Mixed type surfaces with bounded Gaussian curvature in three-dimensional Lorentzian manifolds

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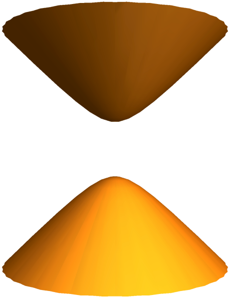
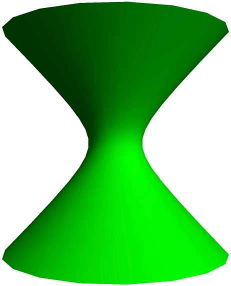
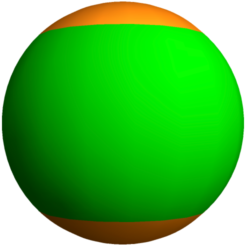


Spacelike surface



Timelike surface



Mixed-type surface

▷ Today's talk :

- **Lightlike points** (= $T_p\Sigma$ is lightlike)
  $\Longleftrightarrow$ **Singular points** of the first fundamental form
- Behavior of the **Gaussian curvature** K at lightlike points

▷ Introduce several invariants κ_L, κ_N at lightlike points,
which looks like '**cuspidal edges**'

Contents

(1) Introduction & Motivation

- What is mixed type surface?
- Mixed type surface with Zero Mean Curvature

(2) Lightlike points of mixed type surfaces

- Lightlike points of the first kind, second kind, L_3 points
- Lightlike singular curvature, lightlike normal curvature

(3) Main result

Honda, Saji, Teramoto,
Mixed type surfaces with bounded Gaussian curvature in three-dimensional Lorentzian manifolds (arXiv:1811.11392).

(1) Introduction & Motivation

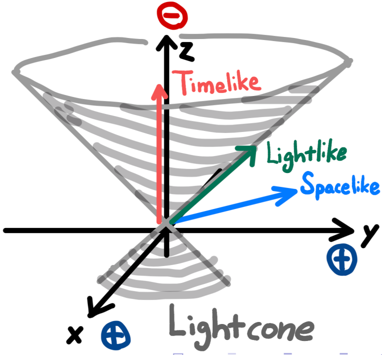
Lorentz-Minkowski space $\boldsymbol{R}^3_1$ (1/2)

- Lorentz-Minkowski 3-space : $\boldsymbol{R}^3_1 = \left(\boldsymbol{R}^3, \langle\ ,\ \rangle\right)$ $(= \boldsymbol{L}^3)$.
- For $\boldsymbol{v} = (x, y, z) \in \boldsymbol{R}^3_1$,

$$\langle \boldsymbol{v}, \boldsymbol{v} \rangle = x^2 + y^2 - z^2.$$

- Vector $\boldsymbol{v} \in \boldsymbol{R}^3_1$ is called
  - spacelike $\overset{\text{def}}{\Longleftrightarrow}$ $\langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0$ or $\boldsymbol{v} = \boldsymbol{0}$
  - timelike $\overset{\text{def}}{\Longleftrightarrow}$ $\langle \boldsymbol{v}, \boldsymbol{v} \rangle < 0$
  - lightlike $\overset{\text{def}}{\Longleftrightarrow}$ $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$
- Set of lightlike vectors :

$\{\boldsymbol{v} \in \boldsymbol{R}^3_1\,;\ \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0\}$: **Lightcone**

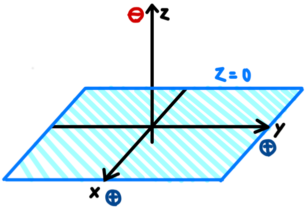


Lorentz-Minkowski space R^3_1 (2/2)

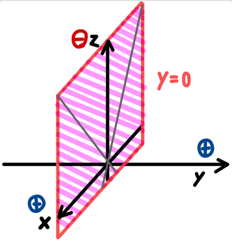
2-dim subspace of $\boldsymbol{R}^3_1$ is written as

$$V = \left\{ \boldsymbol{v} \in \boldsymbol{R}^3_1 \, ; \, \langle \boldsymbol{v}, \boldsymbol{n} \rangle = 0 \right\} \qquad (\text{where } \boldsymbol{n} \in \boldsymbol{R}^3_1 \setminus \{\boldsymbol{0}\})$$

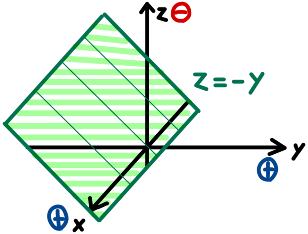
- V : spacelike $\overset{\text{def}}{\iff}$ $\boldsymbol{n}$: timelike $\iff$ $\langle \, , \, \rangle|_V$: positive definite
- V : timelike $\overset{\text{def}}{\iff}$ $\boldsymbol{n}$: spacelike $\iff$ $\langle \, , \, \rangle|_V$: indefinite
- V : lightlike $\overset{\text{def}}{\iff}$ $\boldsymbol{n}$: lightlike $\iff$ $\langle \, , \, \rangle|_V$: degenerate



