Markov Chains, Graph Spectra, and Some Static/Dynamic Scaling Limits

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§1 Introduction — algebraic(-combinatoric) vs random structures

§2 Cut-off Phenomenon and Asymptotic Spectral Analysis

§3 Markov Chains on Young Diagrams

§1 Introduction

Interplay between randomness and algebraic(-combinatoric) structure

Algebraic structure plays twofold essential roles:

- produce specific randomness
- give nice tools for analyzing random phenomena

— frameworks of harmonic analysis

(Bose-Mesner algebra, symmetric functions, Kerov-Olshanski algebra, ...)

As probability model,

temporally homogeneous Markov chain on a finite set

(quite simple!)

- asymptotic behavior as time (step) $\to \infty$

  recurrence, convergence to invariant distribution, ...

- asymptotic behavior as time $\to \infty$ and size of state space $\to \infty$

  appropriate scaling in time/space

- asymptotic behavior as size of state space $\to \infty$

1. Cut-off phenomenon — critical phenomenon in highly symmetric Markov chain (on group)
2. Interface evolution — Markov chain on Young diagrams (dual object of symmetric group)

Both probabilistic models show

macroscopic deterministic aspect (law of large numbers)

$+$ fluctuation (central limit theorem)

§2 Cut-off Phenomenon and Asymptotic Spectral Analysis

§2.1 Markov chain

§2.2 Cut-off phenomenon I: Hamming graph

§2.3 Random walk on association scheme

§2.4 Cut-off phenomenon II

§2.5 Asymptotic spectral analysis (static model) via quantum decomposition

§3 Markov Chains on Young Diagrams

§3.1 Young graph

§3.2 Restriction-induction chain

§3.3 Irreducible characters of $\mathfrak{S}_n$

§3.4 Kerov-Olshanski algebra

§3.5 Limit shape (static model)

§3.6 Interface evolution

§2.1 Markov chain

For finite set S, given

- transition probability $p(x,y)$ $(x, y \in S)$: $p(x,y) \geqq 0$, $\sum_{y \in S} p(x,y) = 1$
- initial distribution $\nu(x) \geqq 0$ $(x \in S)$: $\sum_{x \in S} \nu(x) = 1$

Then, there exist probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of random variables $(X_n)_{n=0,1,2,\cdots}$ $(X_n : \Omega \longrightarrow S)$ s.t.

$$\mathbb{P}(X_{n+1} = y \,|\, X_n = x) = p(x,y), \quad \mathbb{P}(X_0 = x) = \nu(x), \qquad x, y \in S$$

(temporally homogeneous Markov chain on S)

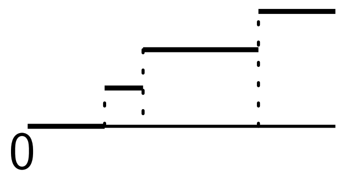
$P = \big(p(x,y)\big)_{x,y \in S}$: transition matrix, $\nu = \big(\nu(x)\big)_{x \in S}$: initial row vector

$$p_n(x,y) = \mathbb{P}(X_n = y \,|\, X_0 = x) = (P^n)_{x,y}, \qquad \mathbb{P}(X_n = x) = (\nu P^n)_x$$

Continuous time Markov chain $(\tilde{X}_s)_{s \geqq 0}$ on S : $\tilde{X}_s = X_{N_s}$

$(N_s)_{s \geqq 0}$: Poisson process, $N_0 = 0$ a.s.

$N_s : \Omega' \longrightarrow \{0, 1, \cdots\}$ for some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) =$



$$\mathbb{P}'(N_s = n) = \frac{e^{-s} s^n}{n!}$$

fig. 1 sample path of Poisson process

$(\Omega, \mathcal{F}, \mathbb{P}) \times (\Omega', \mathcal{F}', \mathbb{P}')$ so that (X_n) and (N_s) are independent

$$\tilde{\mathbb{P}}(\tilde{X}_s = x) = \sum_{n=0}^{\infty} \tilde{\mathbb{P}}(X_{N_s} = x,\, N_s = n) = \sum_{n=0}^{\infty} \tilde{\mathbb{P}}(X_n = x \,|\, N_s = n) \tilde{\mathbb{P}}(N_s = n)$$

$$= \sum_{n=0}^{\infty} (\nu P^n)_x \frac{e^{-s} s^n}{n!} = \left(\nu e^{s(P-I)}\right)_x, \qquad x \in S$$

▷ Ehrenfests' urn (extended)

Imagine n urns and d balls put in them. At each step, pick up a ball among d at random and move it into another urn chosen at random.

$S = \{1, 2, \cdots, n\}^d \ni x,\ y$ ($x = (x_i)$ indicates ith ball is in x_ith urn)

$$p(x, y) = \begin{cases} 1/\,d(n-1) & \text{if } x \text{ and } y \text{ differ at just 1 entry} \\ 0 & \text{otherwise} \end{cases}$$

▷ Bernoulli-Laplace diffusion

Imagine two rooms separated by a partition, one containing d particles and the other $v - d$. At each step, pick up a particle at random from each room and interchange the two.

$S = \big\{ d\text{-subset of } \{1, 2, \cdots, v\} \big\} \ni x,\ y$

$$p(x, y) = \begin{cases} 1/\,d(v-d) & \text{if } x \text{ and } y \text{ have } d-1 \text{ common elements} \\ 0 & \text{otherwise} \end{cases}$$

§2.2 Cut-off phenomenon I: Hamming graph

Illustrate the cut-off phenomenon — certain critical phenomenon for Markov chain in which the process of convergence to stationarity is remarkable.

Ehrenfests' urn (simple random walk on Hamming graph) is a perfect model!

Hamming graph $H(d,n)$:

vertex sets $S = \{1, 2, \cdots, n\}^d$

For $x = (x_i)$, $y = (y_i) \in S$, $\partial(x,y) = \sharp$ of (i's s.t. $x_i \neq y_i$).

adjacency matrix $A_{x,y} = \begin{cases} 1, & \partial(x,y) = 1 \\ 0, & \partial(x,y) \neq 1 \end{cases}$,

valency $\kappa = d(n-1)$

transition matrix $P = \dfrac{1}{\kappa} A$

$\Longrightarrow$ simple random walk on S with uniform invariant distribution

For continuous time simple random walk on $H(d,n)$,

$(e^{s(P-I)})_{x,\,\cdot}$: distribution at time s starting from x

total variation distance between distributions at time s and ∞

$$
\begin{aligned}
D^{(d,n)}(s) &= \frac{1}{2} \Big\| (e^{s(P-I)})_{x,\,\cdot} - \text{(uniform distribution)} \Big\|_{\mathrm{tot}} \\
&= \frac{1}{2} \sum_{y \in S} \Big| (e^{s(P-I)})_{x,y} - \frac{1}{n^d} \Big|
\end{aligned}
$$

