Singularities of parallels to tangent maps of frontals

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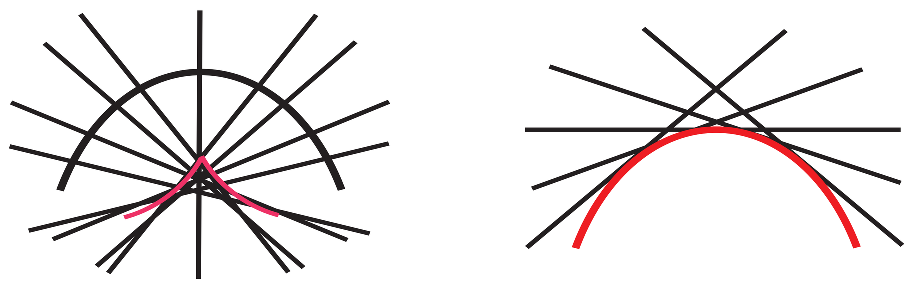
【 Normal maps and tangent maps 】

Given a submanifold in an Euclidean space, we consider two typical mappings: the normal map, ruled by normal spaces along the submanifold, and the tangent map, ruled by tangent spaces:



Normal maps naturally appears in, for example, geometric optics and extrinsic differential geometry ([15, 1, 17, 14]). Their singularities are called caustics or focal sets. Normal maps have always "Lagrangian singularities". It is known that generic Lagrangian singularities occur for normal maps of generic submanifolds. They are studied and classified in Lagrangian singularity theory by, for instance, the method of generating families ([1]). Contrary to normal maps, tangent maps of submanifolds, which are natural subjects as well, have not necessarily Lagrange singularities, and tangential maps have rather degenerate singularities even in generic cases.

【 Frontals 】

The normal map and tangent map are defined also for a frontal, which is a singular submanifold with well-defined tangent bundle.

Let U be an n-dimensional manifold and M an m-dimensional manifold with $n \leq m$. A map-germ $f : (U, a) \to M$ is called frontal if there exists a smooth ($= C^\infty$) subbundle T_f of the pullback bundle $f^*(TM)$ such that the image of the differential $f_*(T_tU) \subseteq (T_f)_{f(t)}$ for any t nearby a. Then we call the associated n-plane field $(T_f)_{f(t)}$ along f, a Legendrian lift of f. The bundle T_f is called the tangent bundle of f. A Legendrian lift of f is regarded as an integral section $\widetilde{f} : (U, a) \to \mathrm{Gr}(n, T\mathbf{R}^{n+p})$ of the Grassmannian bundle over f associated to $T\mathbf{R}^{n+p}$.

A map $f : U \to M$ is called a frontal if any germ of f at any point $a \in U$ is frontal.

If M is a Riemannian manifold and a Legendrian lift T_f over U is given, then the pull-back bundle $f^*(TM)$ is decomposed into the sum $T_f \oplus N_f$ of T_f and the normal bundle $N_f := (T_f)^\perp$ of rank $p = m - n$. Note that the decomposition is uniquely determined by f if f is a proper frontal, i.e. if the singular locus of f is nowhere dense ([10]).

In this presentation, we study the case $M = \mathbf{R}^{n+p}$, $n + p$-dimensional Euclidean space.

【 Normal and flat connections of frontals 】

We recall and generalise the notions of normal and flat connections in differential geometry.

Let $f : U \to \mathbf{R}^{n+p}$ be a frontal endowed with the decomposition $f^*(T\mathbf{R}^{n+p}) = T_f \oplus N_f$.