Spacelike



Timelike



Lightlike

Mixed type surface (1/2)

Immersion $f : \Sigma \to \boldsymbol{R}^3_1$ (Σ : connected 2-mfd) . . . **REGULAR surf**

- $V_p = df(T_p\Sigma)$: 2-dim subspace of $\boldsymbol{R}^3_1$.
- $ds^2 = f^*\langle\ ,\ \rangle$: the first fundamental form of f.

A point $p \in \Sigma$ is

- **spacelike pt** $\overset{\text{def}}{\Longleftrightarrow}$ V_p : spacelike $\Longleftrightarrow$ $(ds^2)_p$: pos. definite
- **timelike pt** $\overset{\text{def}}{\Longleftrightarrow}$ V_p : timelike $\Longleftrightarrow$ $(ds^2)_p$: indefinite
- **lightlike pt** $\overset{\text{def}}{\Longleftrightarrow}$ V_p : lightlike $\Longleftrightarrow$ $(ds^2)_p$: degenerate

- Σ_+ ($\subset \Sigma$) : set of spacelike points
- Σ_- ($\subset \Sigma$) : set of timelike points
- LD ($\subset \Sigma$) : set of lightlike points

Mixed type surface (2/2)

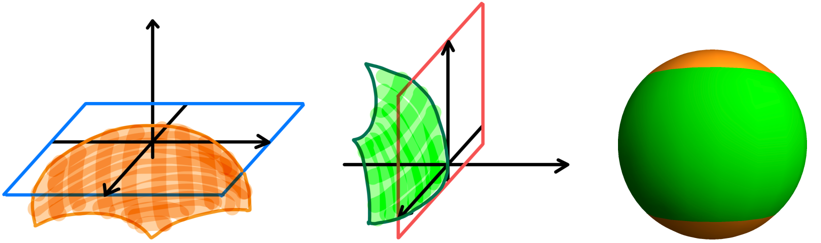
Immersion $f : \Sigma \to \boldsymbol{R}^3_1$ (Σ : connected 2-mfd) ... **REGULAR surf**

- Σ_+ : spacelike point set / Σ_- : timelike point set

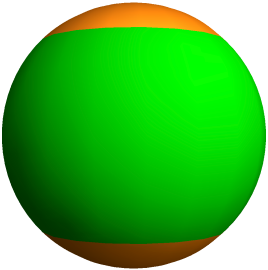
▷ f : spacelike surface $\iff$ $\Sigma = \Sigma_+$

▷ f : timelike surface $\iff$ $\Sigma = \Sigma_-$

▷ f : **mixed type surface** $\overset{\text{def}}{\iff}$ $\Sigma_+ \neq \emptyset, \Sigma_- \neq \emptyset$.



spacelike surface　　timelike surface　　mixed type surface



Example (Unit sphere)

Regard $S^2 := \{(x, y, z)\,;\, x^2 + y^2 + z^2 = 1\}$ as a surface in $\boldsymbol{R}^3_1$

- $\Sigma_+ = \{(x, y, z) \in S^2\,;\, |z| > \frac{1}{\sqrt{2}}\}$
- $\Sigma_- = \{(x, y, z) \in S^2\,;\, |z| < \frac{1}{\sqrt{2}}\}$
- $LD = \{(x, y, z) \in S^2\,;\, |z| = \frac{1}{\sqrt{2}}\}$

Since $\Sigma_+ \neq \emptyset$, $\Sigma_- \neq \emptyset \Longrightarrow S^2$ is a mixed type surface

- Gaussian curvature K, mean curvature H is defined on Σ_+, Σ_-
- In general, K, H is unbounded near LD

Example (graph)

$z = \varphi(x, y)$: graph (where $\Sigma \subset \boldsymbol{R}^2$: a domain on xy-plane)

$$\Sigma_+ = \{(x, y) \in \Sigma\, ;\ 1 - \varphi_x^2 - \varphi_y^2 > 0\}$$
$$\Sigma_- = \{(x, y) \in \Sigma\, ;\ 1 - \varphi_x^2 - \varphi_y^2 < 0\}$$
$$LD = \{(x, y) \in \Sigma\, ;\ 1 - \varphi_x^2 - \varphi_y^2 = 0\}$$