(independent of starting vertex x)

$$D(0) = 1 - \frac{1}{n^d} \approx 1, \quad D(\infty) = 0$$

Theorem (Diaconis-Graham-Morrison 1990)

For simple random walk on $H(d,2)$

$$D^{(d,2)}\left(\frac{1}{4}d(\log d+\tau)\right) \xrightarrow[d\to\infty]{} \sqrt{\frac{2}{\pi}}\int_0^{\frac{1}{2}e^{-\tau/2}} e^{-x^2/2}dx = c(\tau), \qquad \tau\in\mathbb{R}$$

holds. ■

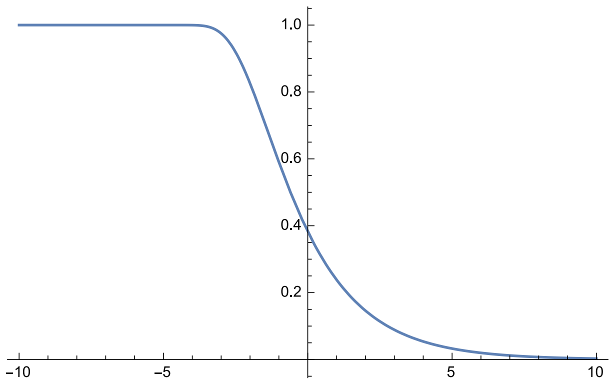


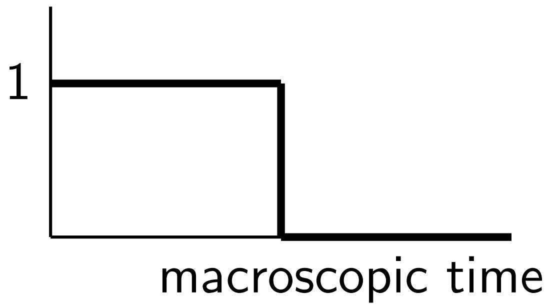
fig. 2 graph of $c(\tau)$

For $\forall\epsilon>0$, $\exists\tau_\epsilon$ s.t. $c(\tau)\begin{cases}>1-\epsilon, & \tau<-\tau_\epsilon\\ <\epsilon, & \tau_\epsilon<\tau\end{cases}$ since $c(\mp\infty)=\begin{cases}1\\0\end{cases}$

Therefore

$$D^{(d,2)}\left(\frac{1}{2}\,\frac{d}{2}\,\log d+\frac{\tau}{2}\,\frac{d}{2}\right)\ \begin{cases} > 1-\epsilon & \text{if } \tau < -\tau_\epsilon \\ < \epsilon & \text{if } \tau_\epsilon < \tau \end{cases}$$

where inverse of spectral gap and multiplicity of 2nd eigenvalue of $\dfrac{1}{\kappa}A$



time axis rescaled by $d \log d$

fig. 3 graph of D in macroscopic time scale

▶ macro time $\ll$ fluctuation d $\ll$ micro time $d \log d$

$\ll$ mean recurrence time 2^d

§2.3 Random walk on association scheme

large multiplicity (degeneration) of 2nd eigenvalue of transition matrix

$\Longleftarrow$ high symmetry for Markov chain $\Longleftarrow$ "random walk"

Let group G act on S transitively, $S \cong G/K$, and

$$p(gx, gy) = p(x, y), \qquad x, y \in S,\ g \in G.$$

Then $\exists \mu \in \mathcal{P}(K \backslash G/K)$ s.t. $P = \mu * \cdot$ (convolution operator), i.e.

the Markov chain is product of independent G-valued random variables with K-bi-invariant distribution

" random walk $\Longleftrightarrow$ spatially symmetric Markov chain"

Natural and fruitful extension is

" random walk $\Longleftrightarrow$ transition matrix belongs to Bose-Mesner algebra of association scheme" finite set S, $S \times S \supset R_i$ $(i = 0, 1, \cdots, d)$

$$i\text{th adjacency matrix} \quad (A_i)_{x,y} = \begin{cases} 1, & (x.y) \in R_i \\ 0, & (x,y) \notin R_i \end{cases}$$

$(S, \{R_i\}_{i=0}^{d})$ is called an association scheme if

(i) $A_0 = I$ (identity matrix), $A_1 + \cdots + A_d = J$ (all entries 1)

(ii) $\forall i$, $\exists i'$ s.t. ${}^t A_i = A_{i'}$

(iii) $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$, $\quad p_{ij}^k \in \mathbb{Z}_{\geqq 0}$: intersection number

$p_{ii'}^0 = \sharp\{y \in S \,|\, (x,y) \in R_i\} = \kappa_i$: ith valency (independent of x)

Furthermore

(iv) $A_i A_j = A_j A_i$: commutative $\qquad$ (v) ${}^t A_i = A_i$: symmetric

$\mathcal{A} = \{$linear combination of $A_0, \cdots, A_d\}$: Bose-Mesner algebra

Markov chain on S is called random walk if transition matrix $P \in \mathcal{A}$

(S,E): finite graph with graph distance ∂, diameter $d = \max_{x,y\in S} \partial(x,y)$

$R_i = \{(x,y) \in S \times S \,|\, \partial(x,y) = i\}$

(S,E) is called distance-regular graph if $(S, \{R_i\}_{i=0}^{d})$ is an association scheme. Then A_i is expressed as polynomial of A_1 (P-polynomial).

Markov chain on S is called simple random walk if transition matrix $P = A/\kappa$

$A = A_1$: adjacency matrix, $\kappa = \kappa_1$: valency (degree).

- Hamming graph $H(d,n)$ see §2.2
- Johnson graph $J(v,d)$: $S = \{d\text{-subset of a } v\text{-set}\} \ni x, y$

  $\partial(x,y) = d - \sharp(x \cap y)$

In commutative association scheme, simultaneously diagonalize A_i's by family of projections $\{E_0, E_1, \cdots, E_d\}$, $E_0 = J/|S|$

$$(A_0 \cdots A_d) = (E_0 \cdots E_d)\mathrm{P}, \qquad \mathrm{P} = \big(p_i(j)\big)_{j,i} \text{ : character table}$$

§2.4 Cut-off phenomenon II

Consider continuous time simple random walk on distance-regular graph S, more precisely, directed family of simple random walks on growing distance-regular graphs

$$D(s) = \frac{1}{2}\left\|(e^{s(P-I)})_{x,\cdot} - \text{(uniform)}\right\|_{\text{tot}} = \frac{1}{2|S|}\sum_{x,y\in S}\left|(e^{s(P-I)} - \frac{1}{|S|}J)_{x,y}\right|$$

