Outer metric Lipschitz classification of definable surface singularities

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All sets and maps are definable in a polynomially bounded o-minimal structure over $\mathbb{R}$ with the filed of exponents $\mathbb{F}$, e.g. semialgebraic or subanalytic with $\mathbb{F} = \mathbb{Q}$.

A **surface singularity** is a germ $(X, 0)$ of a two-dimensional set in $\mathbb{R}^n$ with **outer metric** $\text{dist}_o(x, y) = |y - x|$.

Two germs $(X, 0)$ and $(Y, 0)$ are **Lipschitz equivalent** if there is a bi-Lipschitz homeomorphism $(X, 0) \to (Y, 0)$.

Classification is a canonical decomposition of a germ $(X, 0)$ into normally embedded Hölder triangles T_j, with some additional data, that define a **complete discrete invariant** (no moduli) of its Lipschitz equivalence class.

An **arc** γ is a germ of a map $\gamma : [0, \epsilon) \to X$ such that $|\gamma(t)| = t$. **Tangency order** $\kappa = \text{tord}(\gamma, \gamma') \in \mathbb{F} \cup \{\infty\}$ of γ and γ' is the smallest exponent in $|\gamma - \gamma'| = ct^\kappa + \cdots$.

Let $\tilde{X}$ be the space of all arcs. A **zone** is a set $Z \subset \tilde{X}$ such that for $\gamma, \gamma' \in Z$ any arc in the Hölder triangle $T_{\gamma\gamma'}$ bounded by γ and γ' is in Z.

The **order** $\text{ord}(Z)$ of a zone Z is the minimal tangency order of arcs in Z.

An arc $\gamma \in Z$ is **generic** if there are arcs γ', γ'' in Z such that $\gamma \in T_{\gamma'\gamma''}$ and $\text{tord}(\gamma, \gamma') = \text{tord}(\gamma, \gamma'') = \text{ord}(Z)$.

A zone Z is **perfect** if each $\gamma \in Z$ is generic.

Special case: pizza. Let X be the union of a β-Hölder triangle T in the xy-plane and a graph $\{z = f(x, y)\}$ in $\mathbb{R}^3$ of a continuous function over T, $f(0, 0) = 0$.

For $\gamma \subset T$, define $\text{ord}_\gamma f = \text{tord}(\gamma, \gamma')$ where $\gamma' = (\gamma, f(\gamma))$. Let $Q(T) \subset \mathbb{F} \cup \{\infty\}$ be the set of $q = \text{ord}_\gamma f$ for all $\gamma \subset T$.

T is **elementary** if $Z_q = \{\gamma \subset T,\ \text{ord}_\gamma f = q\}$ is a zone for any $q \in Q(T)$. The **width function** on $Q(T)$ is defined as $\mu(q) = \text{ord}(Z_q)$.

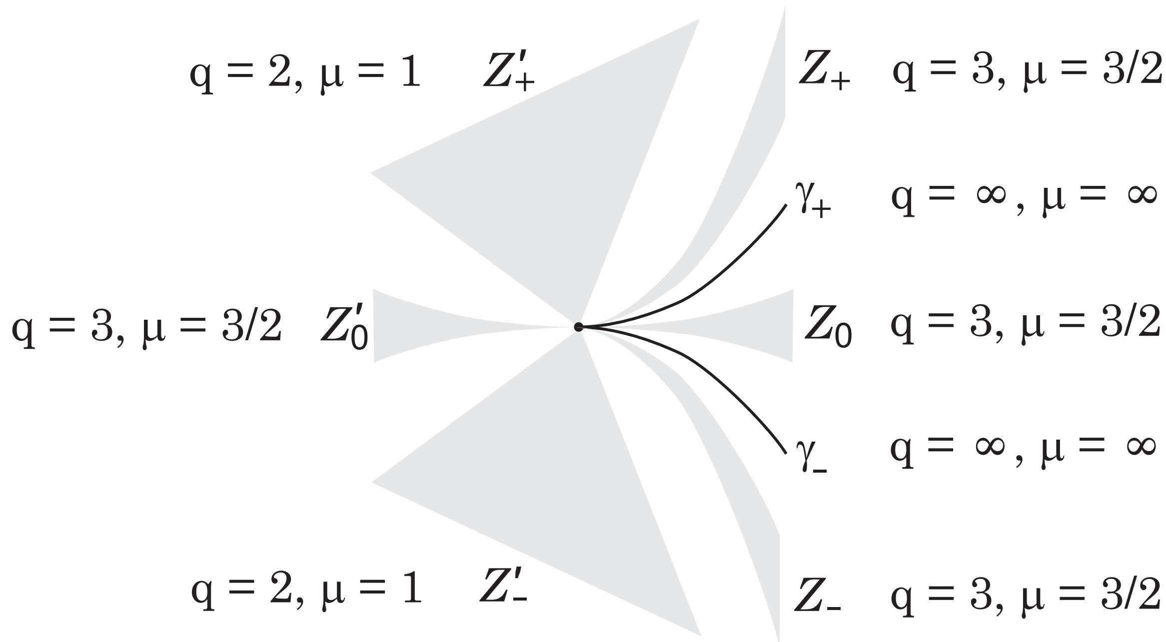
T is a **pizza slice** either if $Q(T)$ is a single point, or if $\mu(q) = aq + b$ is affine, where $a \in \mathbb{F} \setminus \{0\}$ and $b \in \mathbb{F}$. The side γ of T where μ is maximal is its **base side**.

