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Classification of phase singularities for complex scalar waves and their bifurcations

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Abstract

Motivated by the importance and universal character of phase singularities which were clarified recently, we study the local structure of equi-phase loci near the dislocation locus of complex valued planar and spatial waves, from the viewpoint of singularity theory of differentiable mappings, initiated by Whitney and Thom. The classification of phase singularities is reduced to the classification of planar curves by radial transformations due to the theory of du Plessis, Gaffney and Wilson. Then fold singularities are classified into hyperbolic and elliptic singularities. We show that the elliptic singularities are never realized by any Helmholtz waves, while the hyperbolic singularities are realized in fact. Moreover, the classification and realizability of Whitney's cusp, as well as its bifurcation problem, are considered in order to explain the three point bifurcation of phase singularities. In this paper, we treat the dislocation of linear waves mainly, developing the basic and universal method, the method of jets and transversality, which is applicable also to nonlinear waves.

Mathematics Subject Classification: 58K40, 78A40,78A05

(Some figures in this article are in colour only in the electronic version)

1. Introduction

A complex scalar wave has a locus, the dislocation locus, where its phase is not defined. The local structure of equi-phase loci near the dislocation locus is called a *phase singularity* [24]. The phase singularities are called *optical vortices* in optics and are very basic and important objects in any science related to waves and quanta. In this paper we give the exhaustive classification of phase singularities of complex scalar waves of low codimension.

In [23, 24], Nye constructed extensively complex scalar global planar waves satisfying the Helmholtz equation, with a detailed analysis of those examples. Also he gave, by his examples, an explanation of an experimental bifurcation process of phase singularities: one degenerate singular point bifurcates to three singular points and then another singular point annihilates with one of the three. See also [4, 5, 25, 26] for a detailed analysis on topology and bifurcations of phase singularities.

In this paper, we understand phase singularities clearly from the viewpoint of the singularity theory of differentiable mappings [2, 3, 8, 21, 29, 30] and provide basic mathematical support to several new phenomena discovered in the works of Nye, Hajnal and Hannay, and of Berry and Dennis quoted above.

The planar complex scalar wave can be regarded, from the general point of view, simply as a differentiable mapping from the plane to the plane of complex numbers. Then, by a theorem of Whitney [31], the generic singularities of the wave, as a differentiable mapping, are just the *fold singularities* and the *cusp singularities*. The singular values form on the plane of complex numbers an immersed curve called the discriminant, with several cusps. Then generically the discriminant does not hit the zero, so zero is a regular value; generic phase singularities are regular, that is, locally diffeomorphic to the standard radial lines emitted from the origin. However, for a generic time-dependent wave, the curve of singular values moves and momentarily may hit the zero. Thus, a generically momentary wave can have degenerate phase singularities described by the fold singularities. Moreover, for a generic two parameter family of plane waves, the cusp singularity occurs as a more complicated phase singularity.

The above simplified story must be examined twofold: first, in Whitney's theorem, the singularities are classified by means of arbitrary local diffeomorphisms of the source plane and the target plane. However, for the classification of phase singularities, we are concerned with the equi-phase lines and thus need to consider finer classification using only diffeomorphisms which preserve the radial lines on the target plane. Second, because waves must obey several natural conditions given by, say, the Helmholtz equations and the wave equations, more than just the differentiability, we must consider the realizability of singularities and determine generic singularities among waves satisfying those conditions.

Among others, we clarify the equivalence relations for phase singularities, and thus classify all phase singularities of low codimension, and discuss the realizability by the Helmholtz waves of phase singularities. Further, we propose a new explanation for the experimental bifurcation process treated in [23, 24].

In the next section, we formulate our equivalence relation providing the basis of our classification. A natural and refined classification by radial transformations is established on phase singularities for planar and spatial complex scalar waves. Then, we give the exact classification of generic complex planar waves and their bifurcations.

In section 3, the realization of singularities by the Helmholtz waves is examined by concrete examples, which have a different character from Nye's examples in [23].

In section 4, we classify phase singularities of spatial complex scalar waves and consider their realizability.

In section 5, the classification problem of planar waves is reduced to that of planar curves under diffeomorphisms preserving radial lines. Then we give proofs of all results stated in section 2 and several results stated in section 4.

We introduce in section 6 the notion of Helmholtz jet spaces and transversality to discuss genericity of singularities for Helmholtz waves. Then we prove results in section 3 and the remaining results in section 4.

In section 7, as an application of the method developed in this paper, we discuss the bifurcation problem of phase singularities of solutions to nonlinear Schrödinger equations.

In this paper, we consider the local classification problem of phase singularities. For the global topology of the dislocation locus, see [4, 5].

For other applications of the singularity theory to solutions of partial differential equations, see [9, 17, 18] for instance.

2. Phase singularities for planar complex scalar waves

We denote by $\boldsymbol{C}$ the plane of complex numbers and write a complex number as $u + \mathrm{i}w = r\mathrm{e}^{\mathrm{i}\theta}$, u, w being the real part and the imaginary part, respectively, while r, θ are the modulus (or the amplitude) and the argument (or the phase), respectively.

Let us consider a complex scalar wave

$$\Psi = \Psi(x, y, t) = u(x, y, t) + \mathrm{i}w(x, y, t)$$

on the (x, y)-plane depending on the time (or any other single-parameter). First we regard Ψ as just a time-dependent complex valued function. We assume $u(x, y, t)$, $v(x, y, t)$ are differentiable (i.e. C^∞) functions.

If $\Psi(x, y, t) \not= 0$ at a point (x, y) and at a moment t, then we can write $\Psi(x, y, t) = r(x, y, t)\mathrm{e}^{\mathrm{i}\theta(x,y,t)}$ uniquely with $r(x, y, t) > 0$ and $\theta(x, y, t) \bmod 2\pi$. We call $\theta(x, y, t) \bmod 2\pi$ the *phase function*, which is differentiable. Then, we are concerned with the wave dislocation locus at a moment $t = t_0$

$$\{(x, y) \mid \Psi(x, y, t_0) = 0\} = \{(x, y) \mid u(x, y, t_0) = 0, w(x, y, t_0) = 0\}$$

and the equi-phase curves $\{(x, y) \mid \theta(x, y, t_0) = \mathrm{const}\}$ outside the wave dislocation locus. The phase function $\theta(x, y, t_0) \bmod 2\pi$ is well defined off the wave dislocation locus. It may have critical points where the equi-phase curves are singular.

Then, in the framework of singularity theory of differentiable mappings, we introduce the notion of *radial transformations* and give the exact classification results of singularities relatively to the radial transformations.

A *radial transformation* on $\boldsymbol{C}$ near 0 is a diffeomorphism, an invertible differentiable transformation, $\tau(u, w) = (U, W)$, $\tau : (\boldsymbol{C}, 0) \to (\boldsymbol{C}, 0)$ which sends any radial line $\{\theta = \mathrm{const}\}$ to a radial line. In fact, a diffeomorphism $\tau(u, w) = (U, W)$ is a radial transformation if and only if there exists a positive function $\rho(u, w)$ and real numbers a, b, c, d with $ad - bc \neq 0$ such that

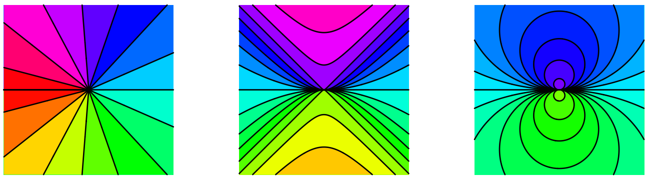
$$U = \rho(u, w)(au + bw), \qquad W = \rho(u, w)(cu + dw).$$

For the classification, we define the equivalence relation on phase singularities: two functions $\Psi(x, y, t)$ and $\Phi(x, y, t) = u'(x, y, t) + \mathrm{i}w'(x, y, t)$ are *radially equivalent* at points and moments (x_0, y_0, t_0) and (x_0', y_0', t_0'), respectively, if there exist a local diffeomorphism $\sigma(x, y) = (X(x, y), Y(x, y))$ on the plane with $X(x_0, y_0) = x_0'$, $Y(x_0, y_0) = y_0'$ and a local radial transformation $\tau(u, w) = (U(u, w), W(u, w))$ near the origin on $\boldsymbol{C}$ such that

$$u(X(x, y), Y(x, y), t_0) = U(u'(x, y, t_0'), w'(x, y, t_0')),$$

$$w(X(x, y), Y(x, y), t_0) = W(u'(x, y, t_0'), w'(x, y, t_0')),$$

namely, that $\Psi(\sigma(x, y), t_0) = \tau(\Phi(x, y, t_0'))$. The radial equivalence preserves the dislocation locus and the locus of critical points of phase functions. We consider the local classification of mappings $\boldsymbol{R}^2 \to \boldsymbol{C}$, namely, the classification of map-germs under the radial equivalence.