▶ $e^{s(P-I)} = I$ at $s = 0$ $\longrightarrow$ $= J/|S|$ at $s = +\infty$

From the argument following Theorem of Diaconis-Graham-Morrison,

Cut-off phenomenon with (macroscopic) critical time s_c

- $s_c \to \infty$ and $s_c/|S| \to 0$
- $\forall\epsilon > 0$, $\exists h_\epsilon$ s.t. $h_\epsilon/s_c \to 0$

$$\inf_{0\leqq s\leqq s_c - h_\epsilon} D(s) \geqq 1-\epsilon, \qquad \sup_{s\geqq s_c + h_\epsilon} D(s) \leqq \epsilon$$

Theorem (2000, formerly DFG-JSPS Proc. 1996)

If a growing family of Q-polynomial distance-regular graphs satisfies certain spectral conditions, simple random walks on them yield cut-off phenomenon with

$$s_c = \frac{1}{2}(1 - \frac{\theta}{\kappa})^{-1} \log m, \quad h_\epsilon \asymp (1 - \frac{\theta}{\kappa})^{-1}.$$

where θ: 2nd eigenvalue and m: its multiplicity of adjacency matrix A. ■

The conditions are far from elegant, however, can be verified for

- $H(d,n)$ under $d \to \infty$ and $n \leqq \mathrm{const.}\, d$
- $J(v,d)$ under $d \to \infty$ and $2d \leqq v \leqq \mathrm{const.}\, d^2$
- q-analogue of them, and many other listed in Bannai-Ito's book

Role of symmetry

• give rise to degeneration of eigenvalues

(eigenspace invariant w.r.t. actions)

• put transition matrix into Bose-Mesner algebra $\Longrightarrow$ functional calculus

(characters, spherical functions, $\cdots$ help diagonalizing transition matrix)

Other models of cut-off phenomenon

★ card shuffling : random walk on symmetric group with various generators

(= various Cayley graphs)

★ framework of hypergroup (finite Gelfand pair, spherical dual) etc.

Theorem (1997)

For simple random walk on $H(d,n)$

- if $n/d \to 0$,

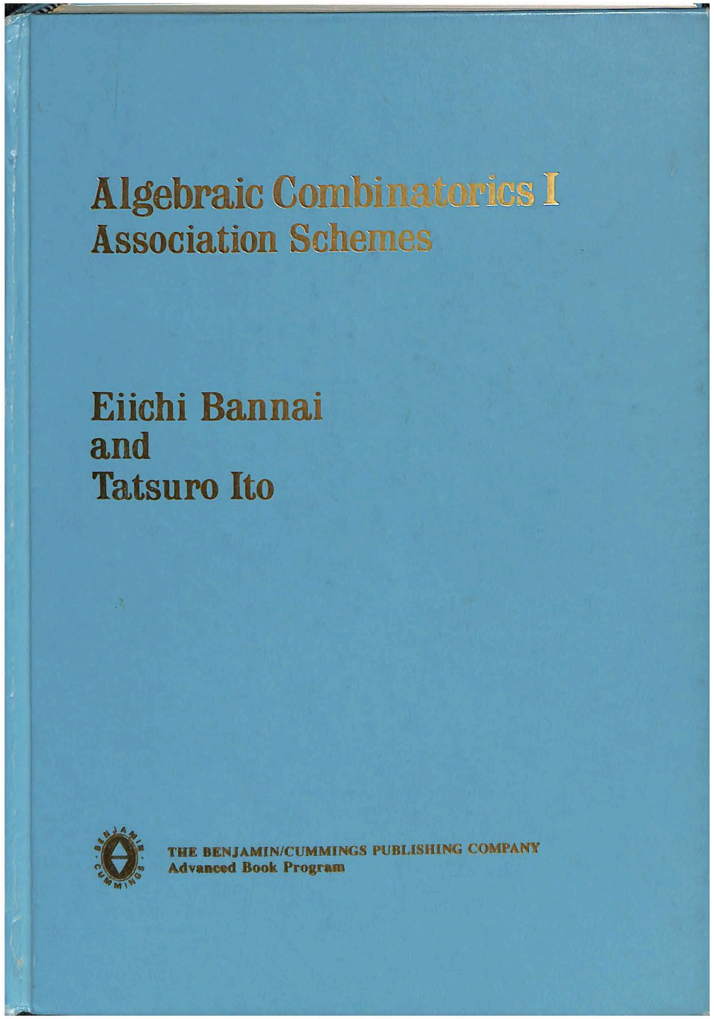
$$D^{(d,n)}\Big(\frac{1}{2}\big(1-\frac{1}{n}\big)d\big(\log(n-1)d+\tau\big)\Big) \xrightarrow[d\to\infty]{} \sqrt{\frac{2}{\pi}}\int_0^{\frac{1}{2}e^{-\tau/2}} e^{-x^2/2}dx$$

- if $n/d \to \alpha \in (0,\infty)$,

$$D^{(d,n)}\Big(\frac{1}{2}\big(1-\frac{1}{n}\big)d\big(\log(n-1)d+\tau\big)\Big) \xrightarrow[d\to\infty]{} \Big\|\mathrm{Poi}(\frac{1}{\alpha})-\mathrm{Poi}(\frac{1}{\alpha}+\frac{e^{-\tau/2}}{\sqrt{\alpha}})\Big\|_{\mathrm{tot}}$$

$(\tau \in \mathbb{R})$ hold. ■

Remark $H(d,n)$: $\kappa = (n-1)d$, $\theta = (n-1)d - n$, $m = (n-1)d$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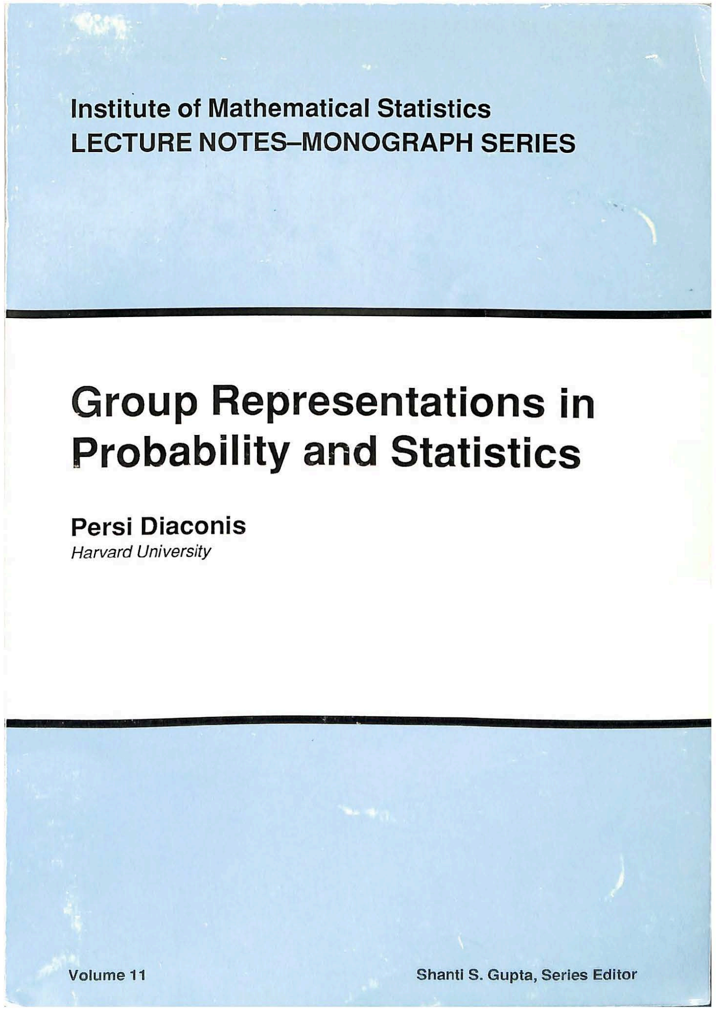
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§2.5 Asymptotic spectral analysis via quantum decomposition