- On $\Sigma_+ \cup \Sigma_-$,

$$K = \frac{\varphi_{xy}^2 - \varphi_{xx}\varphi_{yy}}{(1 - \varphi_x^2 - \varphi_y^2)^2}, \quad H = \frac{(1 - \varphi_y^2)\varphi_{xx} + 2\varphi_x\varphi_y\varphi_{xy} + (1 - \varphi_x^2)\varphi_{yy}}{|1 - \varphi_x^2 - \varphi_y^2|^{3/2}}$$

- $z = \varphi(x, y)$: **Zero Mean Curvature graph** (ZMC graph)

$$\overset{\text{def}}{\Longleftrightarrow} \quad (1 - \varphi_y^2)\varphi_{xx} + 2\varphi_x\varphi_y\varphi_{xy} + (1 - \varphi_x^2)\varphi_{yy} = 0$$

Zero Mean Curvature graphs (1/3)

- $z = \varphi(x, y)$: **Zero Mean Curvature graph** (ZMC graph)

$$\overset{\text{def}}{\Longleftrightarrow} \quad (1 - \varphi_y^2)\varphi_{xx} + 2\varphi_x\varphi_y\varphi_{xy} + (1 - \varphi_x^2)\varphi_{yy} = 0$$

The Calabi-Bernstein theorem

Any **entire spacelike ZMC graph** must be a plane.

- A spacelike ZMC graph $\Longleftrightarrow$ **maximal graph**.
- $\exists$ **Counter example** without **spacelike** condition :

Fact ([O. Kobayashi, Tokyo J. Math., 1983])

$\exists z = \varphi(x, y)$ **: entire mixed type ZMC graph** s.t.

- **non-planar**, but **real analytic.**

$$z = \log(\cosh x / \cosh x), \qquad z = x \tanh y$$

Zero Mean Curvature graphs (2/3)

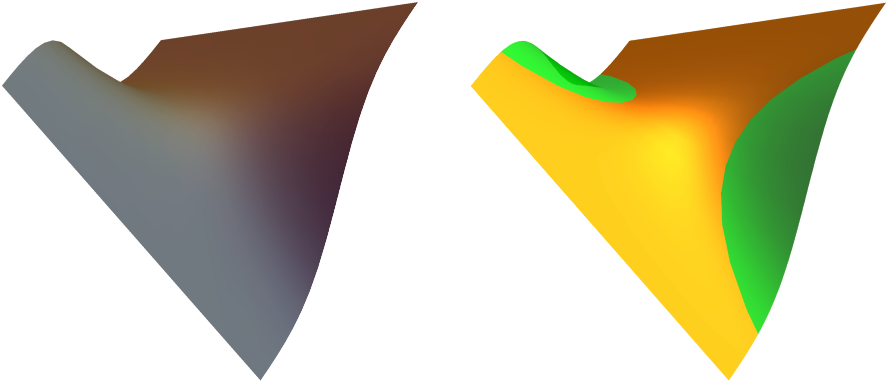


Figure: An entire ZMC graph $z = x \tanh y$ [O. Kobayashi, 1983]

Zero Mean Curvature graphs (3/3)

- Such non-trivial entire ZMC graphs of mixed type were not known other than **Kobayashi's examples**
- **[Fujimori-Kawakami-Kokubu-Rossman-Umehara-Yamada, 2016]** : such examples were constructed (**Kobayashi surface of order n**; $(4n-7)$-parameter family)
- **[Akamine, 2017]** : an entire example foliated by parabolas was given, independently (Kobayashi surface of order 2)
- On the other hand, **[H-Koiso-Kokubu-Umehara-Yamada, 2017]** : $\nexists$ **mixed type surfaces of non-zero constant mean curvature**

Question

Behavior of **the Gaussian curvature** K at lightlike points?

- Regarding **lightlike point** = **singular points** of the first fundamental forms, **we apply the technics of surfaces with singular points (wave fronts)**

(2) Lightlike points of mixed type surfaces

Non-degenerate lightlike points

- A mixed type surface $f : \Sigma \to \boldsymbol{R}^3_1$
- On coordinate neighborhood $(U; u, v)$ of Σ,

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

where

$$E := \langle f_u, f_u \rangle\,, \quad F := \langle f_u, f_v \rangle\,, \quad G := \langle f_v, f_v \rangle$$

- Set $\lambda := EG - F^2$.
  - $p \in \Sigma$: **spacelike point** $\iff \lambda(p) > 0.$
  - $p \in \Sigma$: **timelike point** $\iff \lambda(p) < 0.$
  - $p \in \Sigma$: **lightlike point** $\iff \lambda(p) = 0.$

Definition

A lightlike point $p \in U$ is called **non-degenerate** $\overset{\text{def}}{\iff} d\lambda(p) \neq 0.$

