Stability of non-proper functions

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24, March 2019, AMS meeting@University of Hawaii

(cf. arXiv:1809.02332)

Assume that mfd's are C^∞ & have no ∂ unless otherwise noted.

Here, $f : N \to P$: proper :$\Leftrightarrow$ $\forall K \subset P$: compact, $f^{-1}(K)$: compact

a function is a C^∞–mapping to $\mathbb{R}$ (i.e. $P = \mathbb{R}$).

◊ Plan of the talk

§.1 Introduction (definitions & background)

§.2 Main Result

§.3 Applications

§.1 Introduction

♢ Notations

- $C^\infty(N,P) := \{f : N \to P$: C^∞–mapping$\}$

  We endow $C^\infty(N,P)$ w/ Whitney C^∞–topology

- $\mathrm{Diff}(N) \subset C^\infty(N,N)$: set of self-diffeomorphisms

- $\Sigma(f) := \{x \in N \mid \mathrm{rank}(df_x) < \dim P\}$ for $f \in C^\infty(N,P)$

  $\Delta(f) := f(\Sigma(f))$: discriminant of f

- $\Gamma(E)$: set of sections of E : vect. bdl. over N

  $\Gamma(E)_S$: set of germs of sections of E at $S \subset N$: finite set

◇ Various notions of stability (1/2)

- $f \in C^\infty(N, P)$: **stable**

  $:\Leftrightarrow \exists \mathcal{U} \subset C^\infty(N, P)$: neighborhood of f

  $\exists(\Theta, \theta) : \mathcal{U} \to \mathrm{Diff}(N) \times \mathrm{Diff}(P)$: map

  s.t. $\forall g \in \mathcal{U}, \theta(g) \circ g \circ \Theta(g) = f$.

- $f \in C^\infty(N, P)$: **strongly stable**

  $:\Leftrightarrow \exists \mathcal{U} \subset C^\infty(N, P)$: neighborhood of f

  $\exists(\Theta, \theta) : \mathcal{U} \to \mathrm{Diff}(N) \times \mathrm{Diff}(P)$: **continuous** map

  s.t. $\forall g \in \mathcal{U}, \theta(g) \circ g \circ \Theta(g) = f$.

◇ Various notions of stability (2/2)

- $f \in C^\infty(N,P)$: **infinitesimally stable**

  $:\Leftrightarrow \Gamma(f^*TP) = tf(\Gamma(TN)) + \omega f(\Gamma(TP))$, where

  $tf : \Gamma(TN) \to \Gamma(f^*TP)$, $tf(\xi) = df \circ \xi$,

  $\omega f : \Gamma(TP) \to \Gamma(f^*TP)$, $\omega f(\eta) = \eta \circ f$.

- $f \in C^\infty(N,P)$: **locally stable**

  $:\Leftrightarrow \forall y \in \Delta(f), \forall S \subset f^{-1}(y)$: finite,

  $\Gamma(f^*TP)_S = tf(\Gamma(TN)_S) + \omega f(\Gamma(TP)_{\{y\}})$.

$\diamondsuit$ Stability of proper mappings

- f : proper $\Rightarrow$ all the stabilities are equivalent (Mather).

  (f : proper $:\Leftrightarrow \forall K \subset P$: compact, $f^{-1}(K) \subset N$: compact)

- In general, it is (relatively) easy to check local stability (Mather).

  e.g. $f : N \to \mathbb{R}$: (not necessarily proper) function is locally stable

  $\Leftrightarrow f$: Morse function, that is,

  – $\forall x \in \Sigma(f)$, $\det \left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}(x) \right)_{i,j} \neq 0$

  – $f|_{\Sigma(f)}$: inj.

Thus, it is easy to check stability of proper mappings!!