R H E

Figure 1. Phase singularities.

Then we have the following basic classification.

Theorem 2.1. *For a generic complex valued function* $\Psi(x, y, t)$*, the map-germ at any point and any moment* (x_0, y_0, t_0) *in the dislocation locus,* $\Psi(x_0, y_0, t_0) =$*, is radially equivalent to the regular singularity*

$$\text{R:} \quad \psi(x, y) = x + \mathrm{i}y,$$

the hyperbolic singularity

$$\text{H:} \quad \psi(x, y) = x^2 - y^2 + \mathrm{i}y$$

or to the elliptic singularity

$$\text{E:} \quad \psi(x, y) = x^2 + y^2 + \mathrm{i}y$$

at the origin $(x, y) = (0, 0)$ *(see figure 1.)*

Thus generic phase singularities consist of just three radial equivalence classes: *regular, hyperbolic and elliptic* singularities, and nothing else.

The proof of theorem 2.1 is given in section 5.

The regular singularity corresponds to the case where the map-germ is non-singular, namely, $0 \in \boldsymbol{C}$ is a regular value. Both hyperbolic and elliptic singularities are equivalent to the *fold singularity*

$$\psi : (x, y) \mapsto (u, w) = (x^2, y)$$

under arbitrary diffeomorphisms not necessarily radial transformation, namely, under the right–left equivalence. Each phase singularity of the classification in theorem 2.1 is determined by its two jets actually.

For a momentary complex wave $\psi(x, y) = u(x, y) + \mathrm{i}w(x, y)$ on the plane, the locus in $\boldsymbol{C}$ of complex values $\psi(x_0, y_0)$ for (x_0, y_0) with

$$\det \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \end{pmatrix} (x_0, y_0) = 0$$

is called the *discriminant* of the complex wave ψ.

For a fold singularity of mapping, the discriminant is a regular curve. If $\psi(x_0, y_0) = 0$, then a phase function is not well defined at (x_0, y_0), and we call (x_0, y_0) a phase singular point. If the discriminant is transverse to the radial line at $\psi(x_0, y_0) \neq 0$, then the phase function is well defined and not critical at (x_0, y_0). If the discriminant is tangent to the radial line at $\psi(x_0, y_0) \neq 0$, then the phase function is well defined but critical at (x_0, y_0).

The condition that a point $(x_0, y_0) \in \boldsymbol{R}^2$ is a singular point of a mapping $\boldsymbol{R}^2 \to \boldsymbol{C}$ has codimension one and within this case the condition that the discriminant curve passes through

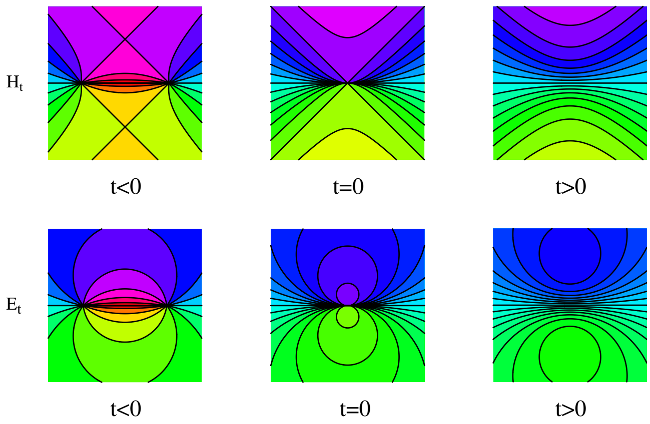


Figure 2. The bifurcations of the hyperbolic phase singularity (top) and the elliptic phase singularity (bottom).

$0 \in \boldsymbol{C}$ also has codimension one; therefore theorem 2.1 concerns cases of codimension two in the space of map-germs. This explains that, for a generic mapping $\boldsymbol{R}^2 \to \boldsymbol{C}$, all deeper singularities are excluded.

Note that the above normal form of fold singularity is never generic as the phase singularity. In fact, the discriminant of ψ in that case is the w-axis in $\boldsymbol{C}$ which has infinite tangency (actually coincides) with the radial lines $\{\theta = \pi/2\}$ and $\{\theta = 3\pi/2\}$. Generically the discriminant must be tangent to the radial lines in a non-degenerate manner, namely, in the second order tangency. Then there are two possibilities of non-degenerate tangency of the discriminant of fold singularities at $0 \in \boldsymbol{C}$; the image of ψ (the value set of ψ) is concave or convex. These correspond, respectively, to the hyperbolic singularity and the elliptic singularity.

Remark 2.2. Besides the wave dislocation, we can classify generic critical points of phase functions defined outside the dislocation locus: the generic critical points are the *non-degenerate maximal (minimal) points*, the *saddle points* and the *cuspidal points*. The last bifurcate to one maximal (minimal) point and one saddle point. Critical points of phase functions are associated with the bifurcation of a phase singularity (see figure 2). In fact, two saddle points are associated with the bifurcation of a hyperbolic singularity into two regular phase singular points. Two maximal (minimal) points are associated with the elimination of an elliptic singularity.

Moreover, the generic bifurcations on t of the hyperbolic singularities and the elliptic singularities are given by

$$\mathrm{H}_t : \Psi(x, y, t) = x^2 - y^2 + t + \mathrm{i}y, \qquad (t \in \boldsymbol{R}),$$
$$\mathrm{E}_t : \Psi(x, y, t) = x^2 + y^2 + t + \mathrm{i}y, \qquad (t \in \boldsymbol{R}).$$

(See figure 2.)

Remark 2.3. Note that the phase singularities and the hyperbolic bifurcations in figures 1 and 2 are found in [24–26] already. See also figure 6 of [10]. However also note that here we have given the exact normal forms for all possible generic phase singularities and their bifurcations and thus we guarantee that any other bifurcations never occur generically.

Remark 2.4. The classification of phase singularities is closely related to the classification of plane curves under diffeomorphisms preserving a given singular foliation on the plane. Then, one of the most delicate cases is the case when the foliation is formed by radial lines, which is given by the Euler vector field $X = x(\partial/\partial x) + y(\partial/\partial y)$ (cf [32]). That is the case we are treating in this paper (see section 5).

The discriminant of the fold singularity is a regular curve. Then degenerate phase singularities are classified as follows.

Proposition 2.5. *The phase singularities arising from fold singularities are classified into*

$$\psi_m(x, y) = x^2 \pm y^m + \mathrm{i}y, \qquad m = 2, 3, 4, \dots,$$

under radial transformations (and diffeomorphisms on the source), provided the discriminant curve has a contact with the tangent line at the origin in a finite multiplicity. The number m is the order of tangency of the discriminant curve and the radial line. Therefore m is an invariant for the radial equivalence.

Remark 2.6. The generic bifurcation of $\psi_m(x, y)$ is described by the family

$$x^2 \pm y^m + t_{m-1}y^{m-1} + t_{m-2}y^{m-2} + \cdots + t_2y^2 + t_0 + \mathrm{i}y,$$

with $(m-1)$-parameters $t_0, t_2, \dots, t_{m-1}$. If $m = 2$, we have $x^2 \pm y^2 + t_0 + \mathrm{i}y$.