A **pizza** is a partition of T into Hölder triangles T_j, each of them a pizza slice, with the **toppings:** exponent β_j of T_j, $Q_j = Q(T_j)$, width function $\mu_j(q)$ on Q_j, base side γ_j of T_j, sign s_j of f on T_j.

Theorem (Birbrair *et al*, 2017). The minimal pizza exists and is unique, up to bi-Lipschitz equivalence, for the Lipschitz contact equivalence class of f.

For a Lipschitz function f, its Lipschitz contact equivalence class is the same as Lipschitz equivalence class of the union of its graph and the xy-plane with respect to the outer metric.

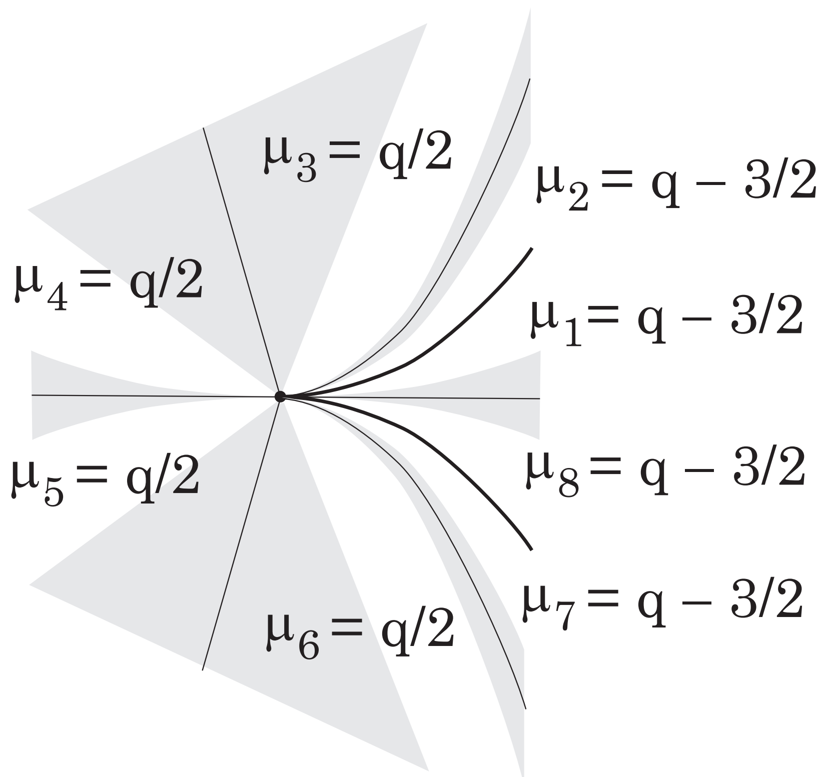
Example. Let $f(x,y) = y^2 - x^3$. Then $f = 0$ on arcs $\gamma_+ = \{x \geq 0,\ y = x^{3/2}\}$ and $\gamma_- = \{x \geq 0,\ y = -x^{3/2}\}$. Each of these two arcs is a "singular" zone of order ∞. There are six other **boundary zones** associated with the **critical exponents** $q = 2$ and $q = 3$ of f:



The set of all arcs γ such that $\text{ord}_\gamma f = 3$ consists of four zones. Three of them, Z_+, Z_- and Z_0, are in the right half-plane, above γ_+, below γ_-, and between γ_+ and γ_-, respectively. The fourth is Z'_0 in the left half-plane. Each of these zones is perfect of order 3/2.