◇ Motivating problem 1

Problem 1

How can we detect (strong) stability of non-proper functions?

e.g. Is $f(x, y) = x^2 - y^2$ stable? (due to Ichiki)

Note that f is **NOT** strongly stable!! (will be seen later)

◇ Remarks on problem 1

Problem 1

How can we detect (strong) stability of non-proper functions?

e.g. Is $f(x, y) = x^2 - y^2$ stable? (due to Ichiki)

- f : inf. stable $\Leftrightarrow$ f : loc. stable & $f|_{\Sigma(f)}$: proper (Mather).

  In particular, infinitesimal stability is easily checked.

  (since it is easy to check local stability.)

  However, it is in general difficult to check (strong) stability!

◇ Remarks on problem 1

Problem 1

How can we detect (strong) stability of non-proper functions?

e.g. Is $f(x,y) = x^2 - y^2$ stable? (due to Ichiki)

- (Dimca) $f \in C^\infty(\mathbb{R}, \mathbb{R})$: stable

  $\Leftrightarrow f$: locally stable & $\Delta(f) \cap (\mathcal{S}(f) \cup L(f)) = \emptyset$, where

  $$L(f) = \left\{ y \in \mathbb{R} \;\middle|\; y = \lim_{x \to \infty} f(x) \text{ or } \lim_{x \to -\infty} f(x) \right\}$$

  $$\mathcal{S}(f) = \left\{ \lim_{i \to \infty} f(x_i) \in \mathbb{R} \;\middle|\; \{x_i\} : \begin{array}{l} \text{sequence in } \Sigma(f) \text{ without} \\ \text{accumulation points} \end{array} \right\}$$

  Thus, it is (somewhat) easy to check stability of $f \in C^\infty(\mathbb{R}, \mathbb{R})$.

Example $f : \mathbb{R} \to \mathbb{R}$, $f(x) := \exp(x) \sin x$.

Since $f^{(k)}(x) = 2^{k/2} \exp(x) \sin\left(x + \frac{k\pi}{4}\right)$, it is easy to see:

- $\Sigma(f) = \left\{ \frac{(4n+3)\pi}{4} \in \mathbb{R} \,\middle|\, n \in \mathbb{Z} \right\}$,

- f : Morse func. (i.e. $f|_{\Sigma(f)}$: inj. & $\forall x \in \Sigma(f)$, $f^{(2)}(x) \neq 0$).

Furthermore, $\mathcal{S}(f) = L(f) = \{0\}$ & $0 \notin \Delta(f) \Rightarrow f$: stable

On the other hand, $(f|_{\Sigma(f)})^{-1}([-1, 1])$: infinite discrete set

$\Rightarrow f$: NOT infinitesimally stable ($\because f|_{\Sigma(f)}$: not proper).

◇ Motivating problem 2

Problem 2

How are the four stabilities related for non-proper functions?

In particular, strongly stable $\Rightarrow$ infinitesimally stable?

$\diamondsuit$ Remarks on problem 2

Problem 2

How are the four stabilities related for non-proper functions?

In particular, strongly stable $\Rightarrow$ infinitesimally stable?

- f : strongly stable $\Rightarrow$ f : stable (obvious).
- f : stable $\Rightarrow$ f : locally stable (Mather).
- f : inf. stable $\Leftrightarrow$ f : loc. stable & $f|_{\Sigma(f)}$: proper (Mather).