Lightlike points of the first & second kind

- ▷ $p \in LD$: **non-degenerate** lightlike point
  - By the implicit function thm, $\exists\gamma(t)$ $(|t| < \varepsilon)$: a regular curve on Σ s.t. $\gamma(0) = p$ & $\mathrm{Im}\gamma = LD$ near p.
  - $\exists\eta(t)$: a vector field along $\gamma(t)$ s.t. $L(t) := df(\eta(t))$: a lightlike vector field of $\boldsymbol{R}^3_1$ (namely, $\eta(t)$ is in the kernel of $ds^2_{\gamma(t)}$)
- ▷ We call $\gamma(t)$ **characteristic curve**, $\eta(t)$ **null vector field**.

Definition

A non-degenerate lightlike point $p \in \Sigma$ is called

- **first kind** $\overset{\text{def}}{\iff}$ $\gamma'(0), \eta(0)$: linearly independent
- **second kind** $\overset{\text{def}}{\iff}$ $\gamma'(0), \eta(0)$: linearly dependent

cf. Criteria for **cuspidal edges** and **swallowtails** (Kokubu-Rossman-Saji-Umehara-Yamada)

L_3 points

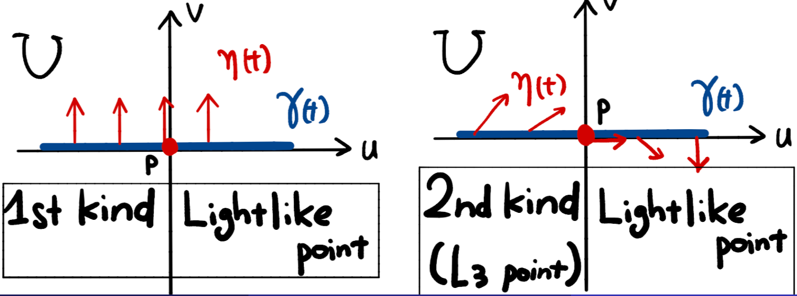
A non-degenerate lightlike point $p \in \Sigma$ is called

- **first kind** $\overset{\text{def}}{\Longleftrightarrow}$ $\gamma'(0), \eta(0)$: linearly independent
- **second kind** $\overset{\text{def}}{\Longleftrightarrow}$ $\gamma'(0), \eta(0)$: linearly dependent

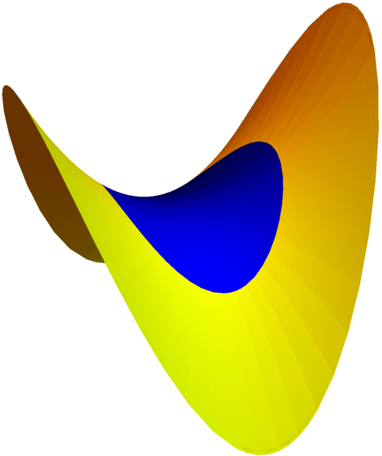
Setting $\delta(t) := \det(\gamma'(t), \eta(t))$,

▷ $p = \gamma(0)$: the first kind $\Longleftrightarrow$ $\delta(0) \neq 0$

▷ $p = \gamma(0)$: L_3 **point** $\overset{\text{def}}{\Longleftrightarrow}$ $\delta(0) = 0, \delta'(0) \neq 0$



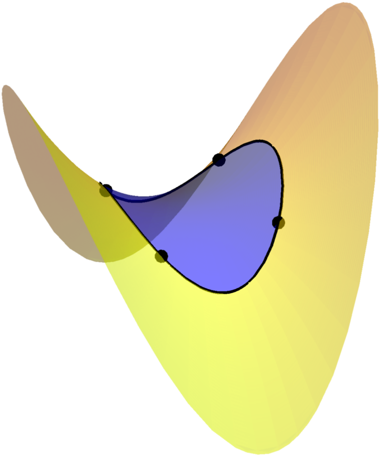
Ex: hyperbolic paraboloid $z = \frac{1}{2}(x^2 - y^2)$



$$f(x,y) = \left(x, y, \tfrac{1}{2}(x^2 - y^2)\right)$$

- $1 - \varphi_x^2 - \varphi_y^2 = 1 - x^2 - y^2$
- $LD = \{(x,y)\,;\, x^2 + y^2 = 1\}$
- $\gamma(t) = (\cos t, \sin t)$
- $\begin{pmatrix} E & F \\ F & G \end{pmatrix}\Big|_{LD} = \begin{pmatrix} \sin^2 t & \cos t \sin t \\ \cos t \sin t & \cos^2 t \end{pmatrix}$
- $\eta(t) = -\cos t \frac{\partial}{\partial u} + \sin t \frac{\partial}{\partial v}$
- $\gamma'(t) = -\sin t \frac{\partial}{\partial u} + \cos t \frac{\partial}{\partial v}$
- $\delta(t) = |\gamma'(t), \eta(t)| = \cos 2t$
- $\gamma(t)$ is of the first kind $\iff t \neq \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$