The set of all arcs γ such that $\text{ord}_\gamma f = 2$ consists of two zones, Z'_+ and Z'_-, in the upper and lower half-planes. Each of them is perfect of order 1.

A minimal pizza for f consists of eight slices obtained by partitioning the xy-plane by the arcs γ_+, γ_-, and any arc selected in each of the six other boundary zones.

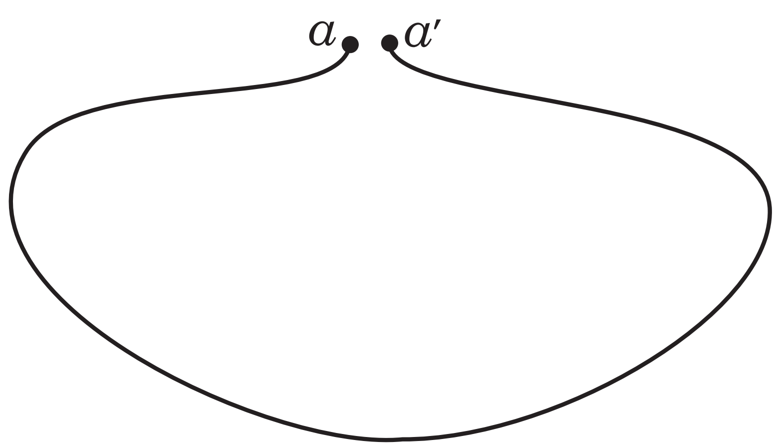


Multi-pizza. If there are several functions $z_\nu = f_\nu(x, y)$ defined on T, we can partition T into triangles T_j each of them a pizza slice for each f_ν, with affine width function $\mu_{\nu,j}(q)$. In addition, we may assume that the base side of T_j (where $\mu_{\nu,j}$ is maximal) is the same for all ν. This is called **multi-pizza**.

Abnormal zones are some new phenomena for general surfaces, which do not appear for graphs of functions.

An arc $\gamma \subset X$ is **abnormal** if there are two normally embedded Hölder triangles T and T' in X such that $\gamma = T \cap T'$ and $T \cup T'$ is not normally embedded. A zone $Z \subset \tilde{X}$ is **abnormal** if it consists of abnormal arcs.

Example. A curve aa' in the Figure below represents β-Hölder triangle T, which is not normally embedded. The boundary arcs γ and γ' of T represented by the points a and a' have tangency order $\alpha > \beta$. "Generic" arcs in T are abnormal, and form an abnormal zone $Z \subset \tilde{T}$.



General surface X strategy:

A pair (T, T') of normally embedded Hölder triangles is **transversal** if $T \cup T'$ is a subset of a normally embedded triangle. A non-transversal pair is **coherent** if it is bi-Lipschitz equivalent to a slice of pizza and a graph of a Lipschitz function over it.

Using critical exponents of the distance function, we identify boundary zones in the space $\tilde{X}$ of arcs in X and show that minimal by inclusion boundary zones are perfect. Any singular curve in X is a boundary zone.

Placing arbitrary arcs in minimal boundary zones (more than one may be needed in an abnormal zone) we decompose X into isolated arcs and normally embedded Hölder triangles so that each pair is either coherent or transversal.

Main Theorem. For a germ $(X, 0)$ of a surface with outer metric, there is a canonical (up to combinatorial equivalence) decomposition of X into isolated arcs and Hölder triangles, such that any two Hölder triangles are either coherent or transversal, with coherent triangles arranged into multi-pizza clusters.

Two such decompositions are **combinatorially equivalent** if there is one-to-one correspondence between their arcs and triangles, preserving all adjacency relations, tangency exponents between all isolated arcs and the boundary arcs of triangles, and all the multi-pizza parameters for the clusters of coherent triangles.

