Recognition Problem of Frontal Singularities

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1 Introduction

This is a survey article on recognition problem of frontal singularities.

First we explain the recognition problem of singularities and its significance.

Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ and $f' : (\mathbb{R}^n, a') \to (\mathbb{R}^m, b')$ be smooth (= C^∞) map-germs. Then f and f' are called *$\mathscr{A}$-equivalent* or *diffeomorphic* if there exist diffeomorphism-germs $\sigma : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a')$ and $\tau : (\mathbb{R}^m, b) \to (\mathbb{R}^m, b')$ such that the diagram

$$\begin{array}{ccc} (\mathbb{R}^n, a) & \xrightarrow{f} & (\mathbb{R}^m, b) \\ \downarrow \sigma & & \downarrow \tau \\ (\mathbb{R}^n, a') & \xrightarrow{f'} & (\mathbb{R}^m, b') \end{array}$$

commutes. By a *singularity* of smooth mappings, we mean an *$\mathscr{A}$-equivalence class* of map-germs.

Suppose that we investigate "singularities" of mappings belonging to some given class. Then the recognition problem of singularities may be understood as the following dual manners:

Problem: Given two map-germs f and f', *belonging to the given class,* determine, *as easily as possible* whether f and f' are equivalent or not.

Problem: Given a singularity, find criteria to determine as easy as possible whether a map-germ f *belonging to some class* has (= falls into) the given singularity or not.

Importance of the recognition problem of singularities can be explained as follows.

Once we establish a classification list of singularities in a *situation A*, we will face (at least) *two kinds of needs*:

1. Given a map-germ in the same *situation A*, we want to know which singularity is it in the list.

2. For another *situation B*, we want to know how similar is the classification list of singularities as A or not.

In both cases, we need to recognize the singularities, *as easily as possible*, by *as many as possible* criteria. For applications of singularity theory, it is indispensable to recognize singularities and to solve classification problems in various situations.

The recognition problem of singularities of smooth map-germs has been treated by the many mathematicians, motivated by differential geometry and other wide area, and its solutions are supposed to have many applications.

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In fact most of known results of recognition of singularities are found under the motivation of geometric studies of singularities appearing in Euclid geometry and various Klein geometries ([21, 3, 19]).

Example 1.1 (Singularities in non-Euclidean geometry) The following is a diagram representing the history of non-Euclidean geometry found in the reference [26]:

Euclid geometry	$\rightarrow$	Riemann geometry
$\downarrow$		$\downarrow$
Klein geometry	$\rightarrow$	Cartan geometry

Then it would be natural to ask
Problem: How are the classification results of singularities in Euclid geometry (resp. in Klein geometry) valid in Riemann geometry (resp. in Cartan geometry)?
In other words,
Problem: Do the classifications of singularities in flat ambient spaces work also for "curved" ambient spaces?

In fact, we applied the several results of recognition ([21, 3]), for instance, to the generic classification of singularities of improper affine spheres and of surfaces of constant Gaussian curvature ([13]), and moreover, to the classification of generic singularities appearing in tangent surfaces which are ruled by geodesics in general Riemannian spaces ([17, 18]). See also §6.

In this paper we will pay our attention to the class of mappings, *frontal mappings*, which is introduced and studied in §2. Then we survey several recognition theorems on them in §3. Note that the recognitions of fronts or frontals $(\mathbb{R}^n, a) \to \mathbb{R}^m$ are studied by many authors ([21, 3, 24, 25, 20]).

To show the theorems given in §3, we introduce the notion of *openings*, relating it with that of frontals, in §4. See also [9, 10]. In fact, in §4, we observe that *any frontal singularity is an opening of a map-germ from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ (Lemma 4.3).

Then we naturally propose:
Problem: Study the *recognition problem* of frontals from the *recognition results* on map-germs $(\mathbb{R}^n, a) \to \mathbb{R}^n$, $(n = m)$, combined with the viewpoint of *openings*.

In this paper, in connection with the above problems, we specify *geometrically* several frontal singularities which we are going to treat (Example 2.2). Then we solve the recognition problem of such singularities, in §3, giving explicit normal forms. In fact we combine the recognition results on $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ by K. Saji ($\sim$2010) and several arguments on openings, which was implicitly performed for the classification of singularities of tangent surfaces (tangent developables) by the author ($\sim$1995) over twenty years, the idea of which traces back to the author's master thesis [5]. We prove recognition theorems in §5.

In the last section §6, as an application of our solutions of recognition problem of frontal singularities, we announce the classification of singularities appearing in tangent surfaces of generic null curves which are ruled by null geodesics in general Lorentz 3-manifolds ([14, 16]), mentioning related recognition results and open problems.

In this paper, all manifolds and mappings are assumed to be of class C^∞ unless otherwise stated.

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2 Frontal singularities

Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a map-germ. Suppose $n \leq m$.

Then f is called a *frontal map-germ* or a *frontal* in short, if there exists a smooth (C^∞) family of n-planes $\widetilde{f}(t) \subseteq T_{f(t)}\mathbb{R}^m$ along f, $t \in (\mathbb{R}^n, a)$, i.e. there exists a smooth lift $\widetilde{f} : (\mathbb{R}^n, a) \to \mathrm{Gr}(n, T\mathbb{R}^m)$ satisfying the "*integrality condition*"

$$T_t f(T_t\mathbb{R}^n) \subset \widetilde{f}(t) \ (\subset T_{f(t)}\mathbb{R}^m),$$

for any $t \in \mathbb{R}^n$ nearby a, such that $\pi \circ \widetilde{f} = f$:

$$\begin{array}{ccc} & & \mathrm{Gr}(n, T\mathbb{R}^m) \\ & \overset{\widetilde{f}}{\nearrow} & \downarrow \pi \\ (\mathbb{R}^n, a) & \xrightarrow[f]{} & (\mathbb{R}^m, b). \end{array}$$

Here $\mathrm{Gr}(n, T\mathbb{R}^m)$ is the *Grassmann bundle* consisting of n-planes $V \subset T_x\mathbb{R}^m (x \in \mathbb{R}^m)$ with the canonical projection $\pi(x, V) = x$, and $T_t f : T_t\mathbb{R}^n \to T_{f(t)}\mathbb{R}^m$ is the differential of f at $t \in (\mathbb{R}^n, a)$.

Then $\widetilde{f}$ is called a *Legendre lift* or an *integral lift* of the frontal f. Actually $\widetilde{f}$ is an integral mapping to the canonical or contact distribution on $\mathrm{Gr}(n, T\mathbb{R}^m)$ (cf. [8]).

Example 2.1 (1) Any *immersion* is a frontal. In fact then the Legendre lift is given by $\widetilde{f}(t) := T_t f(T_t\mathbb{R}^n)$.

(2) Any map-germ $(\mathbb{R}^n, a) \to (\mathbb{R}^n, b), (n = m)$ is a frontal. In fact the Legendre lift is given by $\widetilde{f}(t) := T_{f(t)}\mathbb{R}^n$.

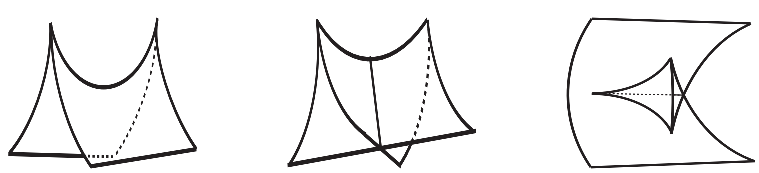
(3) Any *constant map-germ* is a frontal. In fact we can take any lift $\widetilde{f}$ of f.

(4) Any *wave-front* $(\mathbb{R}^n, a) \to (\mathbb{R}^{n+1}, b)$, that is a Legendre projection of a Legendre submanifold in $\mathrm{Gr}(n, T\mathbb{R}^{n+1}) = PT^*\mathbb{R}^{n+1}$, is a frontal. Take the inclusion of the Legendre submanifold as the Legendre lift.

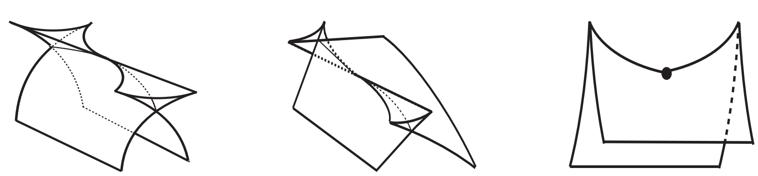
Example 2.2 (Singularities of tangent surfaces) Let $\gamma : (\mathbb{R}, 0) \to \mathbb{R}^m$ be a curve-germ in Euclidean space. Then the tangent surface $\mathrm{Tan}(\gamma) : (\mathbb{R}^2, 0) \to \mathbb{R}^m$ is defined as the ruled surface generated by tangent lines along the curve. Suppose γ is of type $\mathbf{L} = (\ell_1, \ell_2, \ell_3, \dots,), (1 \leq \ell_1 < \ell_2 < \ell_3 < \cdots)$, i.e.

$$\gamma(t) = (t^{\ell_1} + \cdots,\ t^{\ell_2} + \cdots,\ t^{\ell_3} + \cdots,\ \dots)$$

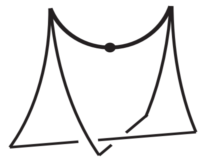
for a system of affine coordinates of $\mathbb{R}^m$ centered at $\gamma(0)$. Then it is known that the singularity of $\mathrm{Tan}(\gamma)$ is *uniquely determined* by the type $\mathbf{L}$ and called cuspidal edge (*CE*) if $\mathbf{L} = (1, 2, 3, \dots)$, folded umbrella (*FU*) or cuspidal cross cap (*CCC*) if $(1, 2, 4)$, swallowtail (*SW*) if $(2, 3, 4)$, Mond (*MD*) or cuspidal beaks (*CB*) if $(1, 3, 4)$, Shcherbak (*SB*) if $(1, 3, 5)$, cuspidal swallowtail (*CS*) if $(3, 4, 5)$, open folded umbrella (*OFU*) if $(1, 2, 4, 5, \dots)$, open swallowtail (*OSW*) if $(2, 3, 4, 5, \dots)$, open Mond (*OMD*) or open cuspidal beaks (*OCB*) if $(1, 3, 4, 5, \dots)$ (see [8]).