A momentary complex wave $\psi\colon (\boldsymbol{R}^2, (x_0, y_0)) \to (\boldsymbol{C}, 0)$ is called a *Whitney cusp* or simply a *cusp* if it is a right–left equivalent (under local diffeomorphisms on $\boldsymbol{R}^2$ and $\boldsymbol{C}$ which are not necessarily radial) to the mapping $\psi(x, y) = x^3 + xy + \mathrm{i}y$. We are interested in this type of phase singularity because there occurs a three point bifurcation by just a translation $\psi_a(x, y) = x^3 + xy + \mathrm{i}(y + a)(a \in \boldsymbol{R})$. (For the classification of more degenerate singularities under the right–left equivalence relations, see [7,27,28].)

The Whitney cusp appears generically in two parameter families of planar complex valued functions.

Then, we have the following proposition.

Proposition 2.7 (The radial classification of Whitney cusps). Any Whitney cusp is equivalent under radial transformations to the standard function $\psi(x, y) = x^3 + xy + \mathrm{i}y$.

The typical bifurcation of the phase singularities for a Whitney cusp is described by $\psi_{a,b}(x, y) = x^3 + xy + b + \mathrm{i}(y + a)$, $(a, b \in \boldsymbol{R})$ (see figure 3.)

Remark 2.8. The bifurcation problem of phase singularities arising from Whitney cusps is related to web geometry [1,11]. In fact, generic two parameter families of Whitney cusps define 3-webs on the plane, and their classification by radial transformations provides functional moduli (remark 5.6).

3. Phase singularities of the Helmholtz waves

Now, we ask about the physical reality of the classification: the instantaneous appearance of singularities for wave functions satisfying the wave equation and the Helmholtz equation. Namely, we assume the wave $\Psi(x, y, t)$ satisfies the wave equation

$$\frac{\partial^2\Psi}{\partial t^2} = c^2\nabla^2\Psi,$$

for a positive real number c, where $\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$ is the Laplacian. Moreover we assume that $\Psi(x, y, t)$ satisfies the Helmholtz equation

$$\nabla^2\Psi + k^2\Psi = 0,$$

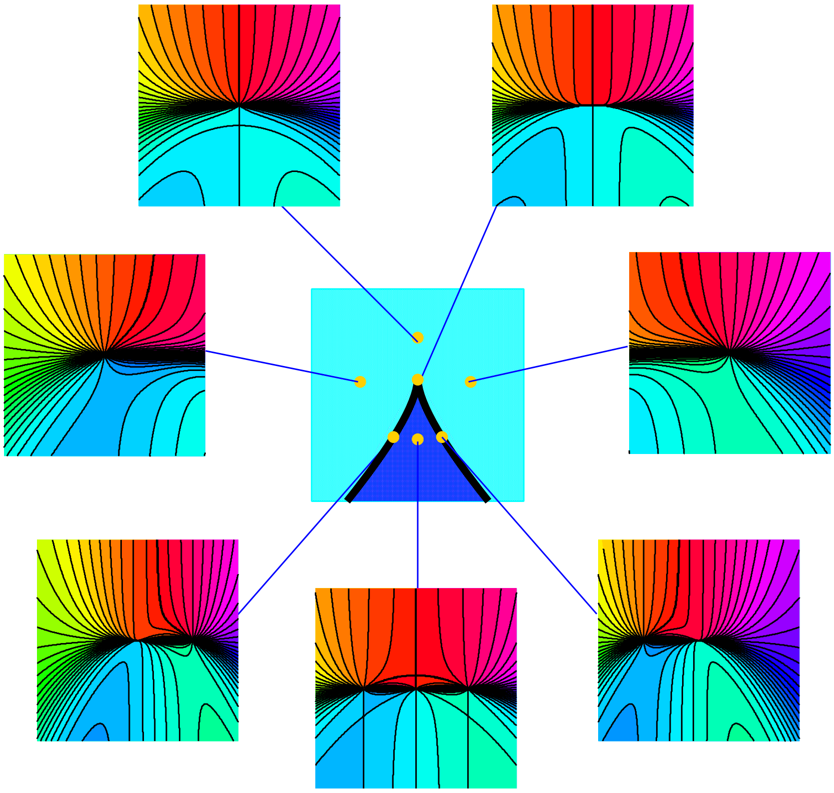


Figure 3. The two parameter bifurcation of the cusp phase singularity.

for a positive real number k as a very natural physical assumption for monochromatic waves. We call a function which satisfies the Helmholtz equation the *Helmholtz function*. Further, we call the Helmholtz function with a parameter t which satisfies the wave equation as well the *Helmholtz wave*. Note that if we have a solution Ψ for $c = 1$, $k = 1$, then, by setting $\widetilde{\Psi}(x, y, t) = \Psi(kx, ky, \frac{c}{k}t)$, we have a solution $\widetilde{\Psi}$ for general c and k.

By solving the Cauchy problem properly, we obtain the following.

Proposition 3.1. *The complex valued function*

$$\psi(x, y) = \cos y - \cos x + \mathrm{i} \sin y$$

satisfies the Helmholtz equation $\psi_{xx} + \psi_{yy} + \psi = 0$ *and has the hyperbolic singularity at the origin. Moreover the hyperbolic singularity with its generic bifurcation is realized by a Helmholtz wave*

$$\Psi(x, y, t) = (\cos y - \cos x + \mathrm{i} \sin y) \cos t + \cos y \sin t,$$

(for $k = 1, c = 1$*).*

We see it is radially equivalent to the normal form simply by observing its Taylor expansion. In contrast, we observe the following proposition.

Proposition 3.2. *Elliptic singularities are not realized as a function satisfying the Helmholtz equation.*

Proof. Suppose a function $\psi(x, y)$ is radially equivalent to the elliptic singularity. Then the image of ψ is convex at the origin where the tangent line supports. Suppose the function $\psi(x, y)$ satisfies the Helmholtz equation with $k = 1$. From the equation $\psi_{xx} + \psi_{yy} + \psi = 0$, we see the Hessian of ψ is traceless at the dislocation locus $\{\psi = 0\}$, so are the real part $\mathrm{Re}(\mathrm{Hess}\psi)$ and the imaginary part $\mathrm{Im}(\mathrm{Hess}\psi)$. Thus, for any real numbers λ, μ,

$$\lambda\mathrm{Re}(\mathrm{Hess}\psi) + \mu\mathrm{Im}(\mathrm{Hess}\psi)$$

is never a definite matrix. However, the linear projection along the tangent line to the image of ψ must be definite. This leads to a contradiction. □

In fact, as for the generic classification of phase singularities for one-parameter families of complex valued functions satisfying the Helmholtz equation, we have the following.

Theorem 3.3. *The generic phase singularities of planar Helmholtz functions are regular singularities and hyperbolic singularities.*

Theorem 3.4. *The generic phase singularities of planar Helmholtz waves are regular singularities and hyperbolic singularities.*

Theorems 3.3 and 3.4 are proved in section 6.

For the cusp singularities, we have the following.

Proposition 3.5. *A Whitney cusp is realized as a Helmholtz wave. In fact,*

$$\psi(x, y) = x^3 \cos y + (x - 3xy) \sin y + \mathrm{i} \sin y$$

is a Whitney cusp satisfying the Helmholtz equation $(k = 1)$: $\psi_{xx} + \psi_{yy} + \psi = 0$. *Moreover,*

$$\Psi(x, y, t) = (x^3 \cos y + (x - 3xy) \sin y + \mathrm{i} \sin y) \cos t + \mathrm{i} \cos y \sin t$$

gives a deformation of ψ *by a Helmholtz wave* $(k = 1, c = 1)$ *describing a three point bifurcation of the phase singularity.*

Remark 3.6. By a similar construction to proposition 3.5, we have another realization

$$\psi(x, y) = x^2 \cos y - y \sin y + \mathrm{i} \sin y$$

of hyperbolic singularities.