Ex: hyperbolic paraboloid $z = \frac{1}{2}(x^2 - y^2)$

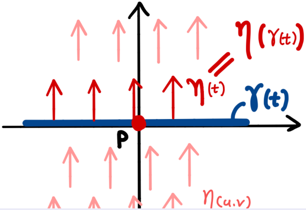


$$f(x,y) = \left(x, y, \tfrac{1}{2}(x^2 - y^2)\right)$$

- $LD = \{(x,y)\,;\, x^2 + y^2 = 1\}$
- $\gamma(t) = (\cos t, \sin t)$
- $\eta(t) = -\cos t \frac{\partial}{\partial u} + \sin t \frac{\partial}{\partial v}$
- $\gamma'(t) = -\sin t \frac{\partial}{\partial u} + \cos t \frac{\partial}{\partial v}$
- $\delta(t) = |\gamma'(t), \eta(t)| = \cos 2t$
- $\gamma(t)$ is of the first kind
  $\iff \quad t \neq \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$
- $\gamma(t)$ is of the second kind
  $\iff \quad t = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ (L_3point)
- $\hat{\gamma}(t) := f \circ \gamma(t)$
  $= (\cos t, \sin t, \frac{1}{2}\cos 2t)$

Invariants of lightlike points of 1st kind (1/3)

- A vector field $\eta = \eta(u, v)$ on U is **(extended) null vector field** $\overset{\text{def}}{\Longleftrightarrow}$ $\eta(t) := \eta(\gamma(t))$ gives a null vector field along $\gamma(t)$.



$$p \in LD \text{ is of the first kind } \Longleftrightarrow \eta_p \langle df(\eta), df(\eta) \rangle \neq 0$$

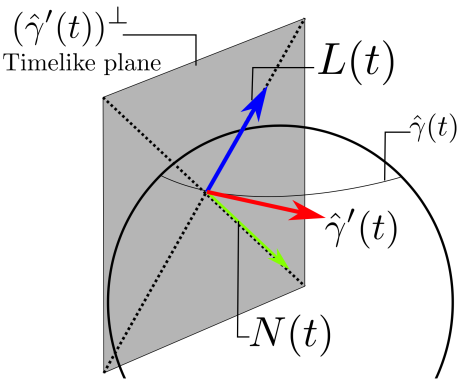
- Set $\hat{\gamma}(t) := f \circ \gamma(t)$: a regular curve in $\boldsymbol{R}^3_1$.
  - $\hat{\gamma}'(0) \in V_p \, (= df(T_p\Sigma))$ is either spacelike or lightlike.

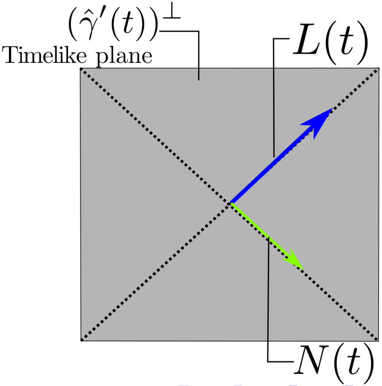
$$p = \gamma(0) \text{ is of the first kind } \Longleftrightarrow \hat{\gamma}'(0) \text{ is spacelike}$$

⇝ $\hat{\gamma}(t)$ **is a spacelike curve in** $\boldsymbol{R}^3_1$ **for sufficiently small** $|t|$

Invariants of lightlike points of 1st kind (2/3)

- **If** $p = \gamma(0)$: **first kind**
  - $\hat{\gamma}(t) = f \circ \gamma(t)$ is a spacelike curve in $\boldsymbol{R}^3_1$
  - $\hat{\gamma}'(t)$: **spacelike vector** $\Longrightarrow (\hat{\gamma}'(t))^{\perp}$: timelike 2-dim subspace.
- $\eta(t)$: null v.f. $\overset{\text{def}}{\Longleftrightarrow}$ $L(t) := df(\eta(t))$: **lightlike v.f.** in $\boldsymbol{R}^3_1$
- $\exists N(t)$: **lightlike v.f.** s.t. $\langle N(t), \hat{\gamma}'(t)\rangle = 0$, $\langle N(t), L(t)\rangle = 1$.