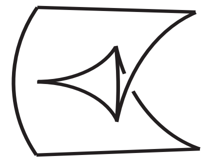
cuspidal edge folded umbrella swallowtail



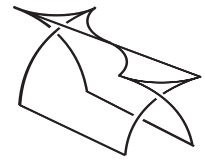
Mond singularity Shcherbak singularity cuspidal swallowtail



open folded umbrella



open swallowtail



open Mond singularity

In general, a frontal $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ is called a *front* if f has an immersive Legendre lift $\widetilde{f}$.

Let $\mathscr{E}_a := \{h : (\mathbb{R}^n, a) \to \mathbb{R}\}$ denote the $\mathbb{R}$-algebra of smooth function-germs on $(\mathbb{R}^n, a)$.

Denote by Γ the set of subsets $I \subseteq \{1, 2, \dots, m\}$ with $\#(I) = n$. For a map-germ $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b), n \leq m$ and $I \in \Gamma$, we set $D_I = \det(\partial f_i / \partial t_j)_{i \in I, 1 \leq j \leq n}$. Then *Jacobi ideal* J_f of f is defined as the ideal generated in $\mathscr{E}_a$ by all n-minor determinants D_I $(I \in \Gamma)$ of Jacobi matrix $J(f)$ of f. Then we have:

Lemma 2.3 (Criterion of frontality) *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a map-germ. If f is a frontal, then the Jacobi ideal J_f of f is principal, i.e. it is generated by one element. In fact J_f is generated by D_I for some $I \in \Gamma$. Conversely, if J_f is principal and the singular locus*

$$S(f) = \{t \in (\mathbb{R}^n, a) \mid \operatorname{rank}(T_t f : T_t\mathbb{R}^n \to T_{f(t)}\mathbb{R}^m) < n\}$$

of f is nowhere dense in $(\mathbb{R}^n, a)$, then f is a frontal.

Proof: Let f be a frontal and $\widetilde{f}$ be a Legendre lift of f. Take $I_0 \in \Gamma$ such that $\widetilde{f}(a)$ projects isomorphically by the projection $\mathbb{R}^m \to \mathbb{R}^n$ to the components belonging to I_0. Let $(p_I)_{I \in \Gamma}$ be the Plücker coordinates of $\widetilde{f}$. Then $p_{I_0}(a) \neq 0$. This implies that for any $I \in \Gamma$, there exists $h_I \in \mathscr{E}_a$ such that $D_I = h_I D_{I_0}$. Set $\lambda = D_{I_0}$. Then the Jacobi ideal J_f is generated by λ.

Conversely suppose J_f is generated by one element $\lambda \in \mathscr{E}_a$. Since J_f is generated by λ, we have that there exists $k_I \in \mathscr{E}_a$ for any $I \in \Gamma$ such that $D_I = k_I \lambda$. Since $\lambda \in J_f$, there exists $\ell_I \in \mathscr{E}_a$ for any $I \in \Gamma$ such that $\lambda = \sum_{I \in \Gamma} \ell_I D_I$. Therefore $(1 - \sum_{I \in \Gamma} \ell_I k_I)\lambda = 0$. Suppose $(\ell_I k_I)(a) = 0$ for any $I \in \Gamma$. Then $1 - \sum_{I \in \Gamma} \ell_I k_I$ is a unit and therefore $\lambda = 0$. Thus we have $J_f = 0$. This contradicts to the assumption that $S(f)$ is nowhere dense. Hence there exists $I_0 \in \Gamma$ such that $(\ell_{I_0} k_{I_0})(a) \neq 0$. Then $k_{I_0}(a) \neq 0$. Therefore J_f is generated by D_{I_0}. Hence $D_I = h_I D_{I_0}$ for any $I \in \Gamma$ with $h_{I_0}(a) = 1$. Then the Legendre lift $\widetilde{f}$ on $\mathbb{R}^n \setminus S(f)$ extends to $(\mathbb{R}^n, a)$, which is given by the Plücker coordinates $(h_I)_{I \in \Gamma}$. $\square$

Example 2.4 Define $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ by $f(t_1, t_2) := (\varphi(t_1),\ \varphi(t_1)t_2,\ \varphi(-t_1))$, where the C^∞ function $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is given by $\varphi(t) = \exp(-1/t^2)(t \geq 0), 0(t \leq 0)$. Then the Jacobi ideal J_f is generated by $\varphi'(t_1)\varphi(t_1)$ and therefore J_f is principal and $J_f \neq 0$. However f is not a frontal. In fact, for $t_1 > 0$, $(T_{(t_1,t_2)} f)(T_{(t_1,t_2)}\mathbb{R}^2)$ is given by the plane $dx_3 = 0$ and for $t_1 < 0$, $(T_{(t_1,t_2)} f)(T_{(t_1,t_2)}\mathbb{R}^2)$ contains the x_3-axis. Therefore f can not be a frontal.

Corollary 2.5 *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a map-germ. Suppose f is analytic and $J_f \neq 0$. Then f is a frontal if and only if J_f is a principal ideal.*

Proof: By Lemma 2.3, if f is frontal, then J_f is principal. If J_f is principal and $J_f \neq 0$, then $D_I \neq 0$ for some $I \in \Gamma$. Since f is analytic, $S(f)$ is nowhere dense. Thus by Lemma 2.3, f is a frontal. $\square$

Example 2.6 Define $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^4, 0)$ by $f(t_1, t_2, t_3) := (t_1^3,\ t_1^2 t_2,\ t_1 t_2^2,\ t_2^3)$. The germ f parametrizes the cone over a non-degenerate cubic in $P(\mathbb{R}^4) = \mathbb{R}P^3$. Then f is analytic and $J_f = 0$ is principal. However f is not a frontal.

Definition 2.7 Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a frontal. Then a generator $\lambda \in \mathscr{E}_a$ of J_f is called a *Jacobian* (or a *singularity identifier*) of f, which is uniquely determined from f up to multiplication of a unit in $\mathscr{E}_a$.

The singular locus $S(f)$ of a frontal f is given by the zero-locus of the Jacobian λ of f.

Definition 2.8 (Proper frontals) A frontal $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ is called *proper* if the singular locus $S(f)$ is nowhere dense in $(\mathbb{R}^n, a)$.

Remark 2.9 Our naming "proper" is a little confusing since its usage is different from the ordinary meaning of properness (inverse images of any compact is compact). Our condition that the singular locus S_f is nowhere dense is easy to handle for the local study of mappings.

Lemma 2.10 *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a proper frontal or $n = m$. Then f has a unique Legendre lift $\widetilde{f} : (\mathbb{R}^n, a) \to \mathrm{Gr}(n, T\mathbb{R}^m)$.*

Proof: On the regular locus $\mathbb{R}^n \setminus S(f)$, there is the unique Legendre lift $\widetilde{f}$ defined by $\widetilde{f}(t) := (T_t f)(T_t \mathbb{R}^n)$. Let f be a proper frontal. Then $\mathbb{R}^n \setminus S(f)$ is dense in $(\mathbb{R}^n, a)$. Therefore the extension of $\widetilde{f}(t)$ is unique. Let $n = m$. Then the unique lift $\widetilde{f}$ is defined by $\widetilde{f}(t) = T_{f(t)}\mathbb{R}^m$ (Example 2.1 (2)). □

Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a frontal (resp. a proper frontal) and $\widetilde{f} : (\mathbb{R}^n, a) \to \mathrm{Gr}(n, T\mathbb{R}^m)$ a Legendre lift of f. Recall that $\widetilde{f}(t), (t \in (\mathbb{R}^n, a))$ is an n-plane field along f. In particular $\widetilde{f}(a) \subseteq T_b \mathbb{R}^m$.

Definition 2.11 A system $(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ of local coordinates of $\mathbb{R}^m$ centered at b is called *adapted* to $\widetilde{f}$ (or, to f) if

$$\widetilde{f}(a) = \left\langle \left(\frac{\partial}{\partial x_1}\right)_b, \dots, \left(\frac{\partial}{\partial x_n}\right)_b \right\rangle_{\mathbb{R}}$$

$$(= \{v \in T_b\mathbb{R}^m \mid dx_{n+1}(v) = 0, \dots, dx_m(v) = 0\}).$$

Clearly we have

Lemma 2.12 *Any frontal $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ has an adapted system of local coordinates on $(\mathbb{R}^m, b)$. In fact any system of local coordinates on $(\mathbb{R}^m, b)$ is modified into an adapted system of local coordinates by a linear change of coordinates.*

Remark 2.13 For an adapted system of coordinates $(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ of f, the Jacobian λ is given by the ordinary Jacobian $\frac{\partial(f_1, \dots, f_n)}{\partial(t_1, \dots, t_n)}$, where $f_i = x_i \circ f$.

Example 2.14 Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be given by

$$(u,t) \mapsto (x_1, x_2, x_3) = (t+u,\ t^3 + 3t^2u,\ t^4 + 4t^3u),$$

which is the tangent surface, Mond surface, of the curve $t \mapsto (t, t^3, t^4)$.

Then the Jacobi matrix $J(f)$ of f is given by

$$J(f) = \begin{pmatrix} 1 & 1 \\ 3t^2 & 3t^2 + 6tu \\ 4t^3 & 4t^3 + 12t^2u \end{pmatrix},$$

and its minors are calculated as

$$\begin{cases} D_{12} & = & 6tu, \\ D_{13} & = & 12t^2u = 2t(6tu), \\ D_{23} & = & 12t^4u = 2t^3(6tu), \end{cases}$$

Then the Jacobi ideal J_f is generated by $\lambda = tu$. Therefore f is a proper frontal with $S(f) = \{(u,t) \mid tu = 0\}$. The unique Legendre lift $\widetilde{f} : (\mathbb{R}^2, 0) \to \mathrm{Gr}(2, T\mathbb{R}^3)$ of f is given, via the Plücker coordinates of fibre components,

$$D_{12}/D_{12} = 1,\ D_{13}/D_{12} = 2t,\ D_{23}/D_{12} = 2t^3.$$

The system of coordinates (x_1, x_2, x_3) is adapted for f in the example.

3 Recognition of several frontal singularities

To give our recognition results we need the notion of "kernel fields" in addition to that of Jacobians of frontals.

Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a map-germ. We denote by $\mathscr{V}_a$ the $\mathscr{E}_a$-module of vector fields over $(\mathbb{R}^n, a)$ and set

$$\mathscr{N}_f := \{\eta \in \mathscr{V}_a \mid \eta f_i \in J_f,\ (1 \leq i \leq m)\},$$

which is an $\mathscr{E}_a$-submodule of $\mathscr{V}_a$.