Remark 3.7. Apart from the classification problem of phase singularities, we can show that the generic Helmholtz function $\psi : \boldsymbol{R}^2 \to \boldsymbol{C}$ is, locally at any point in $\boldsymbol{R}^2$, *right–left equivalent* to a regular point, to a fold point or to a cusp point.

4. The radial classification of phase singularities for spatial complex scalar waves

We study, in this section, the phase singularities of spatial waves $\Psi = \Psi(x, y, z, t) : \boldsymbol{R}^3 \times \boldsymbol{R} \to \boldsymbol{C}$.

The generic singularities of differentiable mappings $\boldsymbol{R}^3 \to \boldsymbol{C}$ consist of the *definite fold singularities*, the *indefinite fold singularities* and the *cusp singularities*. The normal forms of them are given by

$$\begin{aligned}
\text{the definite fold singularity}: \quad \psi(x, y, z) &= x^2 + y^2 + \mathrm{i}z,\\
\text{the indefinite fold singularity}: \quad \psi(x, y, z) &= x^2 - y^2 + \mathrm{i}z,\\
\text{the cusp singularity}: \quad \psi(x, y, z) &= x^3 + xy + z^2 + \mathrm{i}y,
\end{aligned}$$

under the left–right equivalence [15]. For a generic complex valued function $\Psi(x, y, z, t)$, only fold singularities may appear as a phase singularity. Moreover the discriminant curve has non-degenerate tangency with the tangent line at the origin. Thus we have theorem 4.1.

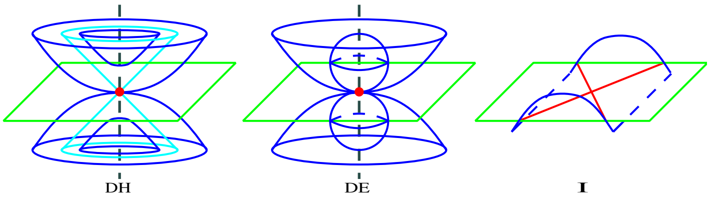


Figure 4. Phase singularities for spatial waves.

Theorem 4.1. *For a generic spatial complex valued function* $\Psi(x, y, z, t)$*, the phase singularity at any point and any moment* (x_0, y_0, z_0, t_0) *is equivalent, under the radial transformation, to the regular singularity*

$$\text{R:} \quad \psi(x, y, z) = x + \mathrm{i}y,$$

to the definite hyperbolic singularity

$$\text{DH:} \quad \psi(x, y, z) = x^2 + y^2 - z^2 + \mathrm{i}z,$$

to the definite elliptic singularity

$$\text{DE:} \quad \psi(x, y, z) = x^2 + y^2 + z^2 + \mathrm{i}z$$

or to the indefinite singularity

$$\text{I:} \quad \psi(x, y, z) = x^2 - y^2 - z^2 + \mathrm{i}z$$

at the origin $(x, y, z) = (0, 0, 0)$ *(see figure 4.)*

The discriminant of a mapping $\psi : \boldsymbol{R}^3 \to \boldsymbol{C}$ is defined as the locus of critical values where the 2×3 Jacobi matrix has rank $\leqslant 1$. The discriminant of a fold singularity is a regular curve.

Similarly to the case of planar complex scalar waves, the generic bifurcations on t of the definite hyperbolic singularities, the definite elliptic singularities and the indefinite singularities are given by

$$\begin{aligned}
\mathrm{DH_t} &: \Psi(x, y, z, t) = x^2 + y^2 - z^2 + t + \mathrm{i}z, \\
\mathrm{DE_t} &: \Psi(x, y, z, t) = x^2 + y^2 + z^2 + t + \mathrm{i}z, \\
\mathrm{I_t} &: \Psi(x, y, z, t) = x^2 - y^2 - z^2 + t + \mathrm{i}z
\end{aligned}$$

(see figure 5.)

In general we have proposition 4.2

Proposition 4.2. *The phase singularities arising from fold map-germs* $(\boldsymbol{R}^3, 0) \to (\boldsymbol{C}, 0)$ *are classified into*

$$\psi_m(x, y, z) = x^2 + y^2 \pm y^m + \mathrm{i}y, \qquad m = 2, 3, 4, \ldots$$

and

$$\psi_m(x, y, z) = x^2 - y^2 - y^m + \mathrm{i}y, \qquad m = 2, 3, 4, \ldots,$$

under the radial equivalence, if the order of tangency of discriminant curve and the radial line is $m < \infty$.

For cusp singularities, we have proposition 4.3.

Proposition 4.3. *The phase singularities which come from cusp singularities are all radially equivalent to*

$$\psi(x, y, z) = x^3 + xy + z^2 + \mathrm{i}y.$$

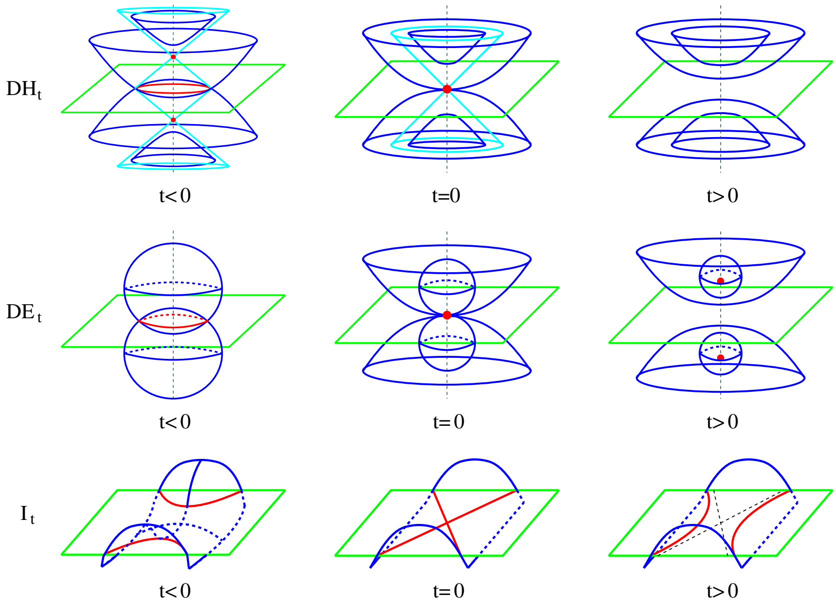


Figure 5. The bifurcations of phase singularities of spatial scalar waves.

A complex valued function $\Psi(x, y, z, t)$ is called a *Helmholtz wave* if it satisfies the wave equation $\Psi_{tt} = c^2(\Psi_{xx} + \Psi_{yy} + \Psi_{zz})$ for a positive real number c and the Helmholtz equation $\Psi_{xx} + \Psi_{yy} + \Psi_{zz} + k^2\Psi = 0$ for a positive real number k.

As for the realizability of the spatial waves as Helmholtz waves, we have the following proposition.

Proposition 4.4. *The definite hyperbolic singularity and the indefinite singularity together with their generic bifurcations are realized by Helmholtz waves* (*for* $k = 1, c = 1$)*:*

$$\mathrm{DH_t} \;:\; \Psi(x, y, z, t) = (-\cos x - \cos y + 2\cos z + \mathrm{i}\sin z)\cos t + \cos z \sin t,$$

$$\mathrm{I_t} \;:\; \Psi(x, y, z, t) = (-2\cos x + \cos y + \cos z + \mathrm{i}\sin z)\cos t + \cos z \sin t.$$

Moreover, in a similar way to proposition 3.2, we have proposition 4.5.

Proposition 4.5. *Definite elliptic singularities are not realized as Helmholtz waves.*

The cusp singularity is realized as a Helmholtz function.