For any vector field η over U and any vector field $X : U \to T\mathbf{R}^{n+p}$ along f, we denote by $\nabla^f_\eta X$ the covariant derivative of X by η induced from the Euclidean metric on $\mathbf{R}^{n+p}$. If $\eta(t) = \sum_{i=1}^n a_i(t)\frac{\partial}{\partial t_i}$ and $X(t) = \sum_{j=1}^{n+p} X_j(t)(\frac{\partial}{\partial x_j} \circ f)$, then

$$\nabla^f_\eta X = \sum_{j=1}^{n+p}\sum_{i=1}^{n}\left\{ a_i \frac{\partial X_j}{\partial t_i}\left(\frac{\partial}{\partial x_j} \circ f\right) + a_i X_j \left(\nabla^f_{\frac{\partial}{\partial t_i}} \frac{\partial}{\partial x_j}\right)\right\} = \sum_{j=1}^{n+p}\sum_{i=1}^{n} a_i \frac{\partial X_j}{\partial t_i}\left(\frac{\partial}{\partial x_j} \circ f\right),$$

since $\nabla^f_{\frac{\partial}{\partial t_i}} \frac{\partial}{\partial x_j} = 0$ in Euclidean case. Note that $\nabla^f_\eta X$ is a section of $f^*(T\mathbf{R}^{n+p})$, i.e. a vector field along f.

By the decomposition $f^*(T\mathbf{R}^{n+p}) = T_f \oplus N_f$, we write

$$\nabla^f_\eta X = \nabla^\top_\eta X + \nabla^\perp_\eta X.$$

Then $\nabla^\perp$ (resp. $\nabla^\top$) defines a connection on the vector bundle N_f (resp. T_f).

The induced connection $\nabla^\perp$ on N_f is called the normal connection or Van der Waerden-Bortolotti connection of the framed frontal f. The connection $\nabla^\top$ on T_f is called the tangential connection or Levi-Civita connection of f.

We call the frontal f normally flat (resp. tangentially flat) if the normal connection on N_f (resp. the tangential connection on T_f) is flat, i.e. there exists locally an orthonormal frame $\{\nu_1, \dots, \nu_p\}$ of N_f, (resp. $\{\tau_1, \dots, \tau_n\}$ of T_f) such that $\nabla^\perp_\eta \nu_i = 0, 1 \leq i \leq p$ (resp. $\nabla^\top_\eta \tau_j = 0, 1 \leq j \leq n$), for any vector field η on U (cf. [18]).

We call $\{\nu_1, \dots, \nu_p\}$ a normally parallel orthonormal frame of N_f or briefly a Bishop frame of the normally flat frontal (see [2][5][16]).

Lemma. ([12]) Any frontal hypersurface $f : (\mathbf{R}^n, 0) \to \mathbf{R}^{n+1}$ is normally flat. In fact, any unit normal is normally parallel. Any frontal curve $f : (\mathbf{R}, 0) \to \mathbf{R}^{1+p}$ is normally flat and tangentially flat.

【 Normally flat frontals and their parallels 】

Let $f : U^n \to \mathbf{R}^{n+p}$ be a frontal endow with the decomposition $f^*(T\mathbf{R}^{n+p}) = T_f \oplus N_f$. Then, for each $t \in U$, the normal space $f(t) + (N_f)_t$ of f through $f(t)$ is well-defined as an affine subspace in $\mathbf{R}^{n+p}$.

A frontal $g : U \to \mathbf{R}^{n+p}$ is said to be parallel to f if two spaces $f(t) + (N_f)_t$ and $g(t) + (N_g)_t$ coincide as affine subspaces of $\mathbf{R}^{n+p}$ for any $t \in U$.

If $p = 1$, i.e. f is a frontal hypersurface, then any parallel g to f is written, for a unit normal ν to f by the form $g(t) = f(t) + s\nu(t)$ for some $s \in \mathbf{R}$ ([3][4]).

In general, suppose f is normally flat and fix any normally parallel orthonormal frame (i.e. Bishop frame) $\nu_1, \nu_2, \dots, \nu_p$ of the normal bundle N_f. Then any parallel to f is given by the form

$$\mathrm{Pal}_\nu(f)(t) := f(t) + \nu(t), \quad \nu(t) := \textstyle\sum_{i=1}^p s_i\nu_i(t), \quad (t \in U, s_1, \dots, s_p \in \mathbf{R}),$$

provided U is connected. Then the parallels are normally flat and have the same Bishop frame with f.

【 Tangent maps of frontal curves 】

In this presentation, we treat frontal curves and their tangent maps.