spectrum of transition matrix $P = \dfrac{1}{\kappa} A$ on distance-regular graph

$$\begin{pmatrix} \theta_0(=\kappa) & \theta_1 & \cdots & \theta_d \\ m_0(=1) & m_1 & \cdots & m_d \end{pmatrix}, \qquad \sum_{j=0}^{d} m_j = |S|$$

rth moment of spectral distribution

$$\sum_{j=0}^{d} \theta_j^r \frac{m_j}{|S|} = \frac{1}{|S|} \mathrm{tr} A^r = (A^r)_{x,x} = \phi_0(A^r) \qquad \text{(independent of } x\text{)}$$

in particular $\quad \phi_0(A) = 0, \quad \phi_0(A^2) = \phi_0(\sum_k p_{11}^k A_k) = \kappa$

asymptotic spectral distribution as central limit theorem

$$\phi_0\Big(\big(\frac{1}{\sqrt{\kappa}} A\big)^r \Big) \xrightarrow[d \to \infty]{} \ ? \ = \int_{\mathbb{R}} x^r \mu(dx) = M_r(\mu)$$

Then, for any $r \in \mathbb{N}$

$$\phi_0(A^r) \sim M_r(\mu)\kappa^{r/2} \qquad \text{as} \quad d \to \infty$$

However, for cut-off phenomenon, one estimates $D(s)$ containing

$$e^{s(P-I)} \sum_{n=0}^{\infty} \frac{e^{-s} s^n}{n!} P^n \qquad \text{(Poisson distribution with mean and variance } s\text{)}$$

i.e. $\phi_0(A^s)$ as $d \to \infty$ and $s = s(d) \to \infty$

Central limit theorem for adjacency matrix (static scaling limit) has different nature from cut-off phenomenon (dynamic scaling limit), however,

– applicable ?

– interesting asymptotics itself

Viewpoint of quantum probability

- quantum decomposition $A = A^+ + A^- \ (+A^o)$
  with certain commutation relation
- limit picture drawn by creation/annihilation operators on appropriate Fock space
- other state than (vacuum) ϕ_0

Hashimoto-Obata-Tabei (2001) : for Hamming graph by using Hermite polynomial, Gauss measure, Boson Fock space

Collaboration with Obata school $\cdots$

A. Hora, N. Obata: Quantum Probability and Spectral Analysis of Graphs, Theoretical and Mathematical Physics, Springer, 2007

Scheme of quantum decomposition approach

$$\begin{array}{ccccccc} A^+, A^-, A^o, \phi & \longleftarrow & A\,(= A^+ + A^- + A^o),\ \phi & \underset{(\sharp)}{\longrightarrow} & \phi(A^r) \\ \downarrow \text{limit} & & & & \downarrow \text{limit} \\ B^+, B^-, B^o, \Phi & \longrightarrow & B = B^+ + B^- + B^o,\ \Phi & \underset{(\natural)}{\longrightarrow} & \Phi(B^r) = M_r(\mu) \end{array}$$

- limit + ($\natural$) is much transparent than ($\sharp$) + limit
- ($\natural$) doesn't need full spectral data of A while ($\sharp$) does
- ($\natural$) is often controlled by well-known orthogonal polynomials and one-mode interacting Fock space

Quantum decomposition of adjacency matrix A on graph (S, ∂)

$S \ni o$, $\quad S_n = \{x \in S \mid \partial(o,x) = n\}$: nth stratum

$S = \bigsqcup_{n=0}^{d} S_n \quad$ (d: diameter)

$$A^+\delta_x = \sum_{y:\, x\uparrow y} \delta_y, \quad A^-\delta_x = \sum_{y:\, x\downarrow y} \delta_y, \quad A^o\delta_x = \sum_{y:\, x\to y} \delta_y$$

$\uparrow$: to upper stratum, $\downarrow$: to lower stratum, $\to$: to the same stratum

For distance-regular graph

$$\mathit{\Gamma} = \text{linear hull of } \{\Phi_0, \cdots, \Phi_d\} \subset \ell^2(S), \quad \Phi_n = \frac{1}{\sqrt{|S_n|}} \sum_{x\in S_n} \delta_x$$

is invariant w.r.t. A^+, A^-, A^0, by using intersection numbers p_{ij}^k,

$$A^+\Phi_n = \sqrt{p_{1,n}^{n+1} p_{1,n+1}^n}\,\Phi_{n+1}, \qquad n = 0, 1, 2, \cdots$$

$$A^-\Phi_0 = 0, \quad A^-\Phi_n = \sqrt{p_{1,n-1}^n p_{1,n}^{n-1}}\,\Phi_{n-1}, \qquad n = 1, 2, \cdots$$

$$A^o\Phi_n = p_{1,n}^n \Phi_n, \qquad n = 0, 1, 2, \cdots$$

“ Theorem ”

Convergence of matrix element of any mixed product of A^+, A^-, A^o

$$\left\langle \Phi_n,\ \frac{A^{\epsilon_1}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_p}}{\sqrt{\kappa}} \Phi_m \right\rangle \longrightarrow \langle \Psi_n,\ B^{\epsilon_1} \cdots B^{\epsilon_p} \Psi_m \rangle$$

B^+, B^-, B^o on one-mode interacting Fock space $\bigoplus_{n=0}^{\infty} \mathbb{C}\Psi_n$

Example (Hashimoto-Hora-Obata 2003; 2003, 2004)

simple random walk on Johnson graph $J(v,d)$ $(2d \leqq v)$

$S = \{d\text{-subset of a } v\text{-set}\} \ni x, y,$ $\qquad \partial(x,y) = d - \sharp(x \cap y)$

$\kappa = d(v-d)$

Limit of data of previous page gives Jacobi coefficients of Laguerre and Meixner polynomials, so as corollary,

$$\phi_0\Big(\big(\frac{1}{\sqrt{d(v-d)}}A^{(v,d)}\big)^r\Big) \xrightarrow[d\to\infty]{2d/v\to p} M_r(\mu)$$

where $$\mu = \begin{cases} e^{-(x+1)}\,1_{[-1,\infty)}(x)dx, & p=1 \\ \displaystyle\sum_{j=0}^{\infty}\frac{2(1-p)}{2-p}\big(\frac{p}{2-p}\big)^j\,\delta_{\frac{2(1-p)}{\sqrt{p(2-p)}}(j-\frac{p}{2(1-p)})}, & 0<p<1 \end{cases}$$