Note that, if $\eta \in \mathscr{N}_f$, then $\eta(t) \in \mathrm{Ker}(T_t f : T_t\mathbb{R}^n \to T_{f(t)}\mathbb{R}^m)$ for any $t \in S(f)$. Moreover note that, if $\lambda \in J_f$, then $\lambda \cdot \mathscr{V}_a \subseteq \mathscr{N}_f$.

A map-germ $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ is called *of corank k* if $\dim_{\mathbb{R}} \mathrm{Ker}(T_a f : T_a\mathbb{R}^n \to T_b\mathbb{R}^m) = k$.

Then we have

Lemma 3.1 *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a map-germ of corank 1. Then $\mathscr{N}_f / J_f \cdot \mathscr{V}_a$ is a free $\mathscr{E}_a$-module of rank 1, i.e. $\mathscr{N}_f / J_f \cdot \mathscr{V}_a$ is isomorphic to $\mathscr{E}_a$ as $\mathscr{E}_a$-modules by $[\eta] \to 1$, for some $\eta \in \mathscr{N}_f$.*

Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a frontal of corank 1 and λ_f the Jacobian of f (Definition 2.7). Then by Lemma 3.1, $\mathscr{N}_f / \lambda_f \cdot \mathscr{V}_a$ is a free module of rank 1.

Definition 3.2 A vector field η over $(\mathbb{R}^n, a)$ is called a *kernel field* (or a *null field*) of f if η generates the free $\mathscr{E}_a$-module $\mathscr{N}_f / \lambda_f \cdot \mathscr{V}_a$.

Remark 3.3 The notion of null fields is introduced first in [21].

Proof of Lemma 3.1: Since f is of corank 1, f is $\mathscr{A}$-equivalent to a map-germ $(\mathbb{R}^n,0) \to (\mathbb{R}^m,0)$ of form

$$g = (t_1,\dots,t_{n-1},\varphi_n(t),\dots,\varphi_m(t)).$$

Note that $\mathscr{N}_f/J_f\mathscr{V}_a$ is isomorphic to $\mathscr{N}_g/J_f\mathscr{V}_0$. Moreover the Jacob ideal of g is generated by

$$\partial\varphi_n(t)/\partial t_n,\dots,\partial\varphi_m(t)/\partial t_n.$$

Let $\eta = \sum_{i=1}^n \eta_i\partial/\partial t_i \in \mathscr{V}_0$. Then $\eta \in \mathscr{N}_g$ if and only if $\eta_1,\dots,\eta_{n-1} \in J_g$. Therefore $\mathscr{N}_g/J_g\mathscr{V}_0$ is freely generated by $\partial/\partial t_n$. Thus we have that $\mathscr{N}_f/J_f\cdot\mathscr{V}_a$ is a free $\mathscr{E}_a$-module of rank 1, □

Now we start to give our recognition theorems on the frontal singularities introduced in Example 2.2. To begin with, we recall the following fundamental recognition result due to Saji ([24]), which is a reformulation of Whitney's original results in [27] for parts (1) and (2).

Theorem 3.4 (Saji[24]) *Let $f : (\mathbb{R}^2,a) \to (\mathbb{R}^2,b)$ be a frontal map-germ of corank 1. Then, for the Jacobian λ and the kernel field η of f, we have*

(1) f is $\mathscr{A}$-equivalent to the fold, i.e. to $(t_1,t_2) \mapsto (t_1,t_2^2)$, if and only if $(\eta\lambda)(a) \neq 0$.

(2) f is $\mathscr{A}$-equivalent to Whitney's cusp, i.e. to $(t_1,t_2) \mapsto (t_1,t_2^3+t_1t_2)$, if and only if $(d\lambda)(a) \neq 0, (\eta\lambda)(a) = 0, (\eta\eta\lambda)(a) \neq 0$.

(3) f is $\mathscr{A}$-equivalent to bec à bec (beak-to-beak), $(t_1,t_2) \mapsto (t_1,t_2^3+t_1t_2^2)$, if and only if λ has an indefinite Morse critical point at a and $(\eta\eta\lambda)(a) \neq 0$.

Remark 3.5 Each condition (1), (2), (3) of Theorem 3.4 is independent of the choice of λ and η, and depends only on $\mathscr{J}$-equivalence class of f which is introduced in Definition 4.13. In fact, if $\mathscr{J}_{f'\circ\sigma} = \mathscr{J}_f$, then f' satisfies the condition for $\lambda' = \lambda\circ\sigma^{-1}$ and $\eta' = (T\sigma)\eta\circ\sigma^{-1}$. (See §4).

Remark 3.6 For a map-germ $f : (\mathbb{R}^2,a) \to (\mathbb{R}^2,b)$ of corank 1, the condition $(d\lambda)(a) \neq 0$ is equivalent to that the Jacobian is $\mathscr{K}$-equivalent to the germ $(t_1,t_2) \mapsto t_1$ at the origin. The condition that λ has an indefinite Morse critical point at a is equivalent to that λ is $\mathscr{K}$-equivalent to the germ $(t_1,t_2) \mapsto t_1t_2$ at the origin.

Remark 3.7 For plane to plane map-germs, the fold (resp. Whitney cusp, bec à bec) is characterized as a"tangent map" of a planar curve of type $(1,2)$ (resp. $(2,3)$, $(1,3)$), which is ruled by tangent lines to the curve ([8, 15]).

Let $f : (\mathbb{R}^2,a) \to (\mathbb{R}^m,b), (m \geq 3)$ be a proper frontal of corank 1. We wish to recognize the singularity, i.e. $\mathscr{A}$-equivalence class of f by the Jacobian $\lambda = \lambda_f$ and the kernel field $\eta = \eta_f$. Moreover we wish to recognize the singularity of f as an opening of a plane-to-plane map-germ. To realize this, we will use an adapted system of coordinates $(x_1,x_2,x_3,\dots,x_m)$ for f and set $f_i = x_i\circ f$. Note that we mention several conditions to recognize singularities in terms of adapted coordinates, however the conditions are, of course, independent of the choice of an adapted coordinates, and therefore any system of adapted coordinates can be taken to simplify the checking of a suitable condition.

In general, we use the following notation:

Definition 3.8 For a germ of vector field $\eta \in \mathscr{V}_a$ over $(\mathbb{R}^n,a)$ and a function-germ $h \in \mathscr{E}_a$ on $(\mathbb{R}^n,a)$, the *vanishing order* $\mathrm{ord}_a^\eta(h)$ of the function h at the point a for the vector-field η is defined by

$$\mathrm{ord}_a^\eta(h) := \inf\{i \in \mathbb{N}\cup\{0\} \mid (\eta^i h)(a) \neq 0\}.$$

Then we characterize the cuspidal edge as an opening of fold map-germ:

Theorem 3.9 (Recognition of cuspidal edge) *For a frontal* $f : (\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ *of corank* 1*, the following conditions are equivalent to each other:*

(1) f *is* $\mathscr{A}$*-equivalent to the cuspidal edge (CE).*

(1') f *is* $\mathscr{A}$*-equivalent to the germ* $(t_1, t_2) \mapsto (t_1, t_2^2, t_2^3)$.

(2) f *is a front and* $\eta\lambda(a) \neq 0$.

(3) $\eta\lambda(a) \neq 0$ *and* $\mathrm{ord}_a^\eta(f_3) = 3$*, for an adapted system of coordinates* (x_1, x_2, x_3) *of* $(\mathbb{R}^3, b)$.

Theorem 3.9 is generalized by

Theorem 3.10 (Recognition of embedded cuspidal edge) *For a frontal* $f : (\mathbb{R}^2, a) \to (\mathbb{R}^m, b), 3 \leq m$ *of corank* 1*, the following conditions are equivalent to each other:*

(1) f *is* $\mathscr{A}$*-equivalent to the cuspidal edge, i.e. the tangent surface to a curve of type* $(1, 2, 3, \dots)$.

(1') f *is* $\mathscr{A}$*-equivalent to the germ* $(t_1, t_2) \mapsto (t_1, t_2^2, t_2^3, 0, \dots, 0)$.

(2) f *is a front and* $\eta\lambda(a) \neq 0$.

(3) $\eta\lambda(a) \neq 0$ *and* $\mathrm{ord}_a^\eta(f_i) = 3$ *for some* $i, 3 \leq i \leq m$*, for an adapted system of coordinates* $(x_1, x_2, x_3, \dots, x_m)$ *of* $(\mathbb{R}^m, b)$.

The following is a recognition of the folded umbrella due to the theory of openings:

Theorem 3.11 (Recognition of folded umbrella (cuspidal cross cap)) *Let* $f : (\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ *be a frontal of corank* 1*. The following conditions are equivalent to each other:*

(1) f *is* $\mathscr{A}$*-equivalent to the folded umbrella (FU), i.e. the tangent surface to a curve of type* $(1, 2, 4)$.

(1') f *is* $\mathscr{A}$*-equivalent to the germ* $(t_1, t_2) \mapsto (t_1,\ t_2^2,\ t_1 t_2^3)$.

(2) $\eta\lambda(a) \neq 0, (\eta^3 f_3)(a) = 0$ *and* $(d\lambda \wedge d(\eta^3 f_3))(a) \neq 0$.

Remark 3.12 It is already known another kind of recognition of folded umbrella by [3].

As for cases of higher codimension, we have

Theorem 3.13 (Recognition of open folded umbrella (open cuspidal cross cap)) *Let* $f : (\mathbb{R}^2, a) \to (\mathbb{R}^m, b), (m \geq 4)$ *be a frontal of corank* 1*. Then the following conditions are equivalent to each other:*

(1) f *is* $\mathscr{A}$*-equivalent to the open folded umbrella, i.e. the tangent surface to a curve of type* $(1, 3, 4, 5, \dots)$.

(1') f *is* $\mathscr{A}$*-equivalent to the germ* $(t_1, t_2) \to (t_1,\ t_2^2,\ t_1 t_2^3,\ t_2^5,\ 0, \dots, 0)$.

(2) $(\eta\lambda)(a) \neq 0$*,* $(\eta^3 f_k)(a) = 0, (3 \leq k \leq m)$*, and there exist* $3 \leq i < j \leq m$ *and* $A \in \mathrm{GL}(2, \mathbb{R})$ *such that, setting* $(f_i, f_j)A = (f_3', f_4')$*,* $(d\lambda \wedge \eta^3 f_3')(a) \neq 0, (d\lambda \wedge \eta^3 f_4')(a) = 0, (\eta^5 f_4')(a) \neq 0$.

