Newton-Okounkov bodies and Segre classes

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Reference: arXiv:1809.07344

Also relevant:

—D. N. Bernstein, *The number of roots of a system of equations,* Funct. Anal. Appl. 9 (1975), 183–185.

—R. Lazarsfeld, M. Mustață, *Convex bodies associated to linear series,* Ann. Sci. Éc. Norm. Supér. (4), 42(5):783–835, 2009.

—K. Kaveh, K. Khovanskii, *Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory,* Ann. of Math. (2), 176(2):925–978, 2012.

—P. Aluffi, *Segre classes as integrals over polytopes,* J. Eur. Math. Soc. 18(12):2849–2863, 2016.

—P. Aluffi, *The Segre-zeta function of an ideal,* Adv. Math. 320:1201–1226, 2017.

Summary

New way to compute Segre classes of subschemes of projective space.

Motivation: Segre classes are key ingredients in intersection theory, and have applications to e.g., singularity theory.
Milnor number/classes, Chern-Schwartz-MacPherson classes of singular varieties may be expressed in terms of Segre classes.

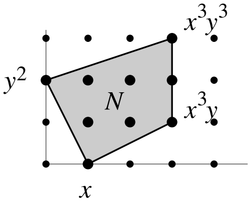
In fact, aim to compute a Segre zeta function $\zeta_I(t)$ for any collection $I = \{f_0, \dots, f_r\}$ of homogeneous polynomials with coefficients in (e.g.) $\mathbb{C}$.

$$\zeta_I(t) = \sum_{i \geq 0} \sigma_i t^i$$

such that for all n

$$\sum_{i=0}^{n} \sigma_i H^i \cap [\mathbb{P}^n] = s(X^{(n)}, \mathbb{P}^n)$$

where H = hyperplane class, $X^{(n)}$ defined by ideal generated by I.



Consider the system of equations

$$\begin{cases} a_{10}x + a_{11}xy + a_{02}y^2 + \cdots + a_{33}x^3y^3 = 0 \\ b_{10}x + b_{11}xy + b_{02}y^2 + \cdots + b_{33}x^3y^3 = 0 \end{cases}$$

where the coefficients a_{ij} and b_{ij} are general.

Question: # solutions with nonzero coordinates?

Answer: 11.

Theorem (Kushnirenko)

 # solutions with nonzero coordinates

$$= \text{'normalized' volume of Newton polytope } N$$
$$= n!\,\mathrm{Vol}_n(N) \quad \text{in dimension } n$$

Proof?

Allegedly, Khovanskii knows about 15 different proofs (as of 2007).
I may know one he does not know, using Segre classes.

Crash course on Segre classes:

- $s(Z, V) \in A_* Z$; often convenient to push-forward to $A_* V$.
- $\pi : \widetilde{V} \to V$ proper birational: $\pi_* s(\pi^{-1}(Z), \widetilde{V}) = s(Z, V)$ ('birational invariance')
- $Z =$ regularly embedded in $V \rightsquigarrow s(Z, V) = c(N_Z V)^{-1} \cap [Z]$.

These are enough to determine $s(Z, V)$ in general!

Segre classes

- Birational invariance
- Z = regularly embedded in $V \rightsquigarrow s(Z,V) = c(N_Z V)^{-1} \cap [Z]$.

For $Z \subsetneq V$, let $\pi : \widetilde{V} = B\ell_Z V \to V$,
$E = \pi^{-1}(Z)$ = exceptional divisor. Then

$$s(Z,V) = \pi_* s(E,\widetilde{V}) = \pi_*(c(N_E\widetilde{V})^{-1} \cap [E]) = \pi_* \frac{E}{1+E}$$
$$= \pi_*(E - E^2 + E^3 - \cdots)$$

There are algorithms implementing this definition, for subschemes of (e.g.) $\mathbb{P}^n$.

Example of computation: $Z =$ Veronese surface in $\mathbb{P}^5$.

$Z =$ image of Veronese embedding $\nu : \mathbb{P}^2 \to \mathbb{P}^5$.

h, H hyperplane classes in $\mathbb{P}^2, \mathbb{P}^5$ resp.: $\nu^*(H) = 2h$.

$\rightsquigarrow c(N_Z\mathbb{P}^5) = \frac{c(T\mathbb{P}^5|_Z)}{c(TZ)} = \frac{\nu^* c(T\mathbb{P}^5)}{c(T\mathbb{P}^2)} = \frac{(1+2h)^6}{(1+h)^3} = 1 + 9h + 30h^2$.