Furthermore

Gibbs state with energy depending on distance from origin o

$$\xrightarrow[d\to\infty\text{ (infinite volume)}]{\beta\to\infty\text{ (zero temperature)}}$$

deformed vacuum state on one-mode interacting Fock space

§3.1 Young graph

vertex set : $\mathbb{Y} = \bigsqcup_{n=0}^{\infty} \mathbb{Y}_n, \quad \mathbb{Y}_0 = \{\varnothing\},$ edge : $\lambda \nearrow \mu$

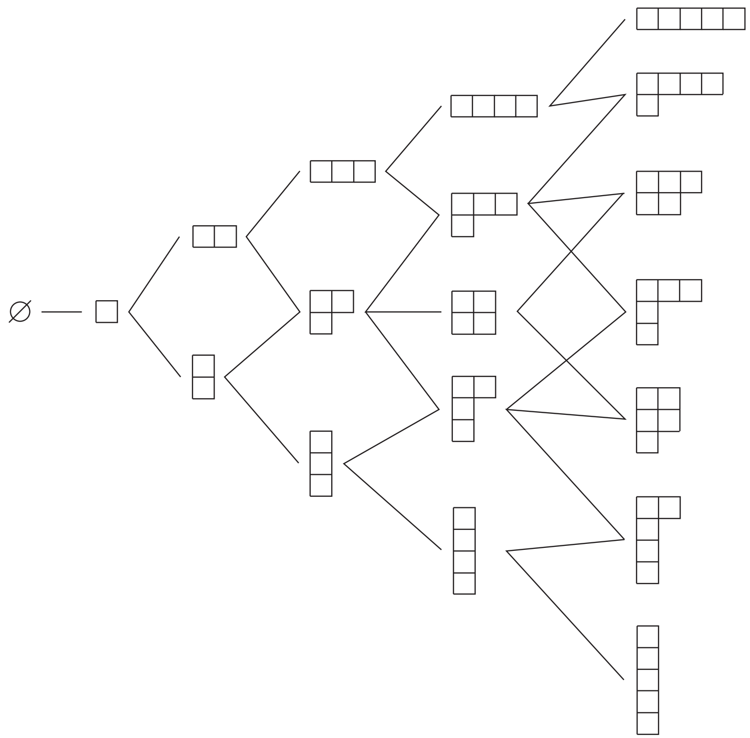


fig. 4 Young graph: dimension in 5th stratum — 1, 4, 5, 6, 5, 4, 1 :
$1^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 1^2 = 5!$

$\mathbb{Y}_n \cong \widehat{\mathfrak{S}_n} \ni \lambda \ni (\pi^\lambda, V^\lambda)$: irreducible representation of $\mathfrak{S}_n$

Irreducible decomposition of restriction/induction of each irreducible representations (branching rule)

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \pi^\lambda \cong \bigoplus_{\nu \in \mathbb{Y}_{n-1}: \nu \nearrow \lambda} \pi^\nu, \qquad \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \pi^\nu \cong \bigoplus_{\mu \in \mathbb{Y}_n : \nu \nearrow \mu} \pi^\mu$$

multiplicity free decomposition $\Longrightarrow$ canonical Young basis of V^λ

$\cong$ {paths from $\varnothing$ to λ}

Theorem!? Irreducible decomposition of representations of a group is a rich source of interesting Markov chains on the dual object of the group.

Proof $\cdots\cdots$ ■

§3.2 Restriction-induction chain

Counting the dimensions of the above irreducible decompositions

(in some sense, putting equal rate for each vector of the Young basis)

$$p^{\downarrow}(\lambda,\nu)=\begin{cases}\frac{\dim\nu}{\dim\lambda}, & \nu\nearrow\lambda,\\ 0, & \text{otherwise}\end{cases}, \qquad p^{\uparrow}(\nu,\mu)=\begin{cases}\frac{\dim\mu}{(|\nu|+1)\dim\nu}, & \nu\nearrow\mu,\\ 0, & \text{otherwise}\end{cases}$$

$$P^{\downarrow}=\big(p^{\downarrow}(\lambda,\nu)\big)_{\lambda,\nu}, \quad P^{\uparrow}=\big(p^{\uparrow}(\nu,\mu)\big)_{\nu,\mu}$$

Res-Ind chain $(X_m^{(n)})_{m=0,1,2,\cdots}$ on $\mathbb{Y}_n$ has transition matrix

$$P^{(n)}=\big(p^{(n)}(\lambda,\mu)\big)_{\lambda,\mu\in\mathbb{Y}_n}\big(=P^{\downarrow}P^{\uparrow}\text{ restricted on }\mathbb{Y}_n\big),$$

$$p^{(n)}(\lambda,\mu)=\sum_{\nu\in\mathbb{Y}_{n-1}:\nu\nearrow\lambda,\nu\nearrow\mu}p^{\downarrow}(\lambda,\nu)p^{\uparrow}(\nu,\mu), \qquad \lambda,\mu\in\mathbb{Y}_n$$

restriction $\leftrightarrow$ removing a box,   induction $\leftrightarrow$ adding a box

restriction-induction $\leftrightarrow$ (non-locally) moving a corner box

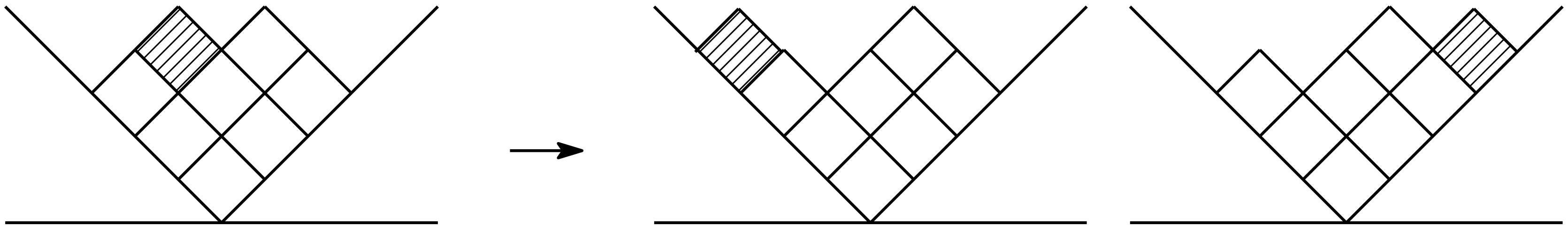


fig. 5   Res-Ind chain: transition from $\lambda = (3, 3, 2)$

Lemma Res-Ind chain is symmetric w.r.t. the Plancherel measure:

$$\mathbb{M}_{\mathrm{Pl}}^{(n)}(\lambda)\, p^{(n)}(\lambda, \mu) = \mathbb{M}_{\mathrm{Pl}}^{(n)}(\mu)\, p^{(n)}(\mu, \lambda), \qquad \lambda, \mu \in \mathbb{Y}_n,$$

hence the Plancherel measure is invariant distribution for Res-Ind chain ■

▷ Plancherel measure on $\mathbb{Y}_n$ is

$$\mathbb{M}_{\mathrm{Pl}}^{(n)}(\lambda) = \frac{(\dim \lambda)^2}{n!}$$

($\leftarrow$ Plancherel formula for Fourier transform on $\mathfrak{S}_n$)

▷ Markov chain (Z_n) on the Young graph with initial distribution $\delta_{\varnothing}$ and transition matrix $P^{\uparrow}$ is called the Plancherel growth process.