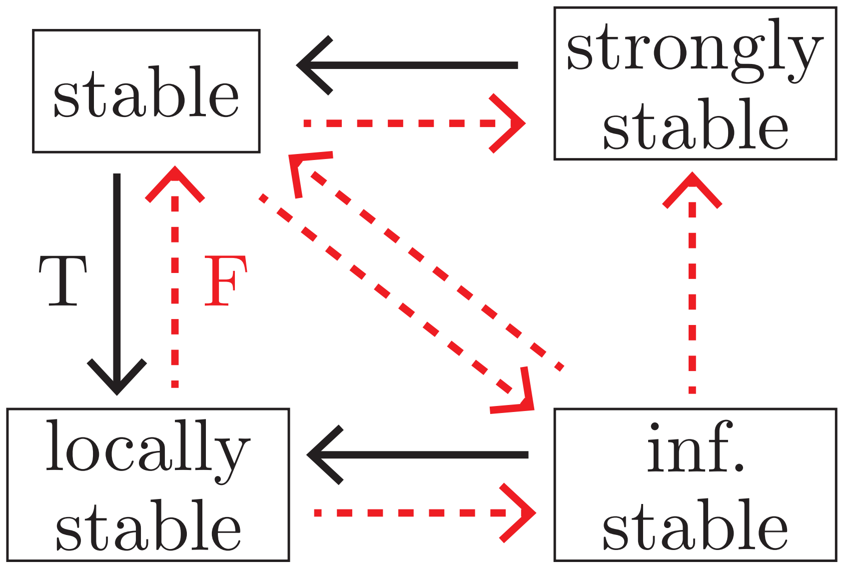
◇ Remarks on problem 2

- f : strongly stable $\Rightarrow$ f : quasi-proper (du Plessis-Vosegaard)

  f : **quasi-proper** $:\Leftrightarrow \exists V \subset P$: neighborhood of $\Delta(f)$ s.t.

  $$f|_{f^{-1}(V)} : f^{-1}(V) \to V \text{ : proper}$$

- Using the results we have explained, we can show:



◇ Motivating problems (Summary)

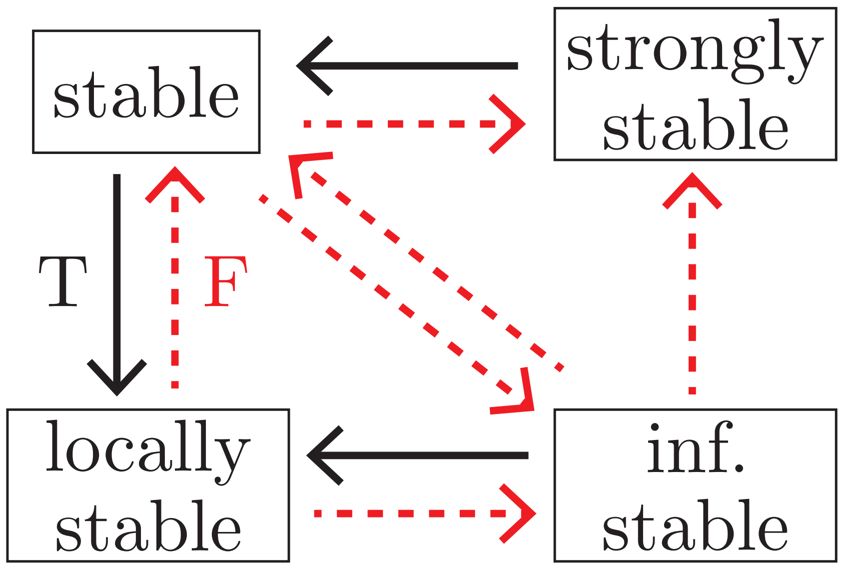
1. **detecting** (strong) **stability of non-proper functions.**

   e.g. **Q.** (due to Ichiki) : Is $f(x, y) = x^2 - y^2$ stable?

   Note that f : **NOT** quasi-proper (thus **NOT** strongly stable).

2. **strongly stable $\Rightarrow$ infinitesimally stable?**

   The other implications are known to be True/False as follows:



§.2 Main result

Theorem (H.)

$f \in C^\infty(N, \mathbb{R})$: Morse function.

$\tau(f) := \{y \in \mathbb{R} \mid f$: "end-trivial" at $y\}$.

(the definition of end-triviality will be given soon...)

1. $\Delta(f) \subset \tau(f) \Rightarrow f$: stable.
2. f : strongly stable $\Leftrightarrow f$: quasi-proper

f : **quasi-proper** $:\Leftrightarrow \exists V \subset P$: neighborhood of $\Delta(f)$ s.t.

$$f|_{f^{-1}(V)} : f^{-1}(V) \to V \text{ : proper}$$

$\diamondsuit$ Remarks on the main result

- As we explained, f : strongly stable $\Rightarrow$ f : quasi-proper for $f \in C^\infty(N, P)$ (du Plessis-Vosegaard)
  We indeed show the converse of it for the case $P = \mathbb{R}$.