$s(Z, \mathbb{P}^5) = (1 + 9h + 30h^2)^{-1} \cap [Z] = (1 - 9h + 51h^2) \cap [Z]$.

Remark: $51 \cdot 64 = 3264$, a famous number.

($=$ number of smooth conics tangent to 5 smooth conics in general position)

Segre classes—Applications

Fulton-MacPherson intersection theory:

- Want to intersect X, Y in V.
  Assume $X \hookrightarrow V$ is a regular embedding, normal bundle N.
- Construct the fiber diagram:

$$\begin{array}{ccc} X \cap Y & \longrightarrow & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ X & \hookrightarrow & V \end{array}$$

- Then $X \cdot Y = \{c(g^* N) \cap s(X \cap Y, Y)\}_{\dim X + \dim Y - \dim V}$.
- The key ingredient here is the Segre class.

$$X \cdot Y = \{c(g^*N) \cap s(X \cap Y, Y)\}_{\dim X + \dim Y - \dim V} .$$

If $Z \subseteq X \cap Y$ is a connected component, the *contribution* of Z to $X \cdot Y$ is $\{c(g^*N) \cap s(Z, Y)\}_{\dim X + \dim Y - \dim V}$.

Applications:

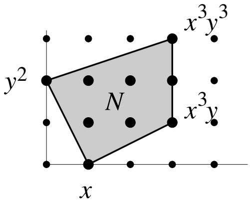
- Enumerative geometry.
  E.g.: *How many curves of degree* d *are tangent to* $d(d+3)/2$ *general lines in the plane?*
  $\leftrightarrow$ Segre class of scheme of nonreduced plane curves.
  (Open! for $d \geq 5$)
  Many open problems in enumerative geometry may be translated into Segre class computations.

- Combinatorics:
  The characteristic polynomial of a hyperplane arrangement may be written in terms of a Segre class.
- Several invariants of singularities are encoded in Segre classes.
  E.g.: *Donaldson-Thomas invariants.*
  E.g.: *Milnor data.* Code to compute topological Euler characteristic of projective varieties is based on Segre classes.
  X: hypersurface in nonsingular compact V, $\mathcal{L} = \mathcal{O}(X)$. Then

$$\chi(X) = \int c(TV) \cap \left(c(\mathcal{L})^{-1} \cap [X] + c(\mathcal{L})^{-1} \left(s(JX, V)^\vee \otimes_V \mathcal{L} \right) \right)$$

  Recent: Generalization of this formula to arbitrary schemes embeddable in a nonsingular variety. (arXiv:1805.11116)

Back to Kushnirenko



We had a system of equations

$$\begin{cases} a_{10}x + a_{11}xy + a_{02}y^2 + a_{21}x^2y + a_{12}xy^2 + a_{31}x^3y + a_{22}x^2y^2 + a_{32}x^3y^2 + a_{33}x^3y^3 = 0 \\ b_{10}x + b_{11}xy + b_{02}y^2 + b_{21}x^2y + b_{12}xy^2 + b_{31}x^3y + b_{22}x^2y^2 + b_{32}x^3y^2 + b_{33}x^3y^3 = 0 \end{cases}$$

where the coefficients a_{ij} and b_{ij} are general.

Kushnirenko's theorem computes the number of solutions with nonzero coordinates:

$\# = n!\,\mathrm{Vol}_n(N)$

- Newton polygon spanned by *monomials* in $x_1, \dots, x_n$.
- Have n general elements in the linear system spanned by these monomials after homogeneization.
- Base locus of linear system: solutions with some coordinate $= 0$.
- $\#$ of 'good' solutions $=$ Bézout number $-$ contribution of base locus. This contribution is evaluated by a Segre class.
- In the example: need Segre class of subscheme $Z \subseteq \mathbb{P}^2$ defined by

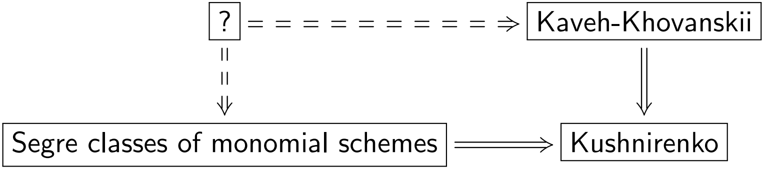
$$(xz^5, xyz^4, y^2z^4, x^2yz^3, xy^2z^3, x^3yz^2, x^2y^2z^2, x^3y^2z, x^3y^3)$$

- There are algorithms computing Segre classes, implemented in Macaulay 2: $s(Z, \mathbb{P}^2) = 25[pt]$.
- Kushnirenko's number $= 6^2 - 25 = 11$.