As for openings of Whitney's cusp mapping, we have

Theorem 3.14 (Recognition of swallowtail) *Let* $f : (\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ *be a frontal of corank* 1*. Then the following conditions are equivalent to each other:*

(1) f *is* $\mathscr{A}$*-equivalent to the swallowtail (SW), i.e. the tangent surface to a curve of type* $(2, 3, 4)$.

(1') f *is* $\mathscr{A}$*-equivalent to the germ* $(t_1, t_2) \mapsto (t_1,\ t_2^3 + t_1 t_2,\ \frac{3}{4} t_2^4 + \frac{1}{2} t_1 t_2^2)$.

(2) f *is a front,* $(d\lambda)(a) \neq 0$ *and* $\mathrm{ord}_a^\eta(\lambda) = 2$.

(3) λ *is* $\mathscr{K}$*-equivalent to the germ* $(t_1, t_2) \mapsto t_1$ *at* 0*,* $\mathrm{ord}_a^\eta(\lambda) = 2$ *and* $\mathrm{ord}_a^\eta(f_3) = 4$*, for an adapted system of coordinates* (x_1, x_2, x_3).

As for cases of higher codimension, we have

Theorem 3.15 (Recognition of open swallowtail) *Let* $f : (\mathbb{R}^2, a) \to (\mathbb{R}^m, b)$ *be a frontal of corank* 1 *with* $m \geq 4$. *Then the following conditions are equivalent to each other:*

(1) *f is $\mathscr{A}$-equivalent to the open swallowtail, i.e. the tangent surface to a curve of type* $(2,3,4,5,\dots)$.

(1') *f is $\mathscr{A}$-equivalent to the germ* $(t_1,t_2) \mapsto (t_1,\ t_2^3 + t_1t_2,\ \frac{3}{4}t_2^4 + \frac{1}{2}t_1t_2^2,\ \frac{3}{5}t_2^5 + \frac{1}{3}t_1t_2^3,\ 0,\dots)$.

(2) *The Jacobian λ is $\mathscr{K}$-equivalent to the germ* $(t_1,t_2) \mapsto t_1$ *at the origin,* $\mathrm{ord}_a^\eta(\lambda) = 2$, $(\eta^3 f_i)(a) = 0, (3 \leq k \leq m)$, *and there exist* $3 \leq i < j \leq m$ *and* $A \in \mathrm{GL}(2,\mathbb{R})$ *such that, setting* $(f_i, f_j)A = (f_3', f_4')$, $\mathrm{ord}_a^\eta(f_3') = 4$, $\mathrm{ord}_a^\eta(f_4') = 5$.

Remark 3.16 Though we treat the open swallowtail as the singularity appeared in tangent surfaces, first it appeared as a singularity of Lagrangian varieties and geometric solutions of differential systems ([1, 4]). The open swallowtail and open folded umbrella appear also in the context of frontal-symplectic versality (Example 12.3 of [12]).

As for openings of bec à bec mapping, we have

Theorem 3.17 (Recognition of Mond singularity (cuspidal beaks), (1)(2) [19]) *Let* $f : (\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ *be a frontal of corank* 1. *Then the following conditions are equivalent to each other:*

(1) *f is $\mathscr{A}$-equivalent to Mond singularity (cuspidal beaks), i.e. the tangent surface to a curve of type* $(1,3,4)$.

(1') *f is $\mathscr{A}$-equivalent to the germ* $(t_1,t_2) \mapsto (t_1,\ t_2^3 + t_1t_2^2,\ \frac{3}{4}t_2^4 + \frac{2}{3}t_1t_2^3)$.

(2) *f is a front, λ is $\mathscr{K}$-equivalent t_1t_2 at the origin, and* $\mathrm{ord}_a^\eta(\lambda) = 2$.

(3) *λ is $\mathscr{K}$-equivalent t_1t_2 at the origin,* $\mathrm{ord}_a^\eta(\lambda) = 2$ *and* $\mathrm{ord}_a^\eta(f_3) = 4$.

Moreover we have:

Theorem 3.18 (Recognition of open Mond singularities (open cuspidal beaks)) *Let* $f : (\mathbb{R}^2, a) \to (\mathbb{R}^m, b)$ *be a frontal of corank* 1 *with* $m \geq 4$. *Then the following conditions are equivalent to each other:*

(1) *f is $\mathscr{A}$-equivalent to the open Mond singularity, i.e. the tangent surface to a curve of type* $(1,3,4,5,\dots)$.

(1') *f is $\mathscr{A}$-equivalent to the germ* $(t_1,t_2) \mapsto (t_1,\ t_2^3 + t_1t_2^2,\ \frac{3}{4}t_2^4 + \frac{2}{3}t_1t_2^3,\ \frac{3}{5}t_2^5 + \frac{1}{2}t_1t_2^4,\ \dots)$.

(2) *λ is $\mathscr{K}$-equivalent to* $(t_1,t_2) \mapsto t_1t_2$ *at the origin,* $\mathrm{ord}_a^\eta(\lambda) = 2$, $(\eta^3 f_i)(a) = 0, (3 \leq k \leq m)$, *and there exist* $3 \leq i \neq j \leq m$ *and* $A \in \mathrm{GL}(2,\mathbb{R})$ *such that, setting* $(f_i, f_j)A = (f_3', f_4')$, $\mathrm{ord}_a^\eta(f_3') = 4$, $\mathrm{ord}_a^\eta(f_4') = 5$.

To conclude this section, we give the result on recognition of Shcherbak singularity:

Theorem 3.19 (Recognition of Shcherbak singularity) *Let* $f : (\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ *be a frontal of corank* 1. *Then the following conditions are equivalent to each other:*

(1) *f is $\mathscr{A}$-equivalent to Shcherbak singularity, i.e. the tangent surface to a curve of type* $(1,3,5)$.

(1') *f is $\mathscr{A}$-equivalent to the germ* $(t_1,t_2) \mapsto (t_1,\ t_2^3 + t_1t_2^2,\ \frac{3}{5}t_2^5 + \frac{1}{2}t_1t_2^4)$ *at the origin.*

(2) *λ is $\mathscr{K}$-equivalent to the germ* $(t_1,t_2) \mapsto t_1t_2$ *at the origin,* $\mathrm{ord}_a^\eta(\lambda) = 2$, $\mathrm{ord}_c^\eta(f_3) \geq 4$ *for any point c on a component of the singular locus $S(f)$, and* $\mathrm{ord}_a^\eta(f_3) = 5$.

Note that Shcherbak singularity necessarily has the $(2,5)$ cuspidal-edge along one component of the singular locus, while it has the ordinary $(2,3)$ cuspidal edge along another component.

4 Frontals and openings

To understand the frontal singularities and to prove the results in the previous section, we introduce the notion of openings and make clear its relation to frontal singularities (see also [11]).

Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a frontal (resp. a proper frontal) and $\widetilde{f} : (\mathbb{R}^n, a) \to \mathrm{Gr}(n, T\mathbb{R}^m)$ any Legendre lift of f. Let $(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ be an adapted system of coordinates to $\widetilde{f}$ (resp. to f) (Definition 2.11). Then, setting $f_i = x_i \circ f, 1 \leq i \leq m$, we have

$$df_i = h_{i1} df_1 + h_{i2} df_2 + \cdots + h_{in} df_n,\ (n+1 \leq i \leq m)$$

for some $h_{ij} \in \mathscr{E}_a, h_{ij}(a) = 0, n+1 \leq i \leq m, 1 \leq j \leq n$.

Definition 4.1 In general, for a map-germ $f = (f_1, \dots, f_m) : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$, we define the $\mathscr{E}_a$-submodule

$$\mathscr{J}_f := \sum_{j=1}^{m} \mathscr{E}_a df_j = \mathscr{E}_a d(f^* \mathscr{E}_b)$$

of the $\mathscr{E}_a$-module of differential 1-forms Ω^1_a on $(\mathbb{R}^n, a)$. We would like to call $\mathscr{J}_f$ the *Jacobi module* of f.

Note that $\mathscr{J}_f$ is determined by the Jacobi matrix $J(f)$ of f. Returning to our original situation, we define the following key notion:

Definition 4.2 We call a map-germ $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ an *opening* of a map-germ $g : (\mathbb{R}^n, a) \to (\mathbb{R}^n, g(a))$ if f is of the form

$$(g_1, \dots, g_n, f_{n+1}, \dots, f_m)$$

with $df_j \in \mathscr{J}_g, (n+1 \leq j \leq m)$ via a system of local coordinates of $(\mathbb{R}^m, b)$.

Then we observe the following:

Lemma 4.3 *Any frontal $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ is an opening of $g := (f_1, \dots, f_n) : (\mathbb{R}^n, a) \to (\mathbb{R}^n, g(a))$ via adapted coordinates to a Legendre lift of f. Conversely, any opening of a map-germ $g : (\mathbb{R}^n, a) \to (\mathbb{R}^n, g(a))$ is a frontal. An opening of g is a proper frontal if and only if g is proper, i.e. $S(g)$ is nowhere dense.*

Proof: The first half is clear. To see the second half, let $f = (g_1, \dots, g_n, f_{n+1}, \dots, f_m)$ be an opening of g. Then

$$df_i = h_{i1} df_1 + h_{i2} df_2 + \cdots + h_{in} df_n,\ (n+1 \leq i \leq m)$$

for some $h_{ij} \in \mathscr{E}_a, n+1 \leq i \leq m, 1 \leq j \leq n$. Then a Legendre lift $\widetilde{f} : (\mathbb{R}^n, a) \to \mathrm{Gr}(n, T\mathbb{R}^m)$ is given, via Grassmannian coordinates of the fiber, by

$$t \mapsto (f(t), \begin{pmatrix} E_n \\ H(t) \end{pmatrix}),$$

where E_n is the $n \times n$ unit matrix and $H(t)$ is given by the $(m-n) \times n$-matrix $(h_{ij}(t))$. Therefore f is a frontal. Note that an adapted system of coordinates for f is given by $(x_1, \dots, x_n, \widetilde{x}_{n+1}, \dots, \widetilde{x}_m)$ with $\widetilde{x}_i = x_i - \sum_{j=n+1}^{m} h_{ij}(a) x_j$ $(n+1 \leq i \leq m)$. The last statement follows clearly. □

Here we recall one of key notion for our approach to the recognition problem of frontal singularities.