- Dimca's condition ($\Delta(f) \cap (\mathcal{S}(f) \cup L(f)) = \emptyset$) is equivalent to ours ($\Delta(f) \subset \tau(f)$). Indeed,
  $\tau(f) = \mathbb{R} \setminus (\mathcal{S}(f) \cup L(f))$ for $f \in C^\infty(\mathbb{R}, \mathbb{R})$, where

$$L(f) = \left\{ y \in \mathbb{R} \;\middle|\; y = \lim_{x \to \infty} f(x) \text{ or } \lim_{x \to -\infty} f(x) \right\},$$

$$\mathcal{S}(f) = \left\{ \lim_{i \to \infty} f(x_i) \in \mathbb{R} \;\middle|\; \{x_i\} : \begin{array}{l} \text{sequence in } \Sigma(f) \text{ without} \\ \text{accumulation points} \end{array} \right\}.$$

◇ End-triviality

$V \subset N$: **neighborhood of the end**:$\Leftrightarrow N \setminus V$: compact

Definition $f \in C^\infty(N, P)$, $y \in P$.

f is **end-trivial** at y if $\forall K \subset N$: compact set,

$\exists W \subset P$: neighborhood of y,

$\exists V \subset N$: open neighborhood of the end with $V \subset N \setminus K$ s.t.

- $f^{-1}(y) \cap V$ contains no critical points of f,
- $\exists \Phi : (f^{-1}(y) \cap V) \times W \to f^{-1}(W) \cap V$: diffeomorphism

  s.t. $f \circ \Phi = p_2 : (f^{-1}(y) \cap V) \times W \to W$: projection

f is **end-trivial** at y if $\forall K \subset N$: compact set, $\exists W \subset P$: nbh. of y,
$\exists V \subset N$: open nbh. of the end with $V \subset N \setminus K$ s.t.

- $f^{-1}(y) \cap V$ contains no critical points of f,
- $\exists \Phi : (f^{-1}(y) \cap V) \times W \to f^{-1}(W) \cap V$: diffeomorphism
  s.t. $f \circ \Phi = p_2 : (f^{-1}(y) \cap V) \times W \to W$: projection

◇ Remarks on end-triviality

- Roughly, end-triviality at y implies that f is the projection "around the end of f^{-1}(nbh. of y) (or $f^{-1}(W)$)".

- Arbitrariness of K merely guarantees that we can take V "as small as we want" (see the blue parts).

$W \subset P$: nbh. of y, $V \subset N$: open nbh. of the end s.t.

- $f^{-1}(y) \cap V$ contains no critical points of f,
- $\exists \Phi : (f^{-1}(y) \cap V) \times W \to f^{-1}(W) \cap V$: diffeomorphism
  s.t. $f \circ \Phi = p_2 : (f^{-1}(y) \cap V) \times W \to W$: projection

