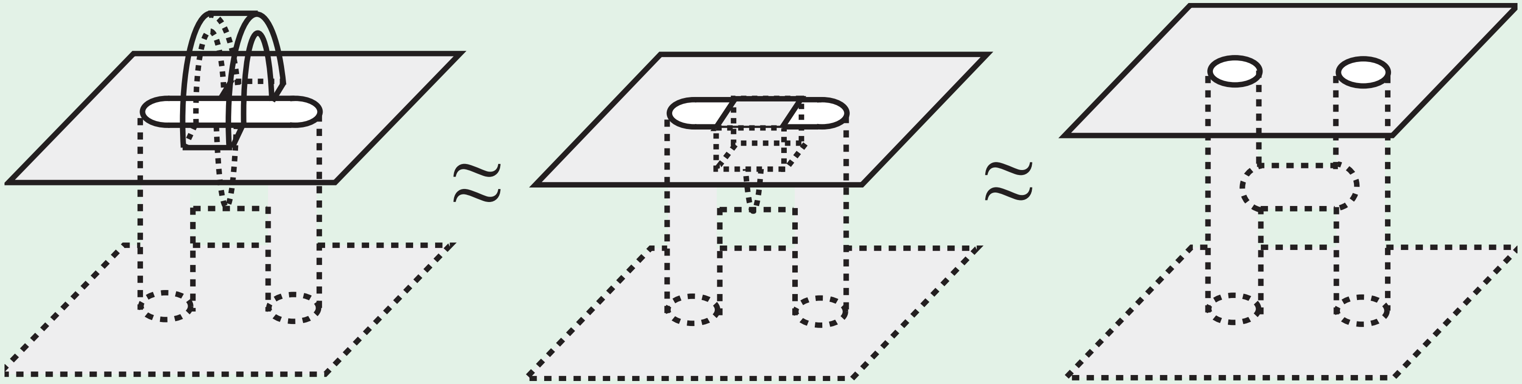


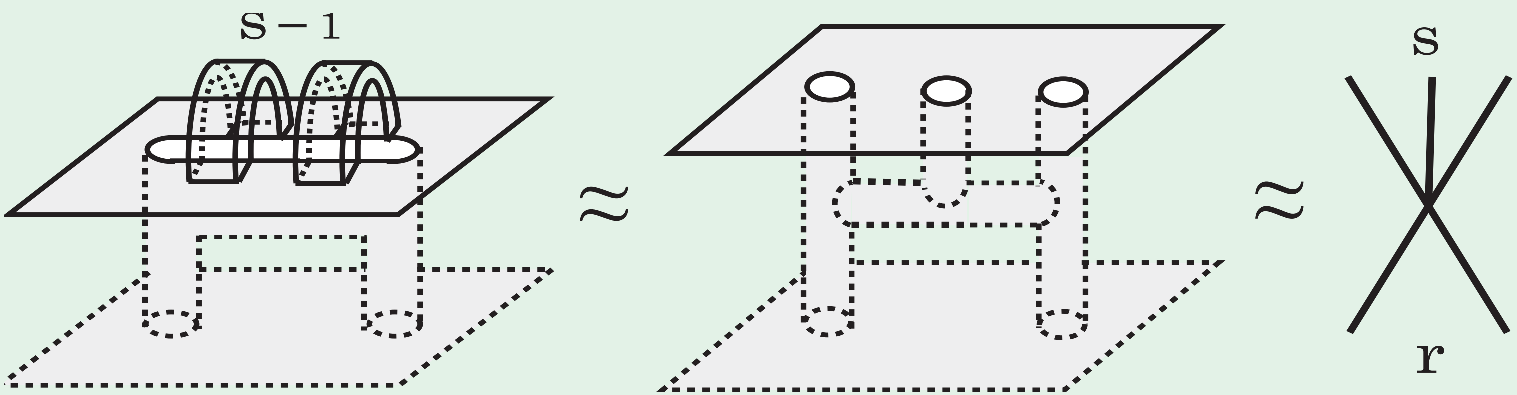
"Digging a tunnel is same as bridging for the topology of ground".

The same argument works for any r. Note that complements to “X” and “H” are diffeomorphic.



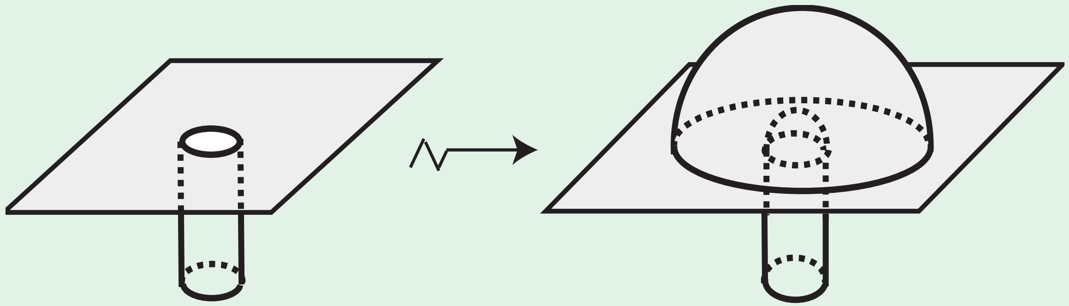
The case $s = 2, r = 2$.

In general, for any $s \geq 2$, the topological change is obtained by attaching trivial $s-1$ handles of index $n-2$.



The case $s = 3, r = 2$.

In the case $s = 0$, contrarily to above, the change of diffeomorphism type is obtained by an attaching not necessarily trivial handle.



Topological change in the case $s = 0$.

【 Proof of Theorem 】

For a $c \in \mathbf{R}$ with $c \ll 0$, $M_{\leq c}$ (resp. $M_{<c}$) is diffeomorphic to the half space $\{x_n \leq c\}$ (resp. $\{x_n < c\}$ deleted d number of half lines. By passing a multiple point of multiplicity i, $M_{\leq c}$ is obtained by attaching $i-1$ number of trivial handles of index $n-2$. After passing all multiple points, $M_{\leq c}$ is diffeomorphic to the space obtained by attaching $\sum_{i=2}^{d}(i-1)t_i$ number of trivial handles of index $n-2$ to the half space deleted d number of half lines. Then $M_{<c}$ is diffeomorphic to the interior of B_g. For a $c \in \mathbf{R}$ with $0 \ll c$, $M_{<c}$ is diffeomorphic to $M(\mathcal{A})$. Hence we have Theorem. □

For details, please see the preprint:

Goo Ishikawa, Motoki Oyama, *Topology of complements to real affine space line arrangements*,
`http://www.math.sci.hokudai.ac.jp/~ishikawa/preprint.html`
or
arXiv:1811.01521 [math.DG].

Thank you for your attention.