The distribution after n step is $\mathbb{P}(Z_n = \lambda) = \frac{(\dim \lambda)^2}{n!} = \mathbb{M}_{\mathrm{Pl}}^{(n)}(\lambda)$

Continuous time Res-Ind chain $\tilde{X}_s^{(n)} = X_{N_s}^{(n)}$ on $\mathbb{Y}_n$ (N_s: Poisson process)

- transition matrix $e^{s(P^{(n)}-I)}$
- invariant distribution $\mathbb{M}_{\mathrm{Pl}}^{(n)}$

§3.3 Irreducible characters of symmetric group

$P^{(n)} = (P^{\downarrow} P^{\uparrow})|_{\mathbb{Y}_n}$: transition matrix of Res-Ind chain on $\mathbb{Y}_n$

Diagonalize $P^{(n)}$ by using irreducible characters of $\mathfrak{S}_n$

(generally available for non-multiplicity-free branching rule also)

For representation (π, V), $\chi(x) = \mathrm{tr}\pi(x)$ $\quad \tilde{\chi} = \chi / \dim V$

$\mathbb{Y}_n$ parametrizes both the equivalence classes of irreducible representations and the conjugacy classes of $\mathfrak{S}_n$

Character table $(\chi^{\lambda}_{\rho})_{\rho, \lambda \in \mathbb{Y}_n}$

Dual approach in asymptotic theory — fix ρ, then $|\lambda| \to \infty$ i.e. consider

$$\tilde{\chi}^{\lambda}_{(\rho, 1^{n-k})}, \qquad \rho \in \mathbb{Y}_k, \quad \lambda \in \mathbb{Y}_n, \quad k \leqq n$$

where $(\rho, 1^{n-k}) = \rho \sqcup (1^{n-k}) \in \mathbb{Y}_n$ so that one can let $n \to \infty$

Lemma For $|\rho| \leqq n$ s.t. $\rho = (1^{m_1(\rho)} 2^{m_2(\rho)} \cdots)$,

$$P^{(n)} \big(\tilde{\chi}^{\lambda}_{(\rho, 1^{n-|\rho|})}\big)_{\lambda \in \mathbb{Y}_n} = \Big(1 - \frac{|\rho| - m_1(\rho)}{n}\Big) \big(\tilde{\chi}^{\lambda}_{(\rho, 1^{n-|\rho|})}\big)_{\lambda \in \mathbb{Y}_n}$$

where $(\,\cdot\,)_{\lambda \in \mathbb{Y}_n}$ is a column vector ■

For transition matrix of continuous time Res-Ind chain,

$$e^{s(P^{(n)} - I)} \big(\tilde{\chi}^{\lambda}_{(\rho, 1^{n-|\rho|})}\big)_{\lambda \in \mathbb{Y}_n} = e^{-(|\rho| - m_1(\rho))s/n} \big(\tilde{\chi}^{\lambda}_{(\rho, 1^{n-|\rho|})}\big)_{\lambda \in \mathbb{Y}_n}$$

Letting ν be an initial distribution on $\mathbb{Y}_n$,

$$\tilde{\mathbb{P}}(\tilde{X}^{(n)}_s = \lambda) = \tilde{\mathbb{P}}^{\tilde{X}^{(n)}_s}(\lambda) = \big(\nu e^{s(P^{(n)} - I)}\big)_{\lambda}, \qquad \lambda \in \mathbb{Y}_n$$

▶ Expectation of irreducible character w.r.t. initial distribution

$\Longrightarrow$ w.r.t. the distribution at time s

§3.4 Kerov-Olshanski algebra

Irreducible characters are (one of) the most important random variables to analyze group-theoretical ensemble of Young diagrams.

For $k = |\rho| \leqq |\lambda| = n$, set a function on $\mathbb{Y}$

$$\Sigma_\rho(\lambda) = n(n-1)\cdots(n-k+1)\tilde{\chi}^\lambda_{(\rho,1^{n-k})} \qquad (= 0 \text{ if } k > n)$$

For one row diagram $\rho = (k)$, $\Sigma_k = \Sigma_{(k)}$

▷ $\mathbb{A} = \{\text{linear combination of } \Sigma_\rho \,|\, \rho \in \mathbb{Y}\}$: Kerov–Olshanski algebra

Considering $\mathbb{A}$ as an algebra of random variables, one can compute many things about random Young diagrams.

Coordinates for a Young diagram $\longrightarrow$ element of $\mathbb{A}$ as a polynomial function

Peak-valley coordinates of $\lambda \in \mathbb{Y}$: $(x_1 < y_1 < x_2 < \cdots < y_{r-1} < x_r)$

$$G_\lambda(z) = \frac{(z-y_1)\cdots(z-y_{r-1})}{(z-x_1)\cdots(z-x_r)} = \frac{\mu_1}{z-x_1} + \cdots + \frac{\mu_r}{z-x_r}$$

Then, $\mu_i > 0$ and $\sum_{i=1}^{r} \mu_i = 1$, so $\mathfrak{m}_\lambda = \sum_{i=1}^{r} \mu_i \delta_{x_i} \in \mathcal{P}(\mathbb{R})$

$\mathfrak{m}_\lambda$: Kerov's transition measure of λ

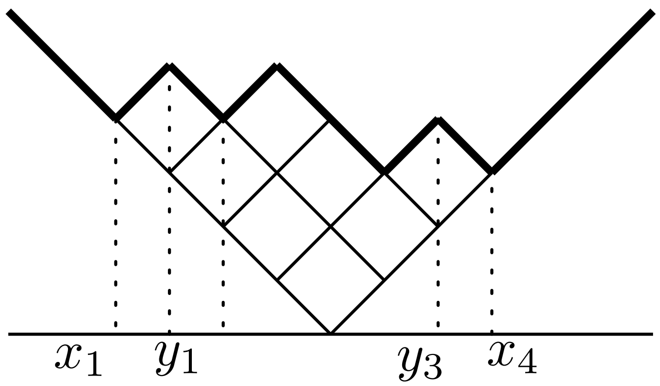


fig. 6 peak-valley coordinates of a Young diagram

$$G_\lambda(z) = \int_{\mathbb{R}} \frac{1}{z-x}\mathfrak{m}_\lambda(dx) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int_{\mathbb{R}} x^n \mathfrak{m}_\lambda(dx), \qquad z \in \mathbb{C}^+$$