Definition 4.4 ([8]) An opening $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b), f = (g; f_{n+1}, \ldots, f_m)$, of a map-germ $g : (\mathbb{R}^n, a) \to (\mathbb{R}^n, g(a))$ is called a *versal opening* if, for any $h \in \mathscr{E}_a$ with $dh \in \mathscr{J}_g$, there exist $k_0, k_1, \ldots, k_{m-n} \in \mathscr{E}_{\mathbb{R}^n, g(a)}$ such that

$$h = g^*(k_0) + g^*(k_1) f_{n+1} + \cdots + g^*(k_{m-n}) f_m.$$

We will use the following result which is proved in Proposition 6.9 of [8].

Theorem 4.5 *Any two versal openings $f, f' : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ (having the same target dimension) of a map-germ g are $\mathscr{A}$-equivalent to each other.*

Recall, for a map-germ $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$, we have defined $\mathscr{J}_f = \mathscr{E}_a d(f^* \mathscr{E}_b)$ (Definition 4.1).

Lemma 4.6 (1) *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$, $f' : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b')$ be map-germs. If f and f' are $\mathscr{L}$-equivalent, i.e. if there exists a diffeomorphism-germ $\tau : (\mathbb{R}^m, b) \to (\mathbb{R}^m, b')$ such that $f' = \tau \circ f$, then $\mathscr{J}_f = \mathscr{J}_{f'}$.*

(2) *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$, $f' : (\mathbb{R}^n, a') \to (\mathbb{R}^m, b)$ be map-germs. If f and f' are $\mathscr{R}$-equivalent, i.e. if there exists a diffeomorphism-germ $\sigma : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a')$ such that $f' = f \circ \sigma$, then $\sigma^*(\mathscr{J}_f) = \mathscr{J}_{f'}$.*

Proof: (1) Since $f^* \mathscr{E}_b = f'^* \mathscr{E}_{b'}$, we have $\mathscr{J}_f = \mathscr{E}_a d(f^* \mathscr{E}_b) = \mathscr{E}_a d(f'^* \mathscr{E}_{b'}) = \mathscr{J}_{f'}$.

(2) Since $f'^* \mathscr{E}_b = \sigma^*(f^* \mathscr{E}_b)$, we have

$$\mathscr{J}_{f'} = \mathscr{E}_{a'} d(f'^* \mathscr{E}_b) = \mathscr{E}_{a'} d(\sigma^*(f^* \mathscr{E}_b)) = \sigma^* \mathscr{E}_a \sigma^* d(f^* \mathscr{E}_b) = \sigma^*(\mathscr{E}_a d(f^* \mathscr{E}_b)) = \sigma^*(\mathscr{J}_f).$$

□

The equality of Jacobi modules $\mathscr{J}_f$ has a simple meaning:

Lemma 4.7 *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$, $f' : (\mathbb{R}^n, a) \to (\mathbb{R}^{m'}, b')$ be map-germs.*

Then the following conditions (i), (ii) *are equivalent:*

(i) *The Jacobi module $\mathscr{J}_f = \mathscr{J}_{f'}$.*

(ii) *There exist an $m' \times m$-matrix P and an $m \times m'$-matrix Q with entries in $\mathscr{E}_a$ such that the Jacobi matrix $J(f') = PJ(f)$ and $J(f) = QJ(f')$.*

In particular, (i) *implies that the Jacobi ideal $J_f = J_{f'}$.*

Moreover, if the target dimension $m = m'$, then the following condition (iii) is equivalent to (i).

(iii) *There exists an invertible $m \times m$-matrix R with entries in $\mathscr{E}_a$ such that $J(f') = RJ(f)$.*

To show Lemma 4.7, we recall the following fact in linear algebra.

Lemma 4.8 (cf. [22]) *Let A, B be $m \times m$-matrices with entries in $\mathbb{R}$. Then there exists an $m \times m$-matrices C with entries in $\mathbb{R}$ such that $C(E_m - BA) + A$ is invertible.*

Proof of Lemma 4.7:

The inclusion $\mathscr{J}_{f'} \subseteq \mathscr{J}_f$ is equivalent to that there exist $p_{ij} \in \mathscr{E}_a$ such that $df'_i = \sum_{j=1^m} p_{ij} df_j, (1 \leq i \leq m)$, namely that $J(f') = PJ(f)$ by setting $P = (p_{ij})$. Similarly, the inclusion $\mathscr{J}_f \subseteq \mathscr{J}_{f'}$ is equivalent to that there exist $q_{ij} \in \mathscr{E}_a$ such that $df_i = \sum_{j=1^m} q_{ij} df'_j, (1 \leq i \leq m)$, namely that $J(f) = QJ(f')$ by setting $Q = (q_{ij})$. Therefore the equivalence between (i) and (ii) is clear.

Suppose $m = m'$. By Lemma 4.8, there exists an $m \times m$-matrix C with entries in $\mathbb{R}$ such that $C(E_m - Q(a)P(a)) + P(a)$ is invertible. Then $R := C(E_m - QP) + P$ is an invertible $m \times m$-matrix with entries in in $\mathscr{E}_a$. Then we have $(E_m - QP)J(f) = J(f) - QJ(f') = O$ and therefore $RJ(f) = C(E_m - QP)J(f) + PJ(f) = J(f')$. $\square$

Remark 4.9 Related to Jacobi modules, we define the ramification module $\mathscr{R}_f \subseteq \mathscr{E}_a$ for a map-germ $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ by

$$\mathscr{R}_f := \{h \in \mathscr{E}_a \mid dh \in \mathscr{J}_f\},$$

using the Jacobi module $\mathscr{J}_f$. Then $\mathscr{R}_f = \mathscr{R}_{f'}$ if and only if $\mathscr{J}_f = \mathscr{J}_{f'}$. See, for details, the series of papers [6, 7, 8, 9, 10, 11].

Lemma 4.10 *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$, $f' : (\mathbb{R}^n, a') \to (\mathbb{R}^{m'}, b')$ be map-germs. If $\mathscr{J}_f = \mathscr{J}_{f'}$, then*

$$J_f = J_{f'}, \quad \mathscr{N}_f = \mathscr{N}_{f'}.$$

Proof: The equality $J_f = J_{f'}$ follows from Lemma 4.7. For any $\eta \in \mathscr{V}_a$, the condition $\eta \in \mathscr{N}_f$ is equivalent to that $\omega(\eta) \in J_f = J_{f'}$ for any $\omega \in \mathscr{J}_f = \mathscr{J}_{f'}$, which is equivalent to that $\eta \in \mathscr{N}_{f'}$. Therefore we have $\mathscr{N}_f = \mathscr{N}_{f'}$. $\square$

Lemma 4.11 *Let $f, f' : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be proper frontals of corank 1. Then the conditions*

$$\lambda_f \cdot \mathscr{E}_a = \lambda_{f'} \cdot \mathscr{E}_a, \quad \mathscr{N}_f = \mathscr{N}_{f'},$$

imply that $\mathscr{J}_f = \mathscr{J}_{f'}$.

Proof: By the assumption we may take $\lambda_f = \lambda_{f'}$ and $\eta_f = \eta_{f'}$. and $\eta_f = \partial/\partial t_n$ for a system of coordinates $t_1, \dots, t_{n-1}, t_n$ of $(\mathbb{R}^n, a)$. Note that, by the assumption, the zero-locus of λ_f is nowhere dense. Then $f_*(\partial/\partial t_1), \dots, f_*(\partial/\partial t_{n-1}), (1/\lambda_f) f_*(\partial/\partial t_n)$ are linearly independent at a as elements of $\mathscr{E}_a^m$. Take additional $\xi_{n+1}, \dots, \xi_m$ to complete a basis of $\mathscr{E}_a^m$. Moreover by the assumption

$$f'_*(\partial/\partial t_1), \dots, f'_*(\partial/\partial t_{n-1}), (1/\lambda_f) f'_*(\partial/\partial t_n)$$

are linearly independent at a as elements of $\mathscr{E}_a^m$. Take additional $\xi'_{n+1}, \dots, \xi'_m$ to complete a basis of $\mathscr{E}_a^m$. Then define $R : (\mathbb{R}^n, a) \to \mathrm{GL}(m, \mathbb{R})$ by

$$Rf_*(\partial/\partial t_i) = f'_*(\partial/\partial t_i), 1 \leq i \leq n-1, \;\; R(1/\lambda_f) f_*(\partial/\partial t_n) = (1/\lambda_f) f'_*(\partial/\partial t_n), \;\; R\xi_j = \xi'_j, n+1 \leq j \leq m.$$

Then $Rf_*(\partial/\partial t_n) = f'_*(\partial/\partial t_n)$ and we have $RJ(f) = J(f')$. By Lemma 4.7, we have $\mathscr{J}_f = \mathscr{J}_{f'}$. $\square$

We utilize the following in the next section:

Lemma 4.12 *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be an opening of $g : (\mathbb{R}^n, a) \to (\mathbb{R}^n, g(a))$ with respect to an adapted system of coordinates $(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$. Then f and g are frontals and $\mathscr{J}_f = \mathscr{J}_g$. They have common Jacobian, same corank, and $\mathscr{N}_f = \mathscr{N}_g$. If they are of corank 1, then they have common kernel field.*

Proof: By Lemma 4.3, we have $\mathscr{J}_f = \mathscr{J}_g$. Then $J_f = J_g$, therefore $\lambda_f = \lambda_g$. Moreover, by Lemma 4.7, $\mathrm{Ker}(T_a f) = \mathrm{Ker}(T_a g) \subseteq T_a\mathbb{R}^n$. Therefore f and g have the same corank. Furthermore, for any $\eta \in \mathscr{V}_a$, the condition that $df_i(\eta) \in J_f, 1 \le i \le m$ is equivalent to that $dg_i(\eta) \in J_f = J_g, 1 \le i \le n$. Hence $\mathscr{N}_f = \mathscr{N}_g$. □

Definition 4.13 Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ and $f' : (\mathbb{R}^n, a') \to (\mathbb{R}^{m'}, b')$ be map-germs. Then f and f' are called *$\mathscr{J}$-equivalent* if there exists a diffeomorphism-germ $\sigma : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a')$ such that $\mathscr{J}_{f'\circ\sigma} = \mathscr{J}_f$. Note that m and m' can be different.