Take-away: Kushnirenko's theorem would follow from results on Segre classes of subschemes defined by *monomial* ideals.

Rest of the talk:

- (1) Explain computation of Segre classes of monomial ideals $\longrightarrow$ Generalization of Kushnirenko's theorem.
- (2) Explain Kaveh-Khovanskii generalization of Kushnirenko's theorem to arbitrary ideals. (*Newton-Okounkov bodies.*)
- (3) Fill the diagram



Integral formula for Segre zeta of a monomial ideal.

Example

Say we want the Segre class of the subscheme $X^{(n)}$ defined by $I = (y^3, x^2y^2)$ in $\mathbb{P}^n$, $n \geq 0$.

Implementations for Segre class computations in Macaulay2 (—, Eklund-Jost-Petersen, Helmer, Harris. . .) essentially implementing the definition:

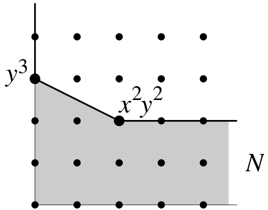
$$s(X^{(n)}, \mathbb{P}^7) = (2H - 2H^2 - 10H^3 + 94H^4 - 538H^5 + 2638H^6 - 12010H^7) \cap [\mathbb{P}^7]$$

This says

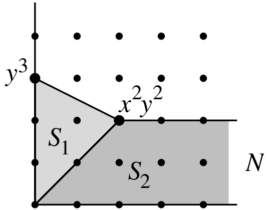
$$\zeta_I(t) = 2t - 2t^2 - 10t^3 + 94t^4 - 538t^5 + 2638t^6 - 12010t^7 + \cdots$$

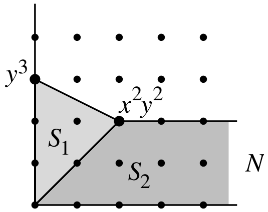
(Not clear how to get the other terms!)

Different approach—Associate a Newton-like region to the ideal:



Subdivide N into 'generalized simplices':





Get a rational function from each simplex, and add up the results

$$\frac{6t^2}{(1+3t)(1+4t)} + \frac{2t}{(1+4t)} = \frac{2t(1+6t)}{(1+4t)}$$

Fact:

$$\frac{2t(1+6t)}{(1+4t)} = 2t - 2t^2 - 10t^3 + 94t^4 - 538t^5 + 2638t^6 - 12010t^7 + 52414t^8 - 222778t^9 + 930478t^{10} - 3840010t^{11} + \cdots$$

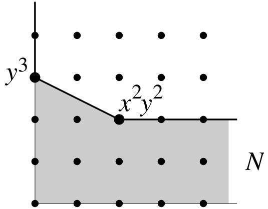
This is nontrivial!
General statement:

- I=set of monomials in $x_1, \ldots, x_n$;
- determine Newton region N in Euclidean n-space, coordinates $a_1, \ldots, a_n$;
- then

Theorem (—, 2013)

$$\zeta_I(t) = \int_N \frac{n! t^n da_1 \cdots da_n}{(1 + (a_1 + \cdots + a_n)t)^{n+1}}$$

Example



$$\int_N \frac{2!\, t^2 da_1\, da_2}{(1+(a_1+a_2)t)^3} = 2t - 2t^2 - 10t^3 + 94t^4 - 538t^5 + \cdots$$

Fact: Integral may be evaluated by sums over simplices.
Contribution of simplex S with vertices $(a_1^{(i)}, \ldots, a_n^{(i)})$:

$$\frac{n!\,\mathrm{Vol}_n(S)t^n}{\prod_i (1+(a_1^{(i)}+\cdots+a_n^{(i)})t)}$$

Proof of the theorem:

- Is independent of Kushnirenko's theorem.
- Full statement of the theorem is more precise: get Segre class in the Chow group, not just after push-forward.
- Also: It works for 'generalized monomial' subschemes in any variety.
- Ordinary monomials: from components of a divisor with simple normal crossings. 'Generalized' monomials: from components of a divisor with 'regular crossings' (much weaker requirement).
- Using: principalization of generalized monomial schemes (C. Harris).
- Main tool: birational invariance of Segre classes, behavior of Newton polytopes under blow-ups. □

Claim: Previous theorem implies Kushnirenko's theorem.