Proposition 4.6. *The complex valued function*

$$\psi(x, y, z) = x^3 \cos y + (x - 3xy)\sin y - \cos y + \cos z + \mathrm{i}\sin y$$

satisfies the Helmholtz equation $\psi_{xx} + \psi_{yy} + \psi_{zz} + \psi = 0$ *and is radially equivalent to the cusp singularities.*

5. Radial classification of planar curves

The classification problem of complex waves under radial transformations is reduced to the classification problem of planar curves under radial transformations by means of du

Plessis, Gaffney and Wilson's theory [12, 14]. The theory reduces the classification to that of discriminants with exceptions. The exceptional cases are hyperbolic and elliptic singularities. Although they have the same discriminant, they are not radially equivalent. It depends on whether the image is convex or concave.

A differentiable map-germ $\psi : (\boldsymbol{R}^n, p) \to (\boldsymbol{R}^m, q), (n \geqslant m)$ is called a *critical normalization* if the ideal of functions vanishing on the set $C(\psi)$ of critical points of f, where the rank of the Jacobi matrix is less than m, is generated by m-minors of the Jacobi matrix and the restriction $\psi : C(\psi) \to \psi(C(\psi))$ is right–left equivalent to an analytic normalization. As special cases of the results in [14], we have proposition 5.1

Proposition 5.1.

(i) *Let two map-germs $\psi : (\boldsymbol{R}^n, p) \to (\boldsymbol{C}, 0)$ and $\psi' : (\boldsymbol{R}^n, p') \to (\boldsymbol{C}, 0)$ $(n = 2, 3)$ be critical normalizations and not definite fold singularities. If there is a radial transformation $\tau : (\boldsymbol{C}, 0) \to (\boldsymbol{C}, 0)$ which maps the discriminant of ψ to that of ψ', then ψ and ψ' are radially equivalent.*

(ii) *Let two map-germs $\psi : (\boldsymbol{R}^n, p) \to (\boldsymbol{C}, 0)$ and $\psi' : (\boldsymbol{R}^n, p') \to (\boldsymbol{C}, 0)$ $(n = 2, 3)$ be definite fold singularities. If there is a radial transformation $\tau : (\boldsymbol{C}, 0) \to (\boldsymbol{C}, 0)$ which maps the discriminant of ψ to that of ψ' and the image of ψ to that of ψ', then ψ and ψ' are radially equivalent.*

Proof.

(i) $\tau \circ \psi$ and ψ' are critical normalizations and they have the same discriminant curve. Then, by the results due to du Plessis, Gaffney and Wilson (e.g. theorem 2.1 of [14]), there exists a germ of diffeomorphism $\sigma : (\boldsymbol{R}^n, p) \to (\boldsymbol{R}^n, p')$ such that $\psi' \circ \sigma = \tau \circ \psi$. Therefore ψ and ψ' are radially equivalent.

(ii) $\tau \circ \psi$ and ψ' are definite fold singularities and they have the same image. Then, by theorem 2.1 of [14], there exists a germ of diffeomorphism $\sigma : (\boldsymbol{R}^n, p) \to (\boldsymbol{R}^n, p')$ such that $\psi' \circ \sigma = \tau \circ \psi$. Therefore ψ and ψ' are radially equivalent.

□

The condition that a map-germ is a critical normalization is a rather mild condition. For instance, stable map-germs are critical normalizations (see [14]). Thus, under this mild condition and with few exceptions, the diffeomorphism class of the pair consisting of the discriminant curve and the phase portrait of the Euler vector field is a complete invariant distinguishing phase singularities.

Remark 5.2. Let $\psi_\ell(x, y) = x^\ell + y^2 + \mathrm{i}y$ $(\ell \geqslant 2)$. Then all ψ_ℓ have the same discriminant curve $u = w^2$, but all ψ_ℓ are not radially equivalent to each other. In fact if $\ell \geqslant 3$, then ψ_ℓ is not a critical normalization.

The following result on curves has been applied to the classification of discriminants and then phase singularities.

Lemma 5.3 (The radial classification of regular curves).
A regular curve through the origin on $\boldsymbol{C}$ is transformed by radial transformations to the curve $u = w^m$ for some integer $m \geqslant 2$ or $u = g(w)$ for some function with null derivatives $g^{(i)}(0) = 0, i = 0, 1, 2, 3, \ldots$.

Proof. Using a linear transformation, we may suppose the regular curve is given by $u = g(w)$ for a function $g(w)$. Suppose $\operatorname{ord} g = m$ at $w = 0$ and $g^{(m)}(0) > 0$. Then we can write

$u = (a(w)w)^m$ for a function $a(w)$ with $a(0) > 0$. Set $\rho(w) = a(w)^{\frac{m}{m-1}}$ and define the radial transformation $U = \rho(w)u,\ W = \rho(w)w$. Then $U = \rho(w)\{a(w)W/\rho(w)\}^m = W^m$. □

Proof of propositions 2.5 and 4.2. Since a fold map-germ $(\boldsymbol{R}^2, p) \to (\boldsymbol{C}, 0)$ (respectively, $(\boldsymbol{R}^3, p) \to (\boldsymbol{C}, 0)$) is a critical normalization and its discriminant is a regular curve by theorem 5.1, the radial classification of phase singularities arising from fold map-germs is reduced to the radial classification of their discriminants, which are a regular curve and their images of map-germs. Therefore we required the results of lemma 5.3. □

Proof of theorems 2.1 and 4.1. First consider the generic one-parameter family of planar mappings $\Psi(x, y, t)$, $\Psi : \boldsymbol{R}^2 \times \boldsymbol{R} \to \boldsymbol{C}$. Then, by a transversality argument, we see that, if $\Psi(x_0, y_0, t_0) = 0$, then $\Psi(x, y, t_0)$ is a regular map-germ or it has a fold singularity at $(x, y) = (x_0, y_0)$. Moreover the discriminant at $\Psi(x, y, t_0)$ has second order tangency at $0 \in \boldsymbol{C}$ with the tangent line, which is one of the radial lines. If $\Psi(x, y, t_0)$ is a regular map-germ, then it is right equivalent to $\psi(x, y) = x + \mathrm{i}y$, by using the inverse mapping theorem. If $\Psi(x, y, t_0)$ has a fold singularity at $(x, y) = (x_0, y_0)$ and the discriminant has second order tangency, then, by proposition 2.5, we see that it is radially equivalent to $\psi(x, y) = x^2 \pm y^2 + \mathrm{i}y$. In the case of the generic one-parameter family of mappings $\Psi(x, y, z, t)$, $\Psi : \boldsymbol{R}^3 \times \boldsymbol{R} \to \boldsymbol{C}$, we use proposition 4.2. Then we have that $\Psi(x, y, z, t_0)$ is right equivalent to $\psi(x, y, z) = x + \mathrm{i}y$ if $\Psi(x, y, z, t_0)$ is not critical at $(x, y, z) = (x_0, y_0, z_0)$ by the implicit mapping theorem. The map-germ $\Psi(x, y, z, t_0)$ is radially equivalent to $\psi(x, y, z) = x^2 + y^2 \pm z^2 + \mathrm{i}z$ if it has a definite fold singularity at $(x, y, z) = (x_0, y_0, z_0)$. The map-germ $\Psi(x, y, z, t_0)$ is radially equivalent to $\psi(x, y, z) = x^2 - y^2 - z^2 + \mathrm{i}z$ if it has an indefinite fold singularity at $(x, y, z) = (x_0, y_0, z_0)$. □

In section 2, we study deformations of phase singularities. For them we observe the following.