By Lemma 4.6 and Lemma 4.11, we have

Corollary 4.14 *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ and $f' : (\mathbb{R}^n, a') \to (\mathbb{R}^{m'}, b')$ be map-germs. If f and f' are $\mathscr{A}$-equivalent, then f and f' are $\mathscr{J}$-equivalent.*

Corollary 4.15 *Let f, f' be proper frontals. If f and f' are $\mathscr{J}$-equivalent, then $(\lambda_f \cdot \mathscr{E}_a, \mathscr{N}_f)$ is transformed to $(\lambda_{f'} \cdot \mathscr{E}_{a'}, \mathscr{N}_{f'})$ by a diffeomorphism-germ $\sigma : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a')$. In particular λ_f and $\lambda_{f'}$ are $\mathscr{K}$-equivalent.*

Moreover if f is of corank 1 and $(\lambda_f \cdot \mathscr{E}_a, \mathscr{N}_f)$ is transformed to $(\lambda_{f'} \cdot \mathscr{E}_{a'}, \mathscr{N}_{f'})$ by a diffeomorphism-germ $\sigma : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a')$, then f and f' are $\mathscr{J}$-equivalent.

On the vanishing order of a function for a vector field introduced in Definition 3.8, we have:

Lemma 4.16 *If $\widetilde{h} = \rho h, \widetilde{\xi} = \nu\xi$ for some $\rho, \nu \in \mathscr{E}_a$ with $\rho(a) \neq 0, \xi(a) \neq 0$, then $\mathrm{ord}_a^{\widetilde{\xi}}(\widetilde{h}) = \mathrm{ord}_a^{\xi}(h)$. If $\overline{h} = h \circ \sigma, \overline{\xi} = (T\sigma^{-1}) \circ \xi \circ \sigma$ for some diffeomorphism-germ $\sigma : (\mathbb{R}^n, a') \to (\mathbb{R}^n, a)$, then $\mathrm{ord}_{a'}^{\overline{\xi}}(\overline{h}) = \mathrm{ord}_a^{\xi}(h)$.*

By Lemma 4.16 we have

Corollary 4.17 *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a proper frontal of corank 1. Then $\mathrm{ord}_a^{\eta}(\lambda)$ is independent of the choices of the Jacobian λ and the kernel field η of f. If $f' : (\mathbb{R}^n, a') \to (\mathbb{R}^{m'}, b')$ is $\mathscr{J}$-equivalent to f, then f' is a proper frontal of corank 1 and $\mathrm{ord}_a^{\eta'}(\lambda')$ is equal to $\mathrm{ord}_a^{\eta}(\lambda)$, for any Jacobian λ' and any kernel filed η' of f'.*

5 Proofs of recognition theorems

In this section we give proofs of Theorems 3.9, 3.10, 3.11, 3.13, 3.14, 3.15, 3.17, 3.18, and 3.19.

Proof of Theorem 3.9: The equivalence of (1) and (1') is classically known (see [6]). The equivalence of (1') and (2) is proved in [21].

To study the condition, we set $g = (f_1, f_2)$. Then for the Jacobian λ and the kernel field η of g we also have $\eta\lambda(a) \neq 0$ (see Lemma4.12). By Theorem 3.4, g is $\mathscr{A}$-equivalent to the fold. Then the condition (3) means that f is a versal opening of the fold g. Since the cuspidal edge is characterized as the (mini)-versal opening of the fold map-germ, we have the equivalence of (3) and (1) by Theorem 4.5. □

Proof of Theorem 3.10: The equivalence of (1) and (1') is proved in Theorem 7.1 of [8]. The condition (3) means that f is a versal opening of the fold g. Since the embedded cuspidal edge is characterized as the versal opening of the fold map-germ, we have the equivalence of (3) and (1) by Theorem4.5. On the other hand, under the condition $\eta\lambda(a) \neq 0$, the condition $\mathrm{ord}_a^{\eta}(f_i) = 3$ for some $i, 3 \leq i \leq m$ is equivalent to that the Legendre lift $\widetilde{f}$ is an immersion i.e. f is a front. Therefore (3) and (2) are equivalent. □

Proof of Theorem 3.11. The equivalence of (1) and (1') is due to Cleave (see [8]).

Suppose the condition (2) is satisfied. Then f is $\mathscr{A}$-equivalent to the germ $g(t_1,t_2) = (t_1, t_2^2, f_3(t_1,t_2))$ at the origin with $\lambda = t_2, \eta = \partial/\partial t_2$, $(\eta^3 f_3)(0) = 0$ and $(d\lambda \wedge d(\eta^3 f_3))(0) \neq 0$. Since $df_3 \in \mathscr{J}_g$, in other word since $f_3 \in \mathscr{R}_g$ (Remark4.9), there exist functions A, B on $(\mathbb{R}^2, 0)$ such that

$$f_3(t_1,t_2) = A(t_1,t_2^2) + B(t_1,t_2^2)t_2^3.$$

Then the condition $(\eta^3 f_3)(0) = 0$ is equivalent to $B(0,0) = 0$, and the condition $(d\lambda \wedge d(\eta^3 f_3))(0) \neq 0$ is equivalent to $\frac{\partial B}{\partial t_1}(0,0) \neq 0$. Define diffeomorphism-germs $\sigma : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ by $\sigma(t_1,t_2) = (B(t_1,t_2^2),t_2)$ and $\tau : (\mathbb{R}^3,0) \to (\mathbb{R}^3,0)$ by $\tau(x_1,x_2,x_3) = (B(x_1,x_2),x_2,x_3 - A(x_1,x_2))$. Then $(t_1,t_2^2,t_1t_2^3) \circ \sigma = \tau \circ (t_1,t_2^2,f_3)$ holds. Therefore f is $\mathscr{A}$-equivalent to folded umbrella. Hence we see that (2) implies (1). Conversely (1) implies (2) for some, so for any, adapted coordinates. □

Proof of Theorem 3.13: The $\mathscr{A}$-determinacy of tangent maps to curves of type $(1,2,4,5,\dots)$ is proved in Theorem 7.2 of [8]. Let $\gamma : (\mathbb{R},0) \to (\mathbb{R}^m,0)$ be the curve $t \mapsto (t,t^2,t^4,t^5,0,\dots)$. Then the tangent map $\mathrm{Tan}(\gamma) : (\mathbb{R}^2,0) \to (\mathbb{R}^m,0)$ is given by

$$\mathrm{Tan}(\gamma)(t,u) = (t+u,\ t^2+2ut,\ t^4+4ut^3,\ t^5+5ut^4,\ 0,\ \dots).$$

Then it is easy to see that $\mathrm{Tan}(\gamma)$ is $\mathscr{A}$-equivalent to $(t_1,t_2) \to (t_1,\ t_2^2,\ t_1t_2^3,\ t_2^5,\ 0,\dots,0)$. Hence we have the equivalence of (1) and (1').

Suppose f satisfies (2). Then f is an opening of (f_1,f_2), which is a fold by Theorem3.4. Therefore f is $\mathscr{A}$-equivalent to a frontal of form $(t_1,t_2^2,f_3,f_4,\dots)$ for an adapted coordinates. The Jacobian is given by $\lambda = t_2$ and the kernel field is given by $\eta = \partial/\partial t_2$. We write $f_i = A_i(t_1,t_2^2) + B_i(t_1,t_2^2)t_2^3$ for some A_i, B_i with $A_i(0,0) = 0, B_i(0,0) = 0$, $(3 \leq i \leq m)$. Then $f_i = \widetilde{A}_i(t_1,t_2^2)t_1t_2^3 + \widetilde{B}_i(t_1,t_2^2)t_2^5$. Then the condition (2) is equivalent to that, for some i,j with $3 \leq i < j \leq m$,

$$\begin{pmatrix} \widetilde{A}_i(0,0) & \widetilde{B}_i(0,0) \\ \widetilde{A}_j(0,0) & \widetilde{B}_j(0,0) \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}).$$

Then f is $\mathscr{A}$-equivalent to $(t_1,\ t_2^2,\ t_1t_2^3,\ t_2^5,\ 0,\dots,0)$. Therefore (2) implies (1'). The converse is clear. □

Proof of Theorem 3.14: The equivalence of (1) and (1') is proved in Theorem 1 of [6]. The equivalence of (1') and (2) is proved in Proposition 1.3 of [21]. The condition that λ is $\mathscr{K}$-equivalent to t_1 and $\mathrm{ord}_a^{\eta}(\lambda) = 2$ is equivalent, by Theorem 3.4, to that f is an opening of Whitney's cusp $g(t_1,t_2) = (t_1,\ t_2^3 + t_1t_2)$. The Jacobian is given by $\lambda = 3t_2^2 + t_1$ and the kernel field is given by $\eta = \partial/\partial t_2$. Set $U_1 = \frac{3}{4}t_2^4 + \frac{1}{2}t_1t_2^2, U_2 = \frac{3}{5}t_2^5 + \frac{1}{3}t_1t_2^3$. Then it is known that the ramification module $\mathscr{R}_g$ is generated by $1, U_1, U_2$ over g^* (see [6]). Since $f_3 \in \mathscr{R}_g$ is the third component for an adapted system of coordinates, f_3 is written as $f_3 = A \circ g + (B \circ g)U_1 + (C \circ g)U_2$, for some functions A, B, C with $A(0,0) = 0, \frac{\partial A}{\partial x_1}(0,0) = 0, \frac{\partial A}{\partial x_2}(0,0) = 0$. By the condition $\mathrm{ord}_a^{\eta}(f_3) = 4$, we have $B(0,0) \neq 0$. Then, by a change of adapted system of coordinates, We may suppose $f = (g, f_3)$ with $f_3 = U_1 + \Phi$, where $\Phi = (\widetilde{B} \circ g)U_1 + (D \circ g)U_2$ with $\widetilde{B}(0,0) = 0$. Then we set the family $F_s = (g, U_1 + s\Phi)$. By the same infinitesimal method used in [6], we can show that the family F_s is trivialized by $\mathscr{A}$-equivalence.