'Proof': Volumes of Newton polytopes
$\rightarrow$ Segre classes of monomial subschemes
$\rightarrow$ evaluation of contribution of base loci to linear systems
$\rightarrow$ intersection numbers
$\rightarrow$ Kushnirenko's theorem.

This gives the bottom of the diagram shown earlier:

Segre classes of monomial schemes $\Longrightarrow$ Kushnirenko

View as a generalization of Kushnirenko: not only computing intersection numbers, rather whole Segre class.

Volumes of Newton-Okounkov bodies

There is a completely different generalization of Kushnirenko's theorem, due to Kiumars Kaveh and Askold Khovanskii.

Kushnirenko: *Monomial* linear systems
KK: *Any* linear system.

L: linear system on (not nec. compact) V, $\dim V = n$.

$[L, \dots, L]$: 'intersection index'.
$[L, \dots, L] = \#$ points of intersection of n general sections of L, *away* from base locus.

Kushnirenko: For monomial L,
$[L, \dots, L] = n! \, \mathrm{Vol}_n(\text{Newton polytope})$.

KK: For arbitrary L, same! but using Newton-Okounkov body.

Newton-Okounkov body of a linear system L: depends on the choice of a *valuation.*

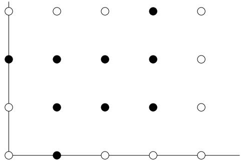
Geometric version:

- Fix a flag of nonsingular subvarieties $V = V_n \supseteq V_{n-1} \supsetneq \cdots \supsetneq V_0$, $\dim V_i = i$;
- For $f \in L$, $f \neq 0$, associate n-tuple of integers:
  - $m_1 :=$ order of vanishing of f along V_{n-1};
  - If $g = 0$ is the equation of V_{n-1}, then $f_1 := fg^{-m_1}|_{V_1}$ does not vanish identically along V_{n-1};
  - $m_2 :=$ order of vanishing of f_1 along V_{n-2};
  - etc. $\rightsquigarrow v(f) = (m_1, \ldots, m_n)$.
- Get set $v(L)$ of tuples, from *all* nonzero $f \in L$.

Linear system L on V; flag $V = V_n \supsetneq V_{n-1} \supsetneq \cdots \supsetneq V_0$, $\dim V_i = i$
$\rightsquigarrow$ set $v(L) \subseteq \mathbb{Z}^n$.

Example: $v(x_1^{m_1} \cdots x_n^{m_n}) = (m_1, \dots, m_n)$
for the flag: $V_i = \{x_1 = \cdots = x_{n-i} = 0\}$.
For $V = \mathbb{P}^2_{(x:\,y:\,z)}$, flag $\mathbb{P}^2 \supseteq \mathbb{P}^1 = \{x = 0\} \supseteq \mathbb{P}^0 = \{x = y = 0\}$,
$L = \langle xz^5, xyz^4, y^2z^4, x^2yz^3, xy^2z^3, x^3yz^2, x^2y^2z^2, x^3y^2z, x^3y^3 \rangle$:

L



Note: Trivially in this case, $\# v(L) = \dim L$.

Fact: For 'all' L, 'all' flags, $\#v(L) = \dim L$.

Idea: The growth of $v(L^k)$ as $k \to \infty$ gives information about the growth of $\dim(L^k)$, hence Hilbert polynomial-type information.

Definition

Newton-Okounkov body of L:

$$NO(L) := \left\{ \text{closed convex hull of } \bigcup_{k>0} \frac{1}{k} v(L^k) \right\}$$

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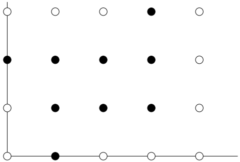
Example: Again with the standard flag, and
$L = \langle xz^5, xyz^4, y^2z^4, x^2yz^3, xy^2z^3, x^3yz^2, x^2y^2z^2, x^3y^2z, x^3y^3 \rangle$:

L



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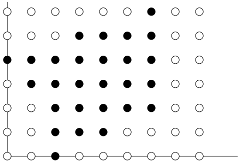
L^2



$$NO(L) := \left\{ \text{closed convex hull of } \cup_{k>0} \frac{1}{k} v(L^k) \right\}$$

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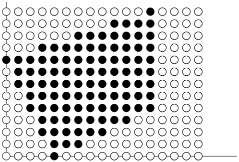
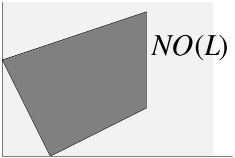
L^4



$$NO(L) := \left\{ \text{closed convex hull of } \cup_{k>0} \frac{1}{k} v(L^k) \right\}$$

Example: Again with the standard flag, and
$L = \langle xz^5, xyz^4, y^2z^4, x^2yz^3, xy^2z^3, x^3yz^2, x^2y^2z^2, x^3y^2z, x^3y^3 \rangle$:

L^{∞}



Fact: Although $NO(L)$ depends on the flag, its *volume* does not!