Invariants of lightlike points of 1st kind (3/3)

- If $p = \gamma(0)$: **first kind**

  $\Longrightarrow$ $\hat{\gamma}(t) = f \circ \gamma(t)$ is a spacelike curve in $\boldsymbol{R}^3_1$ (i.e. $\langle \hat{\gamma}'(t), \hat{\gamma}'(t) \rangle > 0$)
- Take **arclength parameter** t (i.e. $\langle \hat{\gamma}'(t), \hat{\gamma}'(t) \rangle = 1$).

  $\Longrightarrow$ Since $\hat{\gamma}''(t) \perp \hat{\gamma}'(t)$,

$$\hat{\gamma}''(t) = \langle \hat{\gamma}'', N \rangle L(t) + \langle \hat{\gamma}'', L \rangle N(t).$$

Definition (HST)

- $$\kappa_L(p) := \frac{1}{\sqrt[3]{\eta_p \langle df(\eta), df(\eta) \rangle}} \langle \hat{\gamma}''(0), L(0) \rangle$$

  : **lightlike singular curvature**
- $$\kappa_N(p) := \sqrt[3]{\eta_p \langle df(\eta), df(\eta) \rangle} \langle \hat{\gamma}''(0), N(0) \rangle$$

  : **lightlike normal curvature**

cf. Singular curvature κ_s, limiting normal curvature κ_ν for cuspidal edges (Saji-Umehara-Yamada, Ann of Math, 2009)

(3) Main result

Mixed type surfaces in Lorentzian mfd M^3

- M^3 : oriented Lorentzian 3-manifold
- Similar arguments hold if we replace $\boldsymbol{R}^3_1$ with M^3
- Mixed type surface

$$f : \Sigma \longrightarrow M^3$$

  - $p \in LD$: lightlike point of the first kind,
  - $\gamma(t)$: characteristic curve, $\gamma(0) = p$, parametrized by arclength
- ∇ : the Levi-Civita connection of M^3

Definition

- $$\kappa_L(p) := \frac{1}{\sqrt[3]{\eta_p \langle df(\eta), df(\eta)\rangle}} \langle \nabla_{d/dt}\hat{\gamma}'(0), L(0)\rangle$$

  : **lightlike singular curvature**

- $$\kappa_N(p) := \sqrt[3]{\eta_p \langle df(\eta), df(\eta)\rangle} \langle \nabla_{d/dt}\hat{\gamma}'(0), N(0)\rangle$$

  : **lightlike normal curvature**

Main result (1/6)

Let $f : \Sigma \to M^3$ be a mixed type surface.

Theorem A

- $p \in \Sigma$: lightlike point of the second kind,
- $\exists\{p_n\}$: sequence of lightlike point of the first kind s.t. $\lim_{n\to\infty} p_n = p$.

Then, lightlike singular curvature κ_L diverges to $-\infty$ along $\{p_n\}$:

$$\kappa_L(p_n) \longrightarrow -\infty \qquad (n \to \infty).$$

Moreover, if p **is not an** L_3 **point**,
$\Longrightarrow$ lightlike normal curvature κ_N also diverges to $-\infty$ along $\{p_n\}$:

$$\kappa_N(p_n) \longrightarrow -\infty \qquad (n \to \infty).$$

▷ If p **is an** L_3 **point**, then $\kappa_N(p_n) \longrightarrow 0, \pm\infty$.

REF: invariants of Cuspidal Edges

$$\hat{\gamma}''(t) = \kappa_s(t)\boldsymbol{b}(t) + \kappa_\nu(t)\nu(t)$$

κ_s : **singular curvature**

- $-\infty$ **(accumulating to 2nd kind)** [SUY]
- 'geodesic curvature'
- intrinsic invariants [SUY]
- **Gauss-Bonnet type thm** [SUY]
- **affects the shape**

κ_ν : **limiting normal curvature**

- **continuous across 2nd kind** [Martins-SUY]
- 'normal curvature'
- extrinsic invariants [Naokawa-UY]
- Gaussian curvature K : **bdd** $\Longleftrightarrow$ $\kappa_\nu \equiv 0$ [SUY]

Main result (2/6)

K : the Gauss curvature on $\Sigma_+ \cup \Sigma_-$ ($f : \Sigma \to M^3$ mixed type surface)

Theorem B

Let $p \in LD$: non-degenerate lightlike point.