Hence $f = F_1$ is $\mathscr{A}$-equivalent to F_0, that is the normal form of (2). Therefore (3) implies (2). The converse is clear. □

Proof of Theorem 3.15: The equivalence of (1) and (1') is proved in [8]. The condition (2) implies, by Theorem 3.4, that f is an opening of Whitney's cusp. Using the same notations as in the proof of Theorem 3.14, we write f_k as $f_k = A_k \circ g + (B_k \circ g)U_1 + (C_k \circ g)U_2$, for some functions A_k, B_k, C_k with $A_k(0,0) = 0, \frac{\partial A_k}{\partial x_1}(0,0) = 0, \frac{\partial A_k}{\partial x_2}(0,0) = 0$. Then by the condition (2), we see that f is a versal opening (Definition 4.4) of g. On the other hand the map-germ of (1') is a versal opening of g ([8]). By Theorem 4.5, we see that (2) implies (1'). The converse implication (1') to (2) is clear. □

Proof of Theorem 3.17: The outline of the proof is similar to that of Theorem 3.14. The equivalence of (1) and (1') is proved in Theorem 1 of [6]. The equivalence of (1') and (2) is proved in [19]. The condition that λ is $\mathscr{K}$-equivalent to t_1t_2 and $\mathrm{ord}_a^{\eta}(\lambda) = 2$ is equivalent, by Theorem 3.4, to that f is an opening of bec à bec $g(t_1,t_2) = (t_1,\, t_2^3 + t_1t_2^2)$. The Jacobian is given by $\lambda = 3t_2^2 + 2t_1t_2$ and the kernel field is given by $\eta = \partial/\partial t_2$. Set $U_1 = \frac{3}{4}t_2^4 + \frac{2}{3}t_1t_2^3, U_2 = \frac{3}{5}t_2^5 + \frac{1}{2}t_1t_2^4$. Then it is known that the ramification module $\mathscr{R}_g$ is generated by $1, U_1, U_2$ over g^* (see [6]). Since $f_3 \in \mathscr{R}_g$ is the third component for an adapted system of coordinates, f_3 is written as $f_3 = A \circ g + (B \circ g)U_1 + (C \circ g)U_2$, for some functions A, B, C with $A(0,0) = 0, \frac{\partial A}{\partial x_1}(0,0) = 0, \frac{\partial A}{\partial x_2}(0,0) = 0$. By the condition $\mathrm{ord}_a^{\eta}(f_3) = 4$, we have $B(0,0) \neq 0$. Then, by a change of adapted system of coordinates, we may suppose $f = (g, f_3)$ with $f_3 = U_1 + \Phi$, where $\Phi = (\widetilde{B} \circ g)U_1 + (C \circ g)U_2$ with $\widetilde{B}(0,0) = 0$. Then, by the infinitesimal method used in [6], the family $F_s = (g, U_1 + s\Phi)$ is trivialized by $\mathscr{A}$-equivalence. Hence $f = F_1$ is $\mathscr{A}$-equivalent to F_0, that is the normal form of (2). Therefore (3) implies (2). The converse is clear. □

Proof of Theorem 3.18: Open Mond singularities are characterized as versal openings of bec à bec ([8]). Then Theorem3.18 is proved similarly as the proof of Theorem3.15. □

Proof of Theorem 3.19: The equivalence of (1) and (1') is proved in [6]. The condition (2) implies that f is an opening of bec à bec. Using the same notations in the proof of Theorem 3.17, we write f_3 as $f_3 = A \circ g + (B \circ g)U_1 + (C \circ g)U_2$, for some functions A, B, C with $A(0,0) = 0, \frac{\partial A}{\partial x_1}(0,0) = 0, \frac{\partial A}{\partial x_2}(0,0) = 0$. By the condition $\mathrm{ord}_a^{\eta}(f_3) = 5$, we have $B(0,0) = 0$ and $C(0,0) \neq 0$. Moreover, by the assumption, we may assume that $\mathrm{ord}_{(t_1,0)}^{\eta} f_3 \geq 4$ along the component $\{t_2 = 0\}$ of $S(f)$ and then $B(x_1,0) = 0$. Then, by a change of adapted system of coordinates, we may suppose $f = (g, f_3)$ with $f_3 = U_2 + \Phi$, where $\Phi = (B \circ g)U_1 + (\widetilde{C} \circ g)U_2$ with $B(x_1,0) = 0, \widetilde{C}(0,0) = 0$. Then by the same infinitesimal method used in [6], the family $F_s = (g, U_2 + s\Phi)$ turns to be trivial under $\mathscr{A}$-equivalence. Hence $f = F_1$ is $\mathscr{A}$-equivalent to F_0, that is the normal form of (1'). Therefore (2) implies (1'). The converse is clear. □

6 An application to 3-dimensional Lorentzian geometry, and other topics

We announce the following result without explanations of notions. The details will be given in [16].

Theorem 6.1 ([2], [14, 16]) *Any null frontal surface in a Lorentzian* 3*-manifold turns to be a null tangent surface of a (directed) null curve, and any generic null frontal surface has only singularities, along the null curve, of type*

(I) *cuspidal edge* (CE)*,* (II) *swallowtail* (SW)*, or* (III) *Shcherbak singularity* (SB).

Moreover the corresponding dual frontal in the space of null-geodesics has (I) *cuspidal edge* (CE)*,* (II) *Mond singularity* (MD)*, or* (III) *generic folded pleat* (GFP).

The same classification result holds not only for any Lorentzian metric but also for arbitrary non-degenerate (strictly convex) cone structure in any 3*-manifold.*

To show Theorem 6.1, we face the recognition problem on cuspidal edge, swallowtail, Scherbak singularity, Mond singularity, and "generic folded pleat". In fact we will use the recognition theorems introduced in the previous section and the following result on openings of Whitney's cusp. The following recognition result is proved by the same method of the above proof of Theorem3.14. The details will be given in [16].

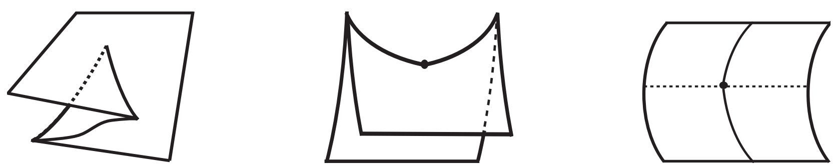
Theorem 6.2 (Recognition of folded pleat) *Let* $f : (\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ *be a frontal of corank* 1. *Then the following conditions are equivalent to each other:*

(1) f *is* $\mathscr{A}$*-equivalent to a folded pleat i.e. the singularity of tangent surface of a curve of type* $(2,3,5)$.

(1') f *is* $\mathscr{A}$*-equivalent to the germ* $(t_1,\ t_2^3+t_1t_2,\ \frac{3}{5}t_2^5+\frac{1}{2}t_1t_2^3+c(\frac{1}{2}t_2^6+\frac{3}{4}t_1t_2^4))$ *at the origin for some* $c \in \mathbb{R}$.

(2) λ *is* $\mathscr{K}$*-equivalent to the germ* $(t_1,t_2) \mapsto t_1$ *at the origin,* $\mathrm{ord}_a^\eta(\lambda)(a) = 2$*,* f *has an injective representative, and* $\mathrm{ord}_p^\eta(f_3) = 5$.

Note that a folded pleat singularity necessarily has an injective representative.



folded pleat   cuspidal swallowtail   cuspidal lips

Remark 6.3 Recall that the diffeomorphism classes (CE), (SW), (SB) and (MD) are exactly characterized as those of tangent surfaces in Euclidean space $\mathbb{R}^3$ of curves of type $(1,2,3)$, $(2,3,4)$, $(1,3,5)$, $(1,3,4)$ respectively. A map-germ $(\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ is called a *folded pleat* (FP) if it is diffeomorphic to the tangent surface of a curve of type $(2,3,5)$ in $\mathbb{R}^3$. The diffeomorphism classes of folded pleats fall into *two* classes, the generic folded pleat and the non-generic folded pleat. In the list of Theorem 6.1, it is claimed that only the generic folded pleat (GFP) appear. Theorem 6.2 do not solve the recognition of a singularity but a class of singularities, which consists of two singularities. Note that the parameter c in (1') of Theorem 6.2 is not a moduli, but provides just two $\mathscr{A}$-equivalence classes. To recognize the generic folded pleat, it is necessary an additional argument to distinguish generic and non-generic folded pleats.

In this occasion we introduce and prove the following two theorems of recognition:

Theorem 6.4 (Recognition of cuspidal swallowtail) *Let* $(\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ *be a frontal of corank* 1. *Then the following conditions are equivalent to each other:*

(1) f *is* $\mathscr{A}$*-equivalent to the cuspidal swallowtail i.e. the singularity of tangent surface of curves of type* $(3,4,5)$.

(1') f *is* $\mathscr{A}$*-equivalent to the germ* $(t_1,t_2) \mapsto (t_1,\ t_2^4+t_1t_2,\ \frac{4}{5}t_2^5+\frac{1}{2}t_1t_2^2)$ *at the origin.*

(2) λ *is* $\mathscr{K}$*-equivalent to the germ* $(t_1,t_2) \mapsto t_1$ *at the origin,* $\mathrm{ord}_a^\eta(\lambda) = 3$ *and* $\mathrm{ord}_a^\eta(f_3) = 5$.