Theorem (Kaveh-Khovanskii)

$$[L, \ldots, L] = n! \, \mathrm{Vol}_n(NO(L))$$

(Rough version; the actual result is more precise.)

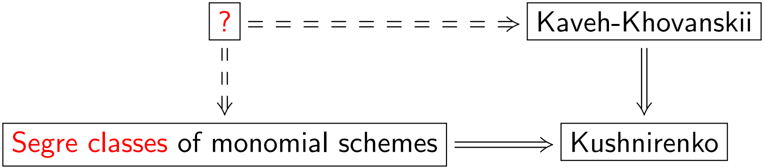
If L is monomial, this is again Kushnirenko's theorem. This is the rightmost arrow in the earlier diagram:

Kaveh-Khovanskii

$$\Downarrow$$

Kushnirenko

Newton-Okounkov bodies and Segre classes



The task: Obtain an integral formula for Segre classes of *arbitrary* projective schemes, in the style of the result for monomial schemes presented earlier.

Natural expectation: It should work in the same way, with the Newton polytope replaced by a suitable Newton-Okounkov body.

$I \subseteq \mathbb{C}[x_0, \ldots, x_n]$: homogeneous ideal
$\mathbb{P}^n = V_n \supsetneq V_{n-1} \supsetneq \cdots \supsetneq V_0$ flag, $\dim V_i = i$.

We will construct a 'Newton-Okounkov body' $NO(I) \subseteq \mathbb{R}^{n+1}$ for which the following will hold.

Theorem (—, 2018)

$$\zeta_I(t) = \int_N \frac{(n+1)!\, t^{n+1} da_0 \cdots da_n}{(1 + (a_0 + \cdots + a_n)t)^{n+2}}$$

where N = complement of $NO(I)$ in positive orthant.

• I monomial, standard flag: Then recover computation of Segre class for monomial schemes

• For arbitrary I, use Segre classes to evaluate contribution of base locus: Then recover Kaveh-Khovanskii.

(However, Kaveh-Khovanskii is used in the proof of main theorem!)

Construction of Newton-Okounkov body of an ideal

—Essentially a special case of a construction of Lazarsfeld-Mustață, 'global Newton-Okounkov body'.

I: homogeneous ideal, so $I = \oplus_{s \geq 0} I_s$.

Each I_s determines a linear system $\rightsquigarrow NO(I_s)$, constructed as before.

Theorem (—, 2018; but really Lazarsfeld-Mustață)

Let $\delta : \mathbb{R}^{n+1} \to \mathbb{R}$, $(a_0, \dots, a_n) \mapsto a_0 + \cdots + a_n$.
There is a naturally defined convex body in $\mathbb{R}^{n+1}$, $NO(I)$, such that for $s \gg 0$ integer, $\delta^{-1}(s) \cap NO(I) = NO(I_s)$.

To define $NO(I)$:

- Dehomogenize I (e.g., set $x_0 = 1$);
- Fix flag, corresponding valuation v on $\mathbb{C}[x_1, \dots, x_n]$;
- $U_I := \left\{(a, s, t) \in \mathbb{R}^n \times \mathbb{R}^2 \,|\, s \in \mathbb{Z}^{\geq 0}, t \in \mathbb{Z}^{\geq 0}, \underline{a} \in v((I^t)_s)\right\}$;
- $\Sigma(U_I) =$ closed convex cone generated by U_I;
- $\Delta(I) = \Sigma(U_I) \cap \{t = 1\} \subseteq \mathbb{R}^n \times \mathbb{R}^1$;
- $NO(I) :=$ image of $\Delta(I)$ in $\mathbb{R}^{n+1}$ via $(a_1, \dots, a_n, s) \mapsto (s - (a_1 + \dots + a_n), a_1, \dots, a_n)$.