Remark 5.4. Let $(u(t, \lambda), w(t, \lambda))$ be a deformation of the curve $(u(t, 0), w(t, 0)) = (t^m, t)$:

$$u(t, \lambda) = \alpha_0(\lambda) + \alpha_1(\lambda)t + \cdots + \alpha_m(\lambda)t^m + \cdots,$$
$$w(t, \lambda) = \beta_0(\lambda) + \beta_1(\lambda)t + \cdots,$$

with $\alpha_0(0) = \cdots = \alpha_{m-1}(0) = \beta_0(0) = 0, \alpha_m(0) = \beta_1(0) = 1$. Then by a family of radial transformations, the family is transformed to

$$u(t, \lambda) = \alpha_0'(\lambda) + \alpha_2'(\lambda)t^2 + \cdots + \alpha_{m-1}'(\lambda)t^{m-1} + t^m,$$
$$w(t, \lambda) = \beta_0'(\lambda) + t,$$

for some functions $\alpha_0'(\lambda), \alpha_2'(\lambda), \ldots, \alpha_{m-1}'(\lambda), \beta_0'(\lambda)$. The latter curve is expressed as

$$u = t_0(\lambda) + t_1(\lambda)w + \cdots + t_{m-1}(\lambda)w^{m-1} + w^m.$$

By a family of linear transformations $(u, w) \mapsto (u - t_1(\lambda)w, w)$, it is reduced to

$$u = t_0(\lambda) + t_2(\lambda)w^2 + \cdots + t_{m-1}(\lambda)w^{m-1} + w^m.$$

In particular if $t_0(\lambda), t_2(\lambda), \ldots, t_{m-1}(\lambda)$ are independent, we obtain the normal form

$$u = t_0 + t_2w^2 + \cdots + t_{m-1}w^{m-1} + w^m.$$

The family of corresponding phase singularities is given by

$$\psi(x, y) = x^2 \pm y^m + t_{m-1}y^{m-1} + \cdots + t_2y^2 + t_0\mathrm{i}y.$$

Thus we have the result in remark 2.6.

In general, any parametrized curve $(u(t), w(t))$ through the origin in $\boldsymbol{C}$ is equivalent by radial transformations and re-parametrizations to

$$u(t) = t^m + O(t^{m+1}), \qquad w(t) = t^n,$$

for some integers m, n with $m > n$. We have $n = 1$ for regular curves. If $n \geqslant 2$, then we call the curve an (n, m)-*cusp*. A $(2, 3)$-cusp is called a *simple cusp*, briefly a *cusp*.

Lemma 5.5 (The radial classification of simple cusps). Any simple cusp is equivalent by radial transformations and re-parametrizations to

$$u(t) = t^3, \qquad w(t) = t^2.$$

Thus, any two simple cusps are radially equivalent to each other.

Proof of propositions 2.7 and 4.3. Since a Whitney cusp $(\boldsymbol{R}^2, p) \to (\boldsymbol{C}, 0)$ (respectively, a cusp map-germ $(\boldsymbol{R}^3, p) \to (\boldsymbol{C}, 0)$) is a critical normalization and its discriminant is a simple cusp by proposition 5.1, the radial classification of Whitney cusps (respectively, cusp map-germs) is reduced to the radial classification of their discriminants. Therefore lemma 5.5 implies propositions 2.7 and 4.3. □

Proof of lemma 5.5. Let $u(t) = t^3 + O(t^4)$, $w(t) = t^2$ be a simple cusp. By a linear transformation on the (u, w)-plane and a re-parametrization of t, we may suppose the curve is given by $u(t) = t^3$, $w(t) = t^2 + O(t^4)$. Set $w(t) = t^2 a(t)$ for a smooth function $a(t)$. Then $a(0) = 1$, $a'(0) = 0$. Then there exists a smooth function $\rho(x, y)$ such that $a(t) = \rho(t^2, t^3)$ by the preparation theorem [15, 20]. Then the curve is radially equivalent to the curve

$$u(t) = \frac{1}{\rho(t^2, t^3)^3} t^3, \qquad w(t) = \frac{1}{\rho(t^2, t^3)^2} t^2,$$

which is radially equivalent to $u(t) = t^3$, $w(t) = t^2$. □

Remark 5.6. Let $C_{a,b}(t) = (u(t, a, b), w(t, a, b))$ be a generic two parameter family of simple cusps. For each (a, b), we draw tangent lines to the simple cusp $C_{a,b}$ from the origin. Then there exists a non-void open subset U such that for $(a, b) \in U$, there are exactly three tangent rays. By the assignment of the corresponding tangent points, we have three functions $\lambda_1, \lambda_2, \lambda_3$ on U; $C_{a,b}(\lambda_1)$, $C_{a,b}(\lambda_2)$, $C_{a,b}(\lambda_3)$ are tangent points. Thus we have a triple of foliations:

$$\lambda_1 = \text{const}, \qquad \lambda_2 = \text{const}, \qquad \lambda_3 = \text{const},$$

that is, a 3-web on U. Moreover, radially equivalent families of simple cusps have isomorphic 3-webs. It is known that the classification of 3-webs has function moduli in general [11].

Remark 5.7. The radial classification of $(3, 4)$-cusps turns out to be different from the right–left classification. For example, the curves

$$E_{6,1}: \quad u(t) = t^4 + t^5, \qquad w(t) = t^3$$

and

$$E_{6,0}: \quad u(t) = t^4, \qquad w(t) = t^3$$

are right–left equivalent. However they are not radially equivalent. In fact, as is well known, any $(3, 4)$-cusps are right–left equivalent to $E_{6,0}$ (see, for instance, [6, 16]). Suppose they are radially equivalent, then we would have

$$(\sigma(t))^4 + (\sigma(t))^5 = \rho(t^4, t^3)(at^4 + bt^3), \qquad (\sigma(t))^3 = \rho(t^4, t^3)(ct^4 + dt^3)$$

for some diffeomorphism-germ $\sigma(t)$, a non-zero smooth function $\rho(u, w)$ and $a, b, c, d \in \boldsymbol{R}$ with $ad - bc \neq 0$. Then, comparing the third, fifth and sixth order terms of the first equation and the fifth order terms of the second equation, we are easily led to a contradiction.

6. Helmholtz jet space and transversality

We introduce the notion of Helmholtz jet spaces and show the transversality theorem in a Helmholtz jet space, as one of the main ideas to show the results in this paper. Note that, in [17], analogous jet spaces are considered for other kinds of Monge–Ampère equations.

Consider the Taylor expansion around $(x, y) = (x_0, y_0)$ of a complex valued function ψ on the (x, y)-plane:

$$\psi(x, y) = a + bX + cY + \frac{e}{2}X^2 + fXY + \frac{g}{2}Y^2 + \frac{h}{6}X^3 + \frac{k}{2}X^2Y + \frac{\ell}{2}XY^2 + \frac{m}{6}Y^3 + \cdots.$$

Here we set $X = x - x_0$, $Y = y - y_0$, and $a, b, c, \ldots$ are complex numbers.

Suppose ψ is a Helmholtz function for $k = 1$, that is, ψ satisfies the Helmholtz equation $\psi_{xx} + \psi_{yy} + \psi = 0$. Then we have

$$e + g + a = 0, \qquad h + \ell + b = 0, \qquad k + m + c = 0.$$

Therefore, we have

$$\begin{aligned}\psi(x, y) = a + bX + cY + \frac{e}{2}X^2 + fXY - \frac{1}{2}(a+e)Y^2 \\ + \frac{h}{6}X^3 + \frac{k}{2}X^2Y - \frac{1}{2}(b+h)XY^2 - \frac{1}{6}(c+k)Y^3 + \cdots.\end{aligned}$$

The Taylor expansion of a function ψ up to order r around a point (x_0, y_0) of $\boldsymbol{R}^2$ is called the *r-jet* of ψ at (x_0, y_0) and denoted by $j^r\psi(x_0, y_0)$. Denote by $J^r(\boldsymbol{R}^2, \boldsymbol{C})$ the space of r-jets of complex valued functions on $\boldsymbol{R}^2$. In it, we denote by $J^r_{\mathrm{Helm}}(\boldsymbol{R}^2, \boldsymbol{C})$ the set of r-jets of planar Helmholtz functions for $k = 1$:

$$J^r_{\mathrm{Helm}}(\boldsymbol{R}^2, \boldsymbol{C}) = \{j^r\psi(x_0, y_0) \mid \psi_{xx} + \psi_{yy} + \psi = 0 \text{ around } (x_0, y_0)\}.$$

We call it the *Helmholtz r-jet space*. For example, $J^3_{\mathrm{Helm}}(\boldsymbol{R}^2, \boldsymbol{C})$ is identified with $\boldsymbol{R}^{16} = \boldsymbol{R}^2 \times \boldsymbol{C} \times \boldsymbol{C}^6$ with coordinates x_0, y_0; $a = a_1 + \mathrm{i}a_2$; $b = b_1 + \mathrm{i}b_2$, $c = c_1 + \mathrm{i}c_2$, $e = e_1 + \mathrm{i}e_2$, $f = f_1 + \mathrm{i}f_2$, $h = h_1 + \mathrm{i}h_2$ and $k = k_1 + \mathrm{i}k_2$.