Proof: In [8] it is proved that the condition (1) is equivalent to that f is $\mathscr{A}$-equivalent to the germ $(t,u) \mapsto (t^3+3u, t^4+4ut, t^5+5ut^2)$, which is $\mathscr{A}$-equivalent to the normal form of (1'). Therefore (1) and (1') are equivalent. In [24], the map-germ which is $\mathscr{A}$-equivalent to the germ $g:(t_1,t_2) \mapsto (t_1,\ t_2^4+t_1t_2)$ at the origin is called a *swallowtail* and it is shown that a map-germ $g:(\mathbb{R}^2,a) \to (\mathbb{R}^2,g(a))$ is a swallowtail if and only if λ is $\mathscr{K}$-equivalent to the germ $(t_1,t_2) \mapsto t_1$ at the origin and $\mathrm{ord}_a^{\eta}(\lambda)=3$. Suppose f satisfies (2). Then f is an opening of swallowtail. Then f is $\mathscr{A}$-equivalent to a frontal of form $f=(g,f_3)$. We have the Jacobian $\lambda = 4t_2^3+t_1$ and $\eta = \partial/\partial t_2$. We follow the method of [6]. Set

$$U = t_2^4+t_1t_2,\ U_1 = \tfrac{4}{5}t_2^5+\tfrac{1}{2}t_1t_2^2,\ U_2 = \tfrac{2}{3}t_2^6+\tfrac{1}{3}t_1t_2^3,\ U_3 = \tfrac{4}{7}t_2^7+\tfrac{1}{4}t_1t_2^4.$$

The third component f_3 is written as

$$f_3 = A\circ g + (B\circ g)U_1 + (C\circ g)U_2 + (D\circ g)U_3.$$

Then the condition $\mathrm{ord}_a^{\eta}(f_3)=5$ implies that $B(0,0)\neq 0$. We may suppose $f=(g,f_3)$ with $f_3 = U_1+\Phi, \Phi = (B\circ g)U_1+(C\circ g)U_2+(D\circ g)U_3, B(0,0)=0$. Then the family $F_s=(g,\ U_1+s\Phi)$ is trivialized by $\mathscr{A}$-equivalence. Thus $f=F_1$ is $\mathscr{A}$-equivalent to F_0 which is the normal form of (1'). Therefore (2) implies (1'). The converse is clear. Hence (1') and (2) are equivalent. □

As for openings of the lips $(t_1,t_2) \to (t_1, t_2^3+t_1^2t_2)$ (see [24]), we have

Theorem 6.5 (Recognition of cuspidal lips) *Let* $(\mathbb{R}^2,a) \to (\mathbb{R}^3,b)$ *be a frontal of corank* 1. *Then the following conditions are equivalent to each other:*

(1) f *is* $\mathscr{A}$*-equivalent to cuspidal lips i.e.* $(t_1,t_2) \to (t_1, t_2^3+t_1^2t_2, \frac{3}{4}t_2^4+\frac{1}{2}t_1^2t_2^2)$.
(2) f *is a front and* λ *is* $\mathscr{K}$*-equivalent to the germ* $(t_1,t_2) \mapsto t_1^2+t_2^2$ *at the origin.*
(3) λ *is* $\mathscr{K}$*-equivalent to the germ* $(t_1,t_2) \mapsto t_1^2+t_2^2$ *at the origin, and* $\mathrm{ord}_a^{\eta}(f_3)=4$.

Proof: The equivalence of (1) and (2) is proved in [19]. Under the condition that λ is $\mathscr{K}$-equivalent to the germ $(t_1,t_2) \mapsto t_1^2+t_2^2$ at the origin, the condition $\mathrm{ord}_a^{\eta}(f_3)=4$ is equivalent to that the Legendre lift $\widetilde{f}$ is an immersion. Thus we have the equivalence of (2) and (3). □

Remark 6.6 Cuspidal lips never appear as singularities of tangent surfaces.

We conclude the paper by presenting open questions:

Question 1. When does $\mathscr{J}$-equivalence imply $\mathscr{A}$-equivalence ?

Remark 6.7 For immersions, folds, cusps, lips, beaks, swallowtails : $(\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$, $\mathscr{J}$-equivalence of frontals of corank 1 implies $\mathscr{A}$-equivalence.

Example 6.8 ([23, 20]) Let $f, f' : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ be defined by $f(t_1,t_2) = (t_1, t_1t_2+t_2^5+t_2^7)$ (butterfly) and $f'(t_1,t_2) = (t_1, t_1t_2+t_2^5)$ (elder butterfly). Then f is not $\mathscr{A}$-equivalent to f' and their recognition by Taylor coefficients is obtained by Kabata [20]. On the other hand we observe, by using the theory of implicit OED of first order, that f is $\mathscr{J}$-equivalent to f' in fact. Therefore we see that it is absolutely impossible to recognize them just in terms of kernel field η and Jacobian λ.

Question 2. When does $\mathscr{J}$-equivalence imply $\mathscr{K}$-equivalence ?

It can be shown, for map-germs of corank 1, that $\mathscr{J}$-equivalence implies $\mathscr{K}$-equivalence under a mild condition:

Lemma 6.9 *Let* $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ *and* $f' : (\mathbb{R}^n, a') \to (\mathbb{R}^{m'}, b')$ *be map-germs of corank* 1*. If* f *and* f' *are* $\mathscr{J}$*-equivalent and* f *is* $\mathscr{K}$*-finite, then* f *and* f' *are* $\mathscr{K}$*-equivalent, i.e.* $(f^*\mathfrak{m}_b)\mathscr{E}_a$ *is transformed to* $(f'^*\mathfrak{m}_{b'})\mathscr{E}_{a'}$ *by a diffeomorphism-germ* $\sigma : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a')$*. Here* $\mathfrak{m}_b \subset \mathscr{E}_b$ *is the maximal ideal. The condition that* f *is* $\mathscr{K}$*-finite means that* $\dim_{\mathbb{R}}(\mathscr{E}_a/(f^*\mathfrak{m}_b)\mathscr{E}_a) < \infty$.

Proof: By the assumption, f is $\mathscr{A}$-equivalent to $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ of form $(t_1, \dots, t_{n-1}, \varphi_n(t), \dots, \varphi_m(t))$ for some $\varphi_i \in \mathscr{E}_0, n \le i \le m$. Then $g^*(\mathfrak{m}_0)\mathscr{E}_0$ is generated by $t_1, \dots, t_{n-1}, t_n^\ell$ for some ℓ and ℓ is uniquely determined by the minimum of orders of $\varphi_n(0, t_n), \dots, \varphi_m(0, t_n)$ for t_n at 0. On the other hand, the Jacobi module $\mathscr{J}_g$ is generated by $dt_1, \dots, dt_{n-1}, (\partial\varphi_n/\partial t_n)dt_n, \dots, (\partial\varphi_n/\partial t_n)dt_n$, and the minimum of orders of $(\partial\varphi_n/\partial t_n)(0, t_n), \dots, (\partial\varphi_m/\partial t_n)(0, t_n)$ for t_n at 0 is invariant under $\mathscr{J}$-equivalence. Therefore $\mathscr{K}$-equivalence class is also invariant under $\mathscr{J}$-equivalence. □

References

[1] V.I. Arnol'd, *Lagrangian manifold singularities, asymptotic rays and the open swallowtail,* Funct. Anal. Appl., **15** (1981). 235–246.

[2] S. Chino, S. Izumiya, *Lightlike developables in Minkowski 3-space,* Demonstratio Mathematica **43–2** (2010), 387–399.

[3] S. Fujimori, K. Saji, M. Umehara, K. Yamada, *Singularities of maximal surfaces,* Math. Z. **259** (2008), 827–848.

[4] A.B. Givental, *Whitney singularities of solutions of partial differential equations,* J. Geom. Phys. **15–4** (1995), 353–368.

[5] G. Ishikawa, *Families of functions dominated by distributions of* $\mathscr{C}$*-classes of mappings,* Ann. Inst. Fourier, **33-2** (1983), 199–217.

[6] G. Ishikawa, *Developable of a curve and its determinacy relatively to the osculation-type*, Quarterly J. Math., **46** (1995), 437–451.

[7] G. Ishikawa, *Symplectic and Lagrange stabilities of open Whitney umbrellas*, Invent. math., **126–2** (1996), 215–234.

[8] G. Ishikawa, *Singularities of tangent varieties to curves and surfaces,* Journal of Singularities, **6** (2012), 54–83.

[9] G. Ishikawa, *Tangent varieties and openings of map-germs,* RIMS Kōkyūroku Bessatsu, **B38** (2013), 119–137.

[10] G. Ishikawa, *Openings of differentiable map-germs and unfoldings,* Topics on Real and Complex Singularities, Proc. of the 4th Japanese-Australian Workshop (JARCS4), Kobe 2011, World Scientific (2014), pp. 87–113.

[11] G. Ishikawa, *Singularities of frontals,* in "Singularities in Generic Geometry", Advanced Studies in Pure Mathematics **78**, Math. Soc. Japan., (2018), pp. 55–106. `http://arxiv.org/abs/1609.00488/`

[12] G. Ishikawa, S. Janeczko, *Symplectic bifurcations of plane curves and isotropic liftings,* Quarterly J. of Math. Oxford, **54** (2003), 73–102.

[13] G. Ishikawa, Y. Machida, *Singularities of improper affine spheres and surfaces of constant Gaussian curvature,* International Journal of Mathematics, **17–3** (2006), 269–293.

[14] G. Ishikawa, Y. Machida, M. Takahashi, *Asymmetry in singularities of tangent surfaces in contact-cone Legendre-null duality,* Journal of Singularities, **3** (2011), 126–143.

[15] G. Ishikawa, Y. Machida, M. Takahashi, *Singularities of tangent surfaces in Cartan's split G_2-geometry,* Asian Journal of Mathematics, **20–2**, (2016), 353–382.

[16] G. Ishikawa, Y. Machida, M. Takahashi, *Null frontal singular surfaces in Lorentzian 3-spaces,* in preparation.

[17] G. Ishikawa, T. Yamashita, *Singularities of tangent surfaces to generic space curves,* Journal of Geometry, **108** (2017), 301–318.

[18] G. Ishikawa, T. Yamashita, *Singularities of tangent surfaces to directed curves,* Topology and its Applications, **234** (2018), 198–208.

[19] S. Izumiya, K. Saji, M. Takahashi, *Horospherical flat surfaces in Hyperbolic 3-space,* J. Math. Soc. Japan **62-3** (2010), 789–849.

[20] Y. Kabata, *Recognition of plane-to-plane map-germs,* Topology and its Applications, **202-1** (2016), 216–238.

[21] M. Kokubu, W. Rossman, K. Saji, M. Umehara, K. Yamada, *Singularities of flat fronts in hyperbolic space,* Pacific J. of Math. **221-2** (2005), 303–351.

[22] J.N. Mather, *Stability of C^∞ mappings III: Finitely determined map-germs,* Publ. Math. I.H.E.S., **35** (1968), 279–308.

[23] J.H. Rieger, *Families of maps from the plane to the plane,* J. London Math. Soc., **36** (1987), 351–369.

[24] K. Saji, *Criteria for singularities of smooth maps from the plane into the plane and their applications,* Hiroshima Math. J. **40** (2010), 229–239.

[25] K. Saji, M. Umehara, K. Yamada, *A_k singularities of wave fronts,* Math. Proc. Camb. Philos. Soc. **146-3** (2009), 731–746.

[26] R.W. Sharp, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*, Graduate Texts in Mathematics **166**, Springer, (2000).

[27] H. Whitney, *On singularities of mappings of Euclidean spaces I, Mappings of the plane into the plane,* Ann. of Math. **62** (1955), 374–410.