Young diagram $\Longleftrightarrow$ peak-valley coordinates

$\Longleftrightarrow$ moment sequence of $\mathfrak{m}_\lambda$: $\{M_n(\mathfrak{m}_\lambda)\}$

$\Longleftrightarrow$ cumulant sequence of $\mathfrak{m}_\lambda$: ordinary $\{C_j(\mathfrak{m}_\lambda)\}$, free $\{R_j(\mathfrak{m}_\lambda)\}$

(polynomial relations by cumulant-moment formula)

▷ $\mathcal{P}(n) = \big\{$partition into subsets of $\{1, 2, \cdots, n\}\big\}$

$\mathcal{P}(n) \ni \pi = \{v_1, \cdots, v_l\}$ (v_i: block in π), $\quad l = b(\pi)$, $\quad \sum_{i=1}^{b(\pi)} |v_i| = n$

$|v_i|$: cardinality of v_i

▷ $\mathcal{NC}(n) = \big\{$non-crossing partition into subsets of $\{1, 2, \cdots, n\}\big\}$

For partition $\pi = \{u_1, \cdots, u_{b(\pi)}\}$ of $\{1, 2, \cdots, n\}$

$$M_{|u_1|} \cdots M_{|u_{b(\pi)}|} = \sum_{\rho=\{v_1,\cdots,v_{b(\rho)}\}\in\mathcal{P}(n):\,\rho\le\pi} C_{|v_1|} \cdots C_{|v_{b(\rho)}|}, \qquad \pi \in \mathcal{P}(n)$$

$$M_{|u_1|} \cdots M_{|u_{b(\pi)}|} = \sum_{\rho=\{v_1,\cdots,v_{b(\rho)}\}\in\mathcal{NC}(n):\,\rho\le\pi} R_{|v_1|} \cdots R_{|v_{b(\rho)}|}, \qquad \pi \in \mathcal{NC}(n)$$

Moebius function of each poset yields inversion respectively, each cumulant expressed by (different) polynomial of moments.

Proposition $\mathbb{A} = \langle \Sigma_k(\lambda) \rangle = \langle M_n(\mathfrak{m}_\lambda) \rangle = \langle C_j(\mathfrak{m}_\lambda) \rangle = \langle R_j(\mathfrak{m}_\lambda) \rangle$ ■

e.g. $\Sigma_1(\lambda) = M_2(\mathfrak{m}_\lambda) = C_2(\mathfrak{m}_\lambda) = R_2(\mathfrak{m}_\lambda) = \dfrac{1}{2}\Big(\sum_{i=1}^{r} x_i^2 - \sum_{i=1}^{r-1} y_i^2\Big)$

Especially, $\{\Sigma_k\}$ vs $\{R_j\}$ is given by Kerov polynomials.

Freeness is a notion for describing relation between random variables.

Free structure often appears in large random matrices/permutations.

In several mathematical contexts,

independence vs freeness for random variables

results in/from interesting contrasts such as

- direct product vs free product (as group or algebra structure)
- lattice vs tree (as Laplacian)
- Gauss vs Wigner (as central limit theorem)
- Boson Fock vs full Fock (as creation and annihilation) etc.

Let a, b be real random variables (typically, self-adjoint elements in function or operator algebra) with distributions μ, ν respectively

$$\mathbb{E}[a^n] = \int_{\mathbb{R}} x^n \mu(dx), \; \mathbb{E}[b^n] = \int_{\mathbb{R}} x^n \nu(dx) \implies \mathbb{E}[(a+b)^n] = \int_{\mathbb{R}} x^n \, ? \, (dx)$$

$a + b \longrightarrow \mu * \nu$	convolution	if a, b are independent
$\longrightarrow \mu \boxplus \nu$	free convolution	if a, b are free

p : projection free to $a \longrightarrow pap$: free compression

$c =$ expectation of $p \in (0, 1)$ i.e. $\mathbb{E}[p] = \mathbb{E}[p^2] = c$

μ_c : distribution of pap (no commutative analogue)

$$\mathbb{E}[(pap)^n] = \int_{\mathbb{R}} x^n \, \mu_c(dx)$$

§3.5 Limit shape (static model)

Putting information on Young diagrams into Kerov-Olshanski algebra, one can compute (scaling limit of) profiles of random Young diagrams.

– macroscopic profile : $1/\sqrt{n}$ both horizontally and vertically

$$\lambda \in \mathbb{Y}_n \quad \longrightarrow \quad \lambda^{\sqrt{n}}(x) = \frac{1}{\sqrt{n}}\lambda(\sqrt{n}x) \quad \in \mathbb{D}_0 \subset \mathbb{D}$$

▷ rectangular diagram

$$\mathbb{D}_0 = \Big\{\lambda : \mathbb{R} \longrightarrow \mathbb{R} \,\Big|\, \text{continuous, piecewise linear,} \\ \lambda'(x) = \pm 1,\ \lambda(x) = |x| \ (|x| \text{ large enough})\Big\}$$

transition measure $\mathfrak{m}_\lambda$ for $\lambda \in \mathbb{D}_0$

▷ continuous diagram

$$\mathbb{D} = \Big\{\omega : \mathbb{R} \longrightarrow \mathbb{R} \,\Big|\, |\omega(x) - \omega(y)| \leqq |x-y|,\ \omega(x) = |x| \ (|x| \text{ large enough})\Big\}$$

▶ Transition measure $\mathfrak{m}_\omega$ for $\omega \in \mathbb{D}$ is defined by approximating ω by elements of $\mathbb{D}_0$

$$\Omega(x) = \begin{cases} \frac{2}{\pi}\left(x \arcsin\frac{x}{2} + \sqrt{4-x^2}\right), & |x| \leqq 2 \\ |x|, & |x| > 2 \end{cases}$$ limit shape

$$\mathfrak{m}_\Omega(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\,1_{[-2,2]}(x)dx$$ semi-circle distribution

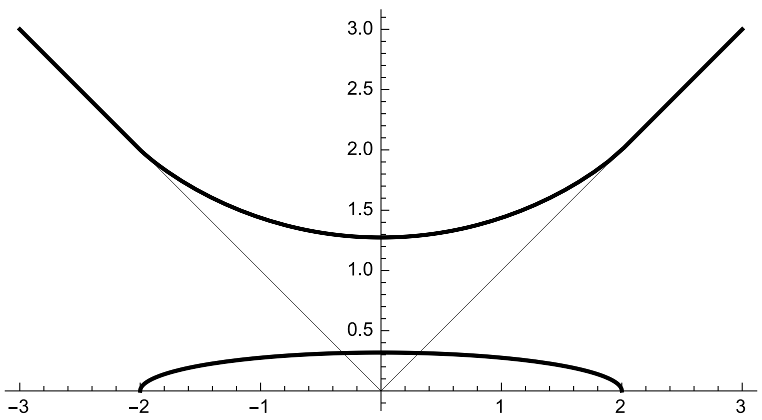


fig. 7 limit shape Ω and its transition measure

The following law of large numbers holds
(static scaling limit for the Plancherel measure)