Then $NO(I) \cap \delta^{-1}(s) = \Delta(I) \cap (\mathbb{R}^n \times \{s\}) = NO(I_s)$ for $s \geq$ max degree of generator of I.
(Proof: Techniques from Lazarsfeld-Mustaţă.)

Construction $\implies$ main theorem

$\Delta(I) \subseteq \mathbb{R}^n \times \mathbb{R}^1 \mapsto \mathbb{R}^1$, $(\underline{a}, s) \mapsto s$.
$\Delta_r :=$ fiber over r.
Also: Define $\sigma_j \in \mathbb{Z}$ by $\sum_{j=0}^n \sigma_j [\mathbb{P}^{n-j}] = [\mathbb{P}^n] - s(X, \mathbb{P}^n)$,
where $X =$ subscheme defined by I.

Lemma

For $r \in \mathbb{R}$, $r >$ max degree of a generator,

$$\mathrm{Vol}_n(\Delta_r) = \sum_{i=0}^n \binom{n}{i} \sigma_{n-i} r^i$$

Main theorem follows from this: The integral extracts the coefficients σ_j. (+ technicalities to get the whole $\zeta_I(t)$.)

Proof of the lemma.

- Kaveh-Khovanskii $\rightsquigarrow$ $\mathrm{Vol}(\Delta_s)$ for $s \gg 0$ integer;
- Formula reduced to showing $[I_s, \dots, I_s] = \sum_{j=0}^n \sigma_j s^{n-j}$ for $s \in \mathbb{Z}$, $s \gg 0$;
- For this: I_s determines a rational map $\varphi_s : \mathbb{P}^n \dashrightarrow \mathbb{P}(I_s^\vee) = \mathbb{P}^{N_s}$;$\rightsquigarrow$ $\Gamma_s \subseteq \mathbb{P}^n \times \mathbb{P}^{N_s}$, graph of φ_s;
- $[\Gamma_s] = g_0^{(s)} H^{N_s} + \cdots + g_n^{(s)} h^n H^{N_s - n}$, where $h, H =$ hyperplane classes in $\mathbb{P}^n, \mathbb{P}^{N_s}$;
- Fact (—, 2003; essentially straightforward): the $g_i^{(s)}$ may be expressed in terms of the Segre class of X in $\mathbb{P}^n$;
- $[I_s, \dots, I_s] = g_n^{(s)}$.

□

Technicalities to get whole Segre zeta function:

- Define $$\sum_{i\geq 0}\rho_i t^i = \int_{NO(I)} \frac{(n+1)!t^{n+1}da_0\cdots da_n}{(1+(a_0+\cdots+a_n)t)^{n+2}} \in \mathbb{Z}[[t]] \quad .$$
  Then $s(X,\mathbb{P}^n) = (1-\sum_{i=0}^n \rho_i h^i)\cap[\mathbb{P}^n]$, X defined by I in $\mathbb{P}^n$;
- In particular, coefficients $\rho_0,\dots,\rho_n$ are independent of the chosen flag; need to deal with ρ_i, $i>n$;
- Key point: If I' = extension of I to $\mathbb{C}[x_0,\dots,x_n,x_{n+1}]$, then may choose flags so that $NO(I') = NO(I)\times\mathbb{R}^{\geq 0}$.
- For such flags,
  $$\int_{NO(I')} \frac{(n+2)!t^{n+2}da_0\cdots da_{n+1}}{(1+(a_0+\cdots+a_{n+1})t)^{n+3}} \equiv \int_{NO(I)} \frac{(n+1)!t^{n+1}da_0\cdots da_n}{(1+(a_0+\cdots+a_n)t)^{n+2}} \mod t^{n+2}$$
- Inductively, extend to $\equiv \mod t^N$ for all N, done. □

Last comments about the proof:

- The proof depends on Kaveh-Khovanskii, and this comes at a price, e.g., the result is 'numerical'. Possible improvements?
- It would be desirable to get $s(X, \mathbb{P}^n)$ as a class in A_*X.
- It would also be desirable to allow more general ambient spaces: get $s(X, Y)$ for arbitrary subschemes X of arbitrary varieties Y.
  (Both points OK for monomial ideals.)
- Blueprint for a stronger result? Extend strategy working for monomial ideals: Use birational invariance, induction on # blow-ups needed to principalize a given ideal.
- Main difficulty: Understand behavior of new Newton-Okounkov body under blow-ups.
  This seems very difficult.

Thank you for your attention!