In general, the Taylor expansion of a Helmholtz function $\psi(x, y)$ defined around (x_0, y_0) is determined by $\psi(x, y_0)$ and $\psi_y(x, y_0)$. Moreover, for any given complex valued analytic functions $\psi_0(x)$ and $\psi_1(x)$ defined around x_0, there exists uniquely a complex valued function $\psi(x, y)$ defined around (x_0, y_0) satisfying the Helmholtz equation, $\psi(x, y_0) = \psi_0(x)$ and $\psi_y(x, y_0) = \psi_1(x)$. The r-jet of ψ at (x_0, y_0) is determined by the r-jet of $\psi_0(x)$ at x_0 and $(r-1)$-jet of $\psi_1(x)$ at x_0. Thus, $J^r_{\mathrm{Helm}}(\boldsymbol{R}^2, \boldsymbol{C})$ is identified with $\boldsymbol{R}^N$ for some natural number N. With any Helmholtz function ψ defined around (x_0, y_0), there is associated a mapping

$$j^r\psi\colon (\boldsymbol{R}^2, (x_0, y_0)) \to J^r_{\mathrm{Helm}}(\boldsymbol{R}^2, \boldsymbol{C})$$

defined by taking the r-jet of ψ at (x, y) for each (x, y) near (x_0, y_0). It is called the *r-jet extension* of ψ. Moreover, to any family $\Psi(x, y, \lambda)\colon \boldsymbol{R}^2 \times \boldsymbol{R}^\ell \to \boldsymbol{C}$ of Helmholtz functions, there corresponds a mapping

$$j^r\Psi\colon (\boldsymbol{R}^2 \times \boldsymbol{R}^\ell, (x_0, y_0, \lambda_0)) \to J^r_{\mathrm{Helm}}(\boldsymbol{R}^2, \boldsymbol{C})$$

by taking the r-jet of $\Psi(x, y, \lambda')$ at (x, y) for each (x, y) near (x_0, y_0) and parameter λ' near λ_0.

By a similar proof to that of the ordinary transversality theorem [15], we have lemma 6.1.

Lemma 6.1. *Suppose a finite number of submanifolds $W_1, W_2, \ldots$ of Helmholtz r-jet space $J^r_{\mathrm{Helm}}(\boldsymbol{R}^2, \boldsymbol{C})$ are given. Then, any Helmholtz function $\psi(x, y)$ defined around (x_0, y_0) is approximated (in C^∞ topology) by a Helmholtz function $\tilde{\psi}(x, y)$ defined around (x_0, y_0) such that the r-jet extension of $\tilde{\psi}(x, y)$ is transversal to any W_i. Moreover, any Helmholtz wave $\Psi(x, y, t)$ is approximated by a Helmholtz wave $\widetilde{\Psi}(x, y, t)$ such that $j^r\Psi$ is transversal to any W_i.*

By using lemma 6.1, we show theorems 3.3 and 3.4. In $J^3_{\mathrm{Helm}}(\boldsymbol{R}^2, \boldsymbol{C})$ with coordinates x_0, y_0; $a = a_1 + \mathrm{i}a_2$; $b = b_1 + \mathrm{i}b_2$, $c = c_1 + \mathrm{i}c_2$, $e = e_1 + \mathrm{i}e_2$, $f = f_1 + \mathrm{i}f_2$, $h = h_1 + \mathrm{i}h_2$ and $k = k_1 + \mathrm{i}k_2$, we set

$$W_1^6 = \{a = 0,\ b = 0,\ c = 0\},$$

which is of codimension 6. Moreover, we define a submanifold W_3^4 of codimension 4 by the equations

$$\begin{vmatrix} \begin{vmatrix} e_1 & c_1 \\ e_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_1 & f_1 \\ b_2 & f_2 \end{vmatrix} & \begin{vmatrix} f_1 & c_1 \\ f_2 & c_2 \end{vmatrix} - \begin{vmatrix} b_1 & a_1 + e_1 \\ b_2 & a_2 + e_2 \end{vmatrix} \\ b_1 & c_1 \end{vmatrix} = 0$$

and

$$\begin{vmatrix} \begin{vmatrix} e_1 & c_1 \\ e_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_1 & f_1 \\ b_2 & f_2 \end{vmatrix} & \begin{vmatrix} f_1 & c_1 \\ f_2 & c_2 \end{vmatrix} - \begin{vmatrix} b_1 & a_1 + e_1 \\ b_2 & a_2 + e_2 \end{vmatrix} \\ b_2 & c_2 \end{vmatrix} = 0,$$

together with $a = 0$, $b_1c_2 - b_2c_1 = 0$, minus a locus W_2^5 of more degenerate singularities, which is of codimension $\geqslant 5$. The definition of W_3^4 is from the idea of the iterated Jacobian [13]. Further, we set

$$W_4^3 = \{a = 0,\ b_1c_2 - b_2c_1 = 0\} \setminus (W_1^6 \cup W_2^5 \cup W_3^4),$$

which is of codimension 3, and

$$W_5^2 = \{a = 0\} \setminus (W_1^6 \cup W_2^5 \cup W_3^4 \cup W_4^3),$$

which is of codimension 2.

From lemma 6.1, any Helmholtz function $\psi(x, y)$ is approximated to a Helmholtz function whose r-jet extension is transversal to the above submanifolds in $J^3_{\mathrm{Helm}}(\boldsymbol{R}^2, \boldsymbol{C})$. This implies the following from the Whitney theory [15]. The transversality of $j^3\psi$ at (x_0, y_0) to W_5^2 implies that ψ has the regular phase singularity at (x_0, y_0). Similarly, the transversality to W_4^3 implies the fold singularity and W_3^4 the cusp singularity as a mapping from a plane to a plane. When a function has fold singularity, generically, there are two possibilities of phase singularities: hyperbolic and elliptic singularities. However, from proposition 3.2, there is no elliptic phase singularity for Helmholtz functions. This shows theorem 3.3. Furthermore, lemma 6.1 claims that the transversality theorem holds even for Helmholtz waves. Therefore, theorem 3.4 is proved in the same way as above.

Remark 6.2. The transversality to W_2^5 and W_1^6 means that $j^3\psi$ does not intersect W_2^5 and W_1^6 for the two-parameter family of Helmholtz functions.

7. Bifurcation problem of phase singularities of nonlinear Schrödinger waves

We can apply our method to study phase singularities appearing in nonlinear waves. Here we actually treat local nonlinear Schrödinger analytic waves or formal waves. A huge number of numerical and experimental results are appearing in various fields (see for instance [22]). However no rigorous mathematical argument from singularity theory seems to be applied yet. Then here we apply, as an example, our basic method introduced in this paper to the study of generic bifurcations of their phase singularities. Note that we also need other methods for the study of global structure of phase singularities of nonlinear waves due to the existence of soliton solutions (see for instance [19]).

Let

$$\mathrm{i}\Psi_t + \tfrac{1}{2}\Psi_{xx} + f(x,\Psi) = 0$$

be a Schrödinger equation for a complex valued function $\Psi = \Psi(t,x)$. Here f is a complex valued real analytic function on $\boldsymbol{R} \times \boldsymbol{C}$, for instance, $f(x,\Psi) = |\Psi|^2\Psi$.