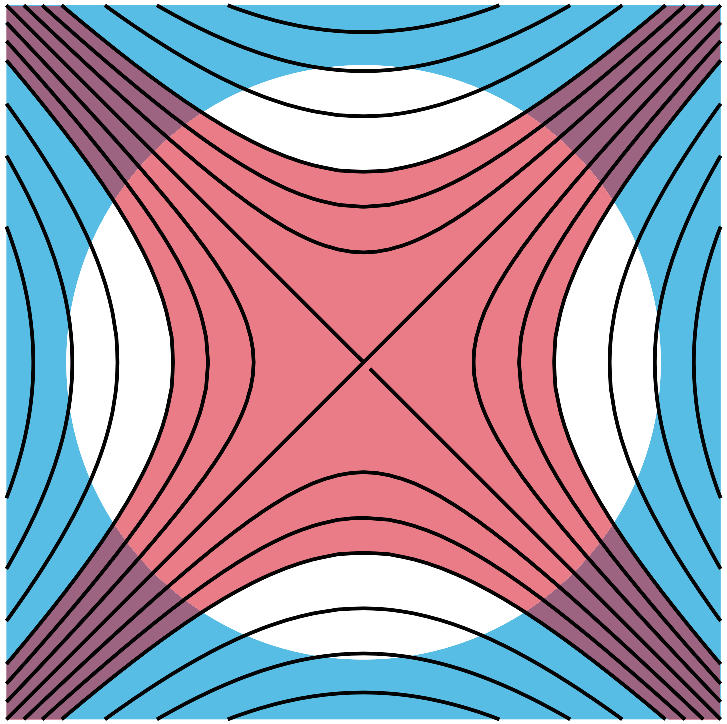
Example The fig. is contours of $f(x, y) := x^2 - y^2$ in $\mathbb{R}^2$.

Blue : outside of (sufficiently large) disk (which is V with $K \subset N \setminus V$)

Red : preimage of nbh. of $0 \in \mathbb{R}$ (which is $f^{-1}(W)$ for $y = 0$)

One can regard $f = p_2$ on Blue$\cap$Red. (i.e. $\exists \Phi$ with the desired property)

Thus, f is end-trivial at 0.



◇ Main result (Again)

Theorem (H.)

$f \in C^\infty(N, \mathbb{R})$: Morse function.

$\tau(f) := \{y \in \mathbb{R} \mid f$: end-trivial at $y\}$.

1. $\Delta(f) \subset \tau(f) \Rightarrow f$: stable.
2. f : strongly stable $\Leftrightarrow f$: quasi-proper

f : **quasi-proper** $:\Leftrightarrow \exists V \subset P$: neighborhood of $f(\mathrm{Crit}(f))$ s.t.

$$f|_{f^{-1}(V)} : f^{-1}(V) \to V \text{ : proper}$$

§.3 Applications

◇ detecting stability

Example $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 - y^2$.

$\Delta(f) = \{0\}$ and $0 \in \tau(f)$ (as we checked) $\Rightarrow f$ is stable.

In general...

Corollary 1 (H.)

$f \in C^\infty(\mathbb{R}^n, \mathbb{R})$: Morse & Nash function
(e.g. polynomial function)

$\nabla f : \mathbb{R}^n \to \mathbb{R}^n$: gradient of f.

Suppose that $\nexists \{x_i\}$: sequence in $\mathbb{R}^n$ w/o accumulation points s.t. $\lim_{i\to\infty} \nabla f(x_i) = 0$. Then f is stable.

Corollary 2 (H.)

$f : \mathbb{R}^n \to \mathbb{R}$: Nash function (not necessarily locally stable)

$\exists \Sigma \subset \mathbb{R}^n$: Lebesgue measure zero set s.t.

$\forall (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \Sigma$ the function

$$f_a(x_1, \ldots, x_n) = f(x_1, \ldots, f_n) + \sum_{i=1}^{n} a_i x_i$$

is stable.

The proof relies on Corollary 1 and Ichiki's result on transversality of generic linear perturbations of mappings (arXiv:1607.03220).

◇ strong & infinitesimal stability

Corollary 3 (H.)

The function $f(x) = \exp(-x^2)\sin x$ is strongly stable but NOT infinitesimally stable.

We indeed show that f : Morse function, quasi-proper

& $f|_{\Sigma(f)}$: NOT proper.

($f \in C^\infty(N, \mathbb{R})$: inf. stable $\Leftrightarrow$ f : Morse & $f|_{\Sigma(f)}$: proper (Mather))

◇ Summary (what we gave)

- a sufficient condition for (strong) stability of $f \in C^\infty(N, \mathbb{R})$.
- the answer to the following questions:

1. Is $f(x, y) = x^2 - y^2$ stable? **Yes!**
2. strongly stable $\Rightarrow$ infinitesimally stable? **No!**

Thank you for your attention!!