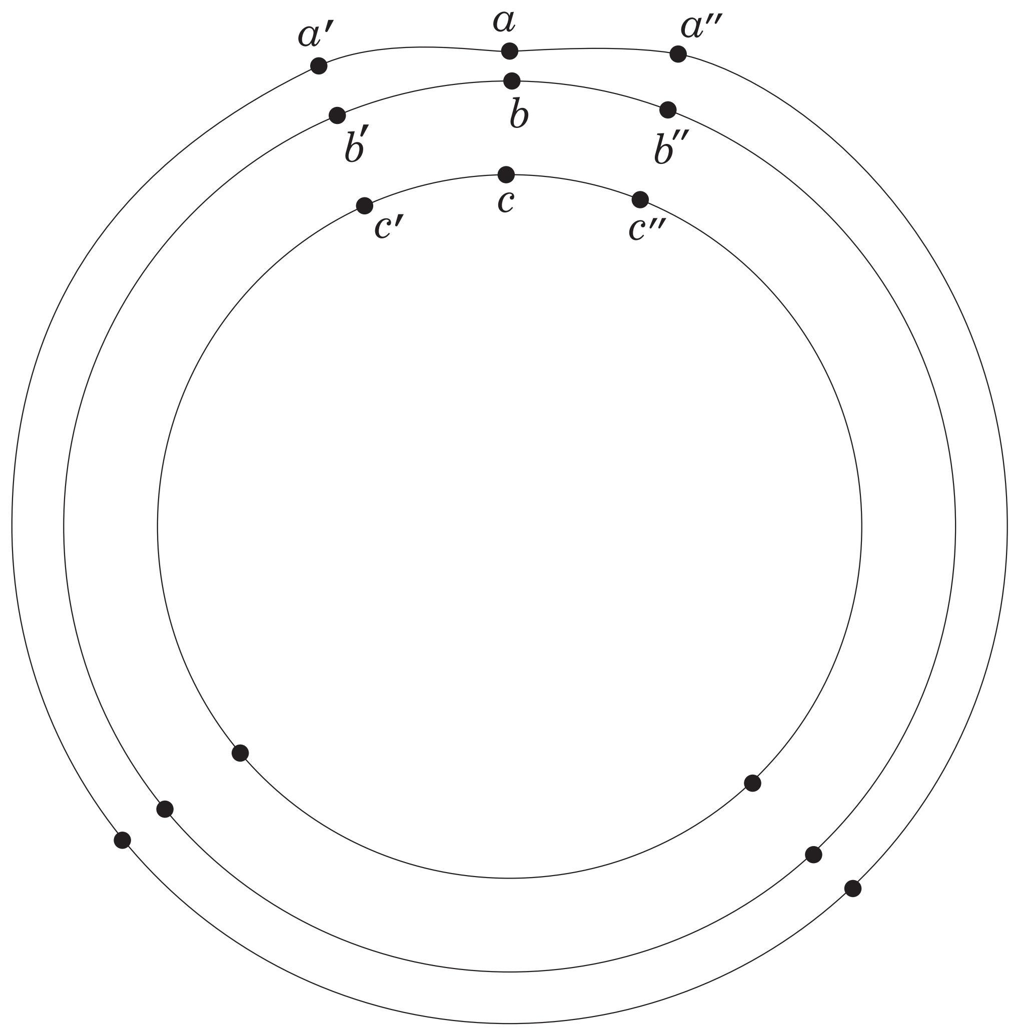
Two surface germs are outer Lipschitz equivalent if and only if their canonical decompositions are combinatorially equivalent.

Example: a complex curve. Let p and q be relatively prime, $p < q$. Then the set $\tilde{X}$ of arcs in a germ $(X, 0)$ of the irreducible complex curve $w^p = z^q$, considered as a surface in $\mathbb{R}^4$, is a single abnormal zone.

Its canonical partition is defined by $3p$ arcs γ_{ij} where $1 \leq i \leq 3$ and $1 \leq j \leq p$, such that $\mathrm{tord}(\gamma_{ij}, \gamma_{kl}) = 1$ for $i \neq k$ and $\mathrm{tord}(\gamma_{ij}, \gamma_{ik}) = q/p$ for $j \neq k$.

The partition consists of three groups of Hölder triangles, with p triangles in each group.

Each group is equivalent to a multi-pizza. Any two triangles in the same group are coherent with $Q = \{q/p\}$ (a single point) and $\mu = 1$, and any two triangles in different groups are transversal.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