Set $\Psi = u + \mathrm{i}w$. Then, in the case $f(x,\Psi) = |\Psi|^2\Psi$ the equation reads

$$\begin{aligned} u_{xx} &= 2w_t - 2(u^2+w^2)u, \\ w_{xx} &= -2u_t - 2(u^2+w^2)w. \end{aligned}$$

Let us denote by

$$J^r_S(\boldsymbol{R}^2,\boldsymbol{C}) := \{j^r\Psi(t_0,x_0) \mid \mathrm{i}\Psi_t + \tfrac{1}{2}\Psi_{xx} + f(x,\Psi) = 0\}$$

the *Schrödinger jet space*. Then we see $J^r_S(\boldsymbol{R}^2,\boldsymbol{C})$ is a submanifold of $J^r(\boldsymbol{R}^2,\boldsymbol{C})$.

In $J^r_S(\boldsymbol{R}^2,\boldsymbol{C})$, the condition $\Psi = 0$ gives a smooth submanifold of $J^r_S(\boldsymbol{R}^2,\boldsymbol{C})$ of codimension 2. Therefore, generically, the phase singularities appear on the (t,x)-plane at isolated points and, on the x-line, the phase singularities appear momentarily. The fold singularities form a submanifold of codimension 3 in $J^2_S(\boldsymbol{R}^2,\boldsymbol{C})$. The fold locus is tangent to the x-line in codimension 4.

The condition $\Psi = \Psi_x = 0$ also gives a smooth submanifold of $J^r_S(\boldsymbol{R}^2,\boldsymbol{C})$ of codimension 4. Therefore, degenerate phase singularities appear in a generic two-parameter family of solutions, where $\Psi = \Psi_x = 0$. Consider the condition $\Psi = \Psi_x = \Psi_{xx} = 0$. If $\Psi(t_0,x_0) = \Psi_x(t_0,u_0) = 0$, then the condition $\Psi_{xx}(t_0,x_0) = 0$ is equivalent to that $\mathrm{i}\Psi_t(t_0,x_0) + f(x_0,0) = 0$. If $f(x_0,0) = 0$, for instance if $f(x,\Psi) = |\Psi|^2\Psi$, then $\Psi_{xx}(t_0,x_0) = 0$ if and only if $\Psi_t(t_0,x_0) = 0$. Thus we conclude that any bifurcation on t of phase singularity with $\Psi(t_0,x_0) = \Psi_x(t_0,x_0) = \Psi_{xx}(t_0,x_0) = 0$ is necessarily degenerate $(\Psi_t(t_0,x_0) = 0)$.

8. Several remarks and open questions

8.1. Degenerate phase singularities

We have classified map-germs from the plane $\boldsymbol{R}^2$ to the complex plane $\boldsymbol{C}$ of low codimension in section 2. They are of corank one, that is, the linear term has rank one. A map-germ of corank 2 appears generically in a four-parameter family. Its singular value hits the origin generically in a six-parameter family; therefore, a phase singularity of corank 2 appears in a six-parameter family generically.

The classification of plane-to-plane singularities after [31] is given in, for instance, [13, 27, 28], under the right–left equivalence. Then it would be fruitful to classify degenerate map-germs $\boldsymbol{R}^2 \to \boldsymbol{C}$ under the radial equivalence, refining those of right–left classification. In remark 5.7, we remark that even the radial classification of (3, 4)-cusps is not a trivial problem.

8.2. Topological classification

A topological radial transformation $\tau\colon(\boldsymbol{C},0) \to (\boldsymbol{C},0)$ is a homeomorphism preserving the foliation defined by radial lines $\{\theta = \text{constant}\}$. The topological radial equivalence of phase singularities is defined by using topological radial transformations on the target and homeomorphisms on the source.

The classifications by radial transformations and by topological radial transformations agree in simpler cases. The question is the difference between two classifications for degenerate singularities. We note that the two curves $E_{6,1}$ and $E_{6,0}$ from remark 5.7 in section 5 are not radially equivalent but topologically radially equivalent. In fact, let $\rho(t)$ be the function satisfying $\rho(t) + t(\rho(t))^2 = 1$. Then the map-germ $\tau: (\boldsymbol{C}, 0) \to (\boldsymbol{C}, 0)$ defined by $\tau(u, w) = (\rho(w^{1/3})^3 u, \rho(w^{1/3})^3 w)$ is a topological radial transformation. Then τ and $\sigma: (\boldsymbol{R}, 0) \to (\boldsymbol{R}, 0)$ defined by $\sigma(t) = t\rho(t)$ give the topological radial equivalence between $E_{6,1}$ and $E_{6,0}$.

8.3. Algebraic radial codimension of phase singularities

Let $\mathcal{E}_{\boldsymbol{C}}(n)$ be the ring of germs of complex valued functions (or locally defined complex valued functions) on $\boldsymbol{R}^n$ at the origin. Let $\psi(\boldsymbol{x}) \in \mathcal{E}_{\boldsymbol{C}}(n)$. Then, we define the *radial codimension* of ψ (or briefly, the *codimension* of ψ) by

$$\operatorname{codim}(\psi) := \dim_{\boldsymbol{R}} \frac{\mathcal{E}_{\boldsymbol{C}}(n)}{\mathcal{R}_{\boldsymbol{C}}(n)},$$

where $\mathcal{R}_{\boldsymbol{C}}(n)$ is the subspace of $\mathcal{E}_{\boldsymbol{C}}(n)$ consisting of functions of the form

$$\sum_{i=1}^{n} a_i(\boldsymbol{x}) \frac{\partial \psi}{\partial x_i} + b(\psi(\boldsymbol{x}))\psi(\boldsymbol{x}) + C\psi(x, y),$$

where $a_i(x_1)$ is a real valued function germ on $\boldsymbol{R}^n$ at $(0, 0)$, $b(u, w)$ a real valued function germ on $\boldsymbol{C}$ at 0 and C a real 2×2-matrix. A real 2×2-matrix $C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ acts on $\psi = u(\boldsymbol{x}) + \mathrm{i}w(\boldsymbol{x})$ by

$$C\psi = (\alpha u(\boldsymbol{x}) + \beta w(\boldsymbol{x})) + \mathrm{i}(\gamma u(\boldsymbol{x}) + \delta w(\boldsymbol{x})).$$

For example, the hyperbolic singularity $\psi : u = \frac{1}{2}(x^2 - y^2)$, $w = y$, $n = 2$, has the radial codimension 1. In fact the quotient space $\mathcal{E}_{\boldsymbol{C}}(2)/\mathcal{R}_{\boldsymbol{C}}(2)$ is spanned by the class of $1 \in \mathcal{E}_{\boldsymbol{C}}(2)$.

To improve the algebraic theory of phase singularities as above is one of the important technical open problems in singularity theory.

8.4. Phase-amplitude singularities

We can also treat *phase-amplitude singularities* based on a finer equivalence than the radial equivalence: a momentary complex wave $\Psi(x, y, t_0)$ can be locally classified by local transformations on the (x, y)-plane and phase-amplitude transformations on $\boldsymbol{C}$. A *phase-amplitude transformation* is a local diffeomorphism on $\boldsymbol{C}$ defined near the origin preserving the lines of polar coordinates, namely, both the equi-phase lines $\{\theta = \text{const}\}$ and the equi-amplitude lines $\{r = \text{const}\}$. In fact, a local diffeomorphism $\tau(u, w) = (U, W)$ is a phase-amplitude transformation if and only if

$$U = \rho(u, w)(au + bw), \qquad W = \rho(u, w)(cu + dw),$$

for a positive function $\rho(u, w)$ and a real 2×2 *orthogonal* matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that a phase-amplitude transformation is a radial transformation.

Thus the *phase-amplitude equivalence* is defined naturally by using phase-amplitude transformations. The classification of phase singularities under the phase-amplitude equivalences remains open.

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