- ▷ **K is bounded on a nbd U of p** $\Longrightarrow$ p is **first kind** lightlike pt.
- ▷ Assume $p \in LD$: **first kind** lightlike point. Then,
  K is bounded on a nbd U of p $\Longleftrightarrow$

$$\kappa_L = 0 \qquad \text{and} \qquad \kappa_N = \kappa_B \qquad \text{on} \quad LD.$$

Here, κ_B is an (intrinsic) invariant called the **balancing curvature**:

$$\kappa_B(p) = \frac{-1}{2E^2(G_v)^{\frac{5}{3}}}\left(G_v\left(EE_{vv} - 2EF_{uv} + E_uF_v\right) - \frac{1}{5}E_v\left(EG_{vv} - 2(F_v)^2\right)\right)\Bigg|_{(0,0)}$$

where (u, v) is a coordinate system s.t. $\gamma(u) = (u, 0)$, $\eta = \partial_v$

Main result (3/6)

Lemma

If $p \in LD$: **1st kind** lightlike pt, $\exists(U; u, v)$: coordinate nbd of p s.t.

$$\gamma(u) = (u, 0), \quad \eta = \partial_v, \quad E(u, 0) = 1, \quad \lambda_v(u, 0) = 1.$$

- Then, $\hat{K} := \lambda^2 K$ is extended to U;

$$\hat{K}(u, v) = C_0(u) + C_1(u)\, v + \frac{1}{2!} C_2(u, v) v^2,$$

where

$$C_0(u) = -\frac{1}{2}\kappa_L(u), \quad C_1(u) = \frac{1}{2}\left\{\kappa_N(u) - \kappa_B(u) + \kappa_L(u)\Phi(u)\right\}.$$

On the other hand, since $\lambda(u, v) = v\,\hat{\lambda}(u, v)$ $(\hat{\lambda} \neq 0)$,
we have (the latter part of) Thm B.

Main result (4/6)

$$\hat{K}(u,v) = -\frac{1}{2}\kappa_L(u) + \frac{1}{2}\{\kappa_N(u) - \kappa_B(u) + \kappa_L(u)\Phi(u)\}\, v + \frac{1}{2!}C_2(u,v)v^2$$

▷ If $\kappa_L(p) > 0$, $\hat{K}(0,0) < 0$.

- Since $\hat{K} = \lambda^2 K$, $\mathrm{sgn}(\hat{K}) = \mathrm{sgn}(K)$.
- If $M^3 = \boldsymbol{R}^3_1$, $\mathrm{sgn}(K) = -\mathrm{sgn}(K_{\mathrm{Euc}})$.
- Namely, $\kappa_L(p) > 0 \Longrightarrow K_{\mathrm{Euc}}(0,0) > 0$.

Corollary

Let $f : \Sigma \to \boldsymbol{R}^3_1$ be a mixed type surface.
Assume $p \in LD$: 1st kind.

- If $\kappa_L(p) > 0 \Longrightarrow$ the image of f is **dome-like** near $f(p)$.
- If $\kappa_L(p) < 0 \Longrightarrow$ the image of f is **saddle-shaped** near $f(p)$.

▷ If $p \in LD$: 2nd kind s.t. 1st kind lightlike pts accumulating to p, $\kappa_L < 0$ (Thm A) $\Longrightarrow f$ is **saddle-shaped** near such $p \in LD$.

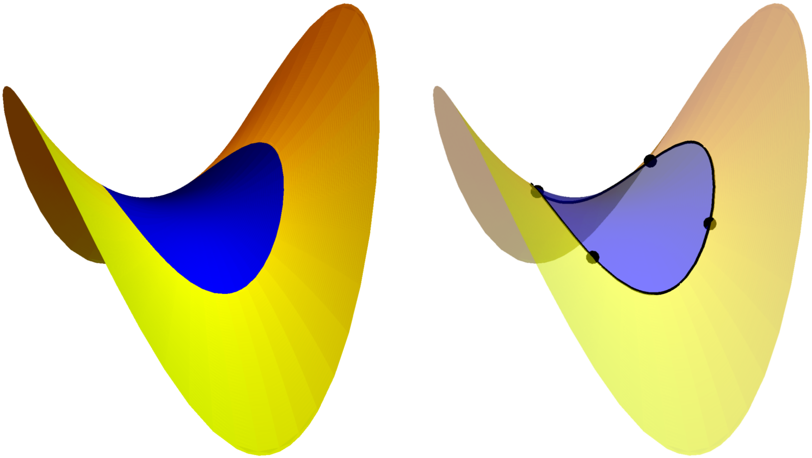


Figure: Hyperbolic paraboloid

Since $\kappa_L < 0$ near $t = \pm\pi/4, \pm 3\pi/4$ (lightlike points of 2nd kind), the surface is saddle-shaped.

Main result (5/6)

Applying Thm B to the results by **Pelletier** and **Steller**, we obtain the Gauss-Bonnet type thm:

Corollary

Let $f : \Sigma \to M^3$ be a mixed type surface
(where Σ : connected, compact, oriented 2-manifold).
Assume that

- every lightlike point of f is non-degenerate, and
- f has bounded Gaussian curvature.