Let $f : (\mathbf{R}, 0) \to \mathbf{R}^{1+p}$ be a frontal curve-germ. Let τ be a frame of T_f over $(\mathbf{R}, 0)$. Then the tangential map $\mathrm{Tan}(f) : (\mathbf{R}, 0) \times \mathbf{R} \to \mathbf{R}^{1+p}$ of f is defined by $\mathrm{Tan}(f)(t, s) := f(t) + s\tau$ for $t \in (\mathbf{R}, 0)$ and $s \in \mathbf{R}$. The right equivalence class of f is independent of the choice of a tangent frame.

Theorem 1. (Tangential maps of frontal curves are normally and tangentially flat)[12]. Let $f : (\mathbf{R}, 0) \to \mathbf{R}^{1+p}$ be a frontal curve. Suppose $\mathrm{Tan}(f) : (\mathbf{R}, 0) \times \mathbf{R} \to \mathbf{R}^{1+p}$ is a proper frontal. Then $\mathrm{Tan}(f)$ is a normally and tangentially flat frontal.

The following result provides an interrelation of "normal" and "tangent".

Theorem 2. (Parallels to a tangent map of a frontal curve are tangent maps of a frontal curve)[12]. Let $f : (\mathbf{R}, 0) \to \mathbf{R}^{1+p}$ be a frontal curve-germ without inflection point. Then $\mathrm{Tan}(f) : (\mathbf{R}, 0) \times \mathbf{R} \to \mathbf{R}^{1+p}$ is a proper frontal and any parallel $\mathrm{Pal}_\nu(\mathrm{Tan}(f))$ of $\mathrm{Tan}(f)$ is right equivalent to the tangent map $\mathrm{Tan}(g)$ for a frontal curve, called the directrix or the edge of regression, $g : (\mathbf{R}, 0) \to \mathbf{R}^{1+p}$.

【 Classifications of generic singularities of parallels to tangent maps of frontal curves 】

We will give the classification results of parallels to tangent maps of frontal curves.

Let I be an interval or a circle and $f : I \to \mathbf{R}^{1+p}$ a curve. Define the $(n+p) \times k$-Wronskian matrix of f by

$$W_k(f)(t) := \left(\frac{df}{dt}(t), \frac{d^2f}{dt^2}(t), \frac{d^3f}{dt^3}(t), \cdots, \frac{d^kf}{dt^k}(t)\right),$$

for any positive integer k. Note that the rank of $W_k(f)(t)$ is independent of local coordinates of I.

Let $a_1, a_2, \dots, a_{1+p}$ be an strictly increasing sequence of positive integers, $1 \leq a_1 < a_2 < \cdots < a_{1+p}$. A curve $f : I \to \mathbf{R}^{1+p}$ is said to be of type $(a_1, a_2, \dots, a_{1+p})$ at $t \in I$ if $a_i = \min\{k \mid \mathrm{rank}(W_k(f)(t)) = i\}, i = 1, 2, \dots, 1+p$ ([6, 7]).

We give the classification of types of singularities appearing in parallels to a generic frontal curve generated by its generic normally parallel normal field.

Theorem 3. (Classification of types of directrixes for generic parallels to frontal curves)[13]
Let $p \geq 2$. Then the types of directrixes for parallels to tangent maps of generic frontal curves in $\mathbf{R}^{1+p}$ along generic normally parallel normal field are given by the following list:

type	codimension
$(1, 2, \dots, p, 1+p)$	0
$(1, 2, \dots, p, p+2)$	1
$(2, 3, \dots, 1+p, 2+p)$	1
$(2, 3, \dots, 1+p, 3+p)$	2
$(3, 4, \dots, 2+p, 3+p)$	2

Each number in the second column means the codimension of the locus of the type in $I \times \mathbf{R}$.