Theorem (Vershik-Kerov, Logan-Shepp 1977)

$$\mathbb{M}_{\mathrm{Pl}}^{(n)}\Big(\Big\{\lambda \in \mathbb{Y}_n \;\Big|\; \sup_{x\in\mathbb{R}} |\lambda^{\sqrt{n}}(x) - \Omega(x)| \geqq \epsilon\Big\}\Big) = \mathbb{P}\big(\|Z_n^{\sqrt{n}} - \Omega\|_{\sup} \geqq \epsilon\big) \xrightarrow[n\to\infty]{} 0 \qquad (\forall \epsilon > 0)$$

Namely, distribution of $Z_n^{\sqrt{n}}$ converges to δ_Ω as $n \to \infty$. ■

Strong law of large numbers also holds by considering the Plancherel measure on the path space of the Young graph.

§3.6 Interface evolution

Dynamic scaling limit

s: microscopic time, t: macroscopic time $\quad s = tn$

– spectral gap of transition matrix of Res-Ind chain is $2/n$ (§3.3 Lemma)

Given any initial macroscopic profile $\omega_0 \in \mathbb{D}$ s.t. $\displaystyle\int_{\mathbb{R}} (\omega_0(x) - |x|)dx = 2$,

Take a sequence $\{\lambda^{(n)}\}_{n\in\mathbb{N}}$ s.t. $\lambda^{(n)} \in \mathbb{Y}_n$, $\lambda^{(n)\,\sqrt{n}} \to \omega_0$ in $\mathbb{D}$ $\quad$ i.e.

$$\lim_{n\to\infty} \left\| \lambda^{(n)\,\sqrt{n}} - \omega_0 \right\|_{\sup} = 0.$$

Continuous time Res-Ind chain $\tilde{X}_s^{(n)}$ with initial distribution on $\mathbb{Y}_n$:

$$\tilde{\mathbb{P}}(\tilde{X}_0^{(n)} = \cdot\,) = \delta_{\lambda^{(n)}}$$

▶ $\tilde{X}_{tn}{}^{(n)\,\sqrt{n}} \xrightarrow[n\to\infty]{} \,?$ (deterministic macroscopic profile depending on t)

Theorem (2015, SpringerBriefs Math-Phys. 2016)

For $\forall t > 0$, there exists macroscopic profile $\omega_t \in \mathbb{D}$ s.t.

$$\tilde{\mathbb{P}}\Big(\big\| \tilde{X}_{tn}{}^{(n)\,\sqrt{n}} - \omega_t \big\|_{\sup} \geqq \epsilon \Big) \xrightarrow[n\to\infty]{} 0 \quad (\forall \epsilon > 0)$$

holds (law of large numbers). Here ω_t is determined by

$$\mathfrak{m}_{\omega_t} = (\mathfrak{m}_{\omega_0})_{e^{-t}} \boxplus (\mathfrak{m}_{\Omega})_{1-e^{-t}}$$

(free convolution of free compressions of transition measures).

Furthermore time evolution is described through the Stieltjes transform of transition measures $G(t,z) = \int_{\mathbb{R}} \frac{1}{z-x} \mathfrak{m}_{\omega_t}(dx)$:

$$\frac{\partial G}{\partial t} = -G \frac{\partial G}{\partial z} + \frac{1}{G} \frac{\partial G}{\partial z} + G, \qquad t > 0,\ z \in \mathbb{C}^+$$

■

initial distribution $\delta_{\lambda^{(n)}}$	$\longrightarrow$	ω_0
$\downarrow$	$1/\sqrt{n}$, $n \to \infty$	$\downarrow$
distribution at time tn $\tilde{\mathbb{P}}(\tilde{X}_{tn}^{(n)} = \cdot\,)$	$\longrightarrow$	ω_t
$\downarrow$		$\downarrow$
invariant distribution $\mathbb{M}_{\mathrm{Pl}}^{(n)}$	$\longrightarrow$	Ω

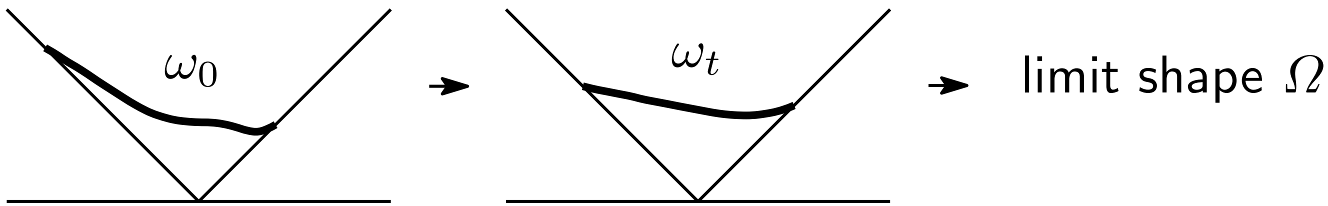
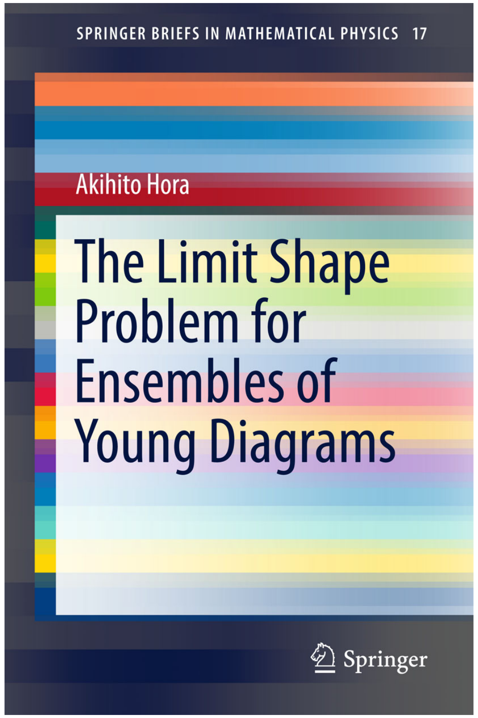


fig. 8 evolution of macroscopic profile: the area kept invariant $\int_{\mathbb{R}}(\omega_t(x) - |x|)dx = 2$ for $\forall t$

Reference for §3



A. Hora: *The Limit Shape Problem for Ensembles of Young Diagrams*, Springer Briefs in Mathematical Physics 17, Springer, 2016

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