Then

$$\int_\Sigma K\, dA = 2\pi\chi(\Sigma).$$

Intrinsic & Extrinsic invariants

Definition Let $f : \Sigma \to M^3$ be mixed type surface.

- A function $I : \Sigma \to \boldsymbol{R}$, or $I : LD \to \boldsymbol{R}$, is called an **invariant**. (Namely, I does not depend on the choice of the coordinate system of Σ or LD).

▷ An invariant $I : \Sigma \to \boldsymbol{R}$, or $I : LD \to \boldsymbol{R}$, is called **intrinsic** $\overset{\text{def}}{\iff}$

- I is a function of E,F, G, and their derivatives,
- where $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$, and
- (u, v) is a coordinate system defined by ds^2 only.

▷ An invariant $I : \Sigma \to \boldsymbol{R}$, or $I : LD \to \boldsymbol{R}$, is called **extrinsic** $\overset{\text{def}}{\iff}$

- $\exists$ another mixed type surface $\tilde{f}$ s.t.
- $\tilde{f}$ is isometric to f (namely, $ds^2_f = ds^2_{\tilde{f}}$), but $I(f) \neq I(\tilde{f})$.

- **Gaussian curvature** K : intrinsic invariant
- **Mean curvature** H : extrinsic invariant

Natural Problem.
To determine a given invariant is **intrinsic** or **extrinsic**.

- ▷ **lightlike singular curvature** κ_L is intrinsic:

$$\kappa_L(u) = -\frac{E_v(u,0)}{2E(u,0)\sqrt[3]{G_v(u,0)}}$$

where (u, v) is a coordinate system s.t. $\gamma(u) = (u, 0)$, $\eta = \partial_v$

- ▷ On the other hand, **lightlike normal curvature** κ_N is extrinsic:

Theorem [H]

- $f : \Sigma \to \boldsymbol{R}^3_1$: a mixed type surface, C^ω (real analytic),
- $p \in \Sigma$: lightlike point of 1st kind satisfying $\kappa_L(p) \neq 0$.

Then, $\exists\, U$: nbd of p, $\exists \tilde{f} : U \to \boldsymbol{R}^3_1$: C^ω-mixed type surface

s.t. $\tilde{f}$ is isometric to f, $\kappa_N(p) \neq \tilde{\kappa}_N(p)$.

- ▷ Apply the Cauchy-Kowalevski thm to Gauss-Codazzi type eq.

Main result (6/6)

If $\kappa_L \neq 0 \Longrightarrow$ lightlike normal curvature κ_N is **extrinsic**

Corollary

$p \in LD$: lightlike pt of 1st kind.
If $\kappa_L = 0$ on $LD \Longrightarrow$ lightlike normal curvature κ_N is **intrinsic**

- ▷ On a coordinate system $(U; u, v)$ s.t.

$$\gamma(u) = (u, 0),\ \eta = \partial_v,\ E(u, 0) = 1,\ \lambda_v(u, 0) = 1,$$

$$\hat{K}(u, v) = -\frac{1}{2}\kappa_L(u) + \frac{1}{2}\{\kappa_N(u) - \kappa_B(u) + \kappa_L(u)\Phi(u)\}v + \frac{1}{2!}C_2(u, v)v^2$$

- ▷ If $\kappa_L(u) \equiv 0$,

$$\kappa_N(u) = \kappa_B(u) + 2\hat{K}_v(u, 0)$$

holds, hence **intrinsic**.

Summary

- We introduced **lightlike singular curvature** κ_L, **lightlike normal curvature** κ_N at **lightlike points of 1st kind.**
  - κ_L : invariant like 'geodesic curvature'
  - κ_N : invariant like 'normal curvature'
- **Thm A.** Accumulating to a lightlike pt p of 2nd kind,
  - $\kappa_L \longrightarrow -\infty$
  - $\kappa_N \longrightarrow -\infty$ (if p is not L_3 point)
- **Thm B.** For a non-degenerate lightlike point p,
  - If Gauss curvature K is bounded $\Longrightarrow$ p must be **1st kind**.
  - Assume p : 1st kind. K is bounded $\Longleftrightarrow$ $\kappa_L = 0$, $\kappa_N = \kappa_B$ on LD.
- **Cor.** Gauss-Bonnet type thm : $\int_\Sigma K\, dA = 2\pi\chi(\Sigma)$
- While κ_L : **intrinsic invariant**,
  - κ_N is **extrinsic** (if $\kappa_L \neq 0$)
  - κ_N is **intrinsic** (if $\kappa_L \equiv 0$).

Than you very much for your attention!

Honda, Saji, Teramoto,
Mixed type surfaces with bounded Gaussian curvature in three-dimensional Lorentzian manifolds (arXiv:1811.11392).