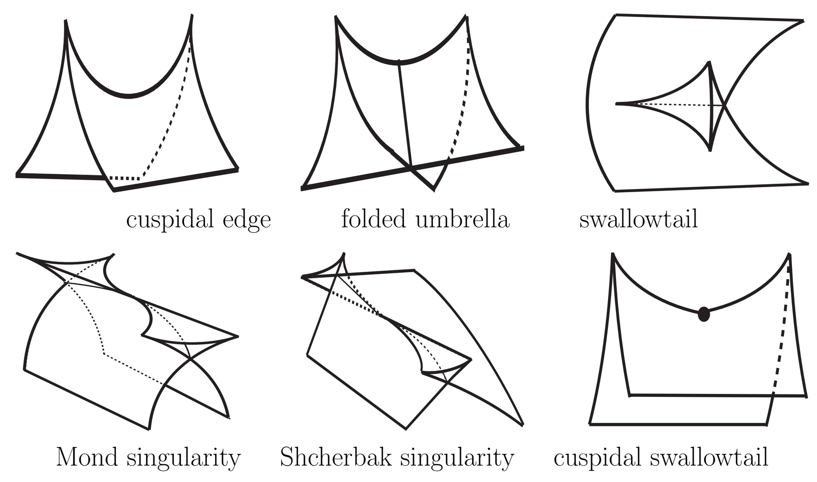
In several cases, singularities of tangent surfaces are described by types of curves ([8, 9]). By Theorem 2, singularities of parallels of $\mathrm{Tan}(f)$ are described as tangent maps $\mathrm{Tan}(g)$ of of directrixes g.

We give the exact generic classification of parallels to tangent maps of frontal curves in $\mathbf{R}^3$ and in $\mathbf{R}^4$.

Theorem 4. (Classification of singularities of generic parallels to tangent maps to frontal curves in $\mathbf{R}^3$)[13].
Let $p = 2$. The list of singularities appearing in parallels $(t, u) \to f(t) + u\tau(t) + s\nu(t)$ by a generic normally parallel normal ν to the tangent map $f(t) + u\tau(t)$ along $u = 0$ of a generic frontal curve $f : I \to \mathbf{R}^3$, for any $(t, u; s) \in I \times \mathbf{R} \times \mathbf{R}$ is given by

type	singularities	codimension
$(1, 2, 3)$	cuspidal edge ($\mathrm{CE}_{2,3}$)	1
$(1, 2, 4)$	folded umbrella ($\mathrm{FU}_{2,3}$)	2
$(2, 3, 4)$	swallowtail ($\mathrm{SW}_{2,3}$)	2
$(2, 3, 5)$	folded pleat ($\mathrm{FP}_{2,3}$)	3
$(3, 4, 5)$	cuspidal swallowtail ($\mathrm{CSW}_{2,3}$)	3

It is known that the diffeomorphism type of tangent maps to curves in $\mathbf{R}^3$ of types listed in Theorem 4 is uniquely determined ([7, 8, 9, 11]).



Theorem 5. (Classification of types of generic parallels to frontal curves in $\mathbf{R}^4$)[13]. Let $p = 3$. The list of singularities along $u = 0$ appearing in parallels $f + u\tau + s\nu$ for a generic normally parallel normal ν of the tangent map $f + v\tau$ to a generic frontal curve $f : I \to \mathbf{R}^4$, for any $(t, u; s) \in I \times \mathbf{R} \times \mathbf{R}$, is given by

type	singularity	codimension
$(1, 2, 3, 4)$	cuspidal edge ($\mathrm{CE}_{2,4}$)	1
$(1, 2, 3, 5)$	cuspidal edge ($\mathrm{CE}_{2,4}$)	2
$(2, 3, 4, 5)$	open swallowtail ($\mathrm{OSW}_{2,4}$)	2
$(2, 3, 4, 6)$	unfurled swallowtail ($\mathrm{USW}_{2,4}$)	3
$(3, 4, 5, 6)$	cuspidal swallowtail ($\mathrm{CSW}_{2,4}$)	3

The diffeomorphism type of tangent maps to curves in $\mathbf{R}^3$ of types listed in Theorem 5 is uniquely determined, except for the case of type $(2, 3, 4, 6)$, where the unfurled swallowtails have exactly two diffeomorphism classes ([13]).

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