Dynamic model for limit profiles and their Gaussian fluctuations in Young diagram ensembles

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Mathematical Aspects of Quantum Fields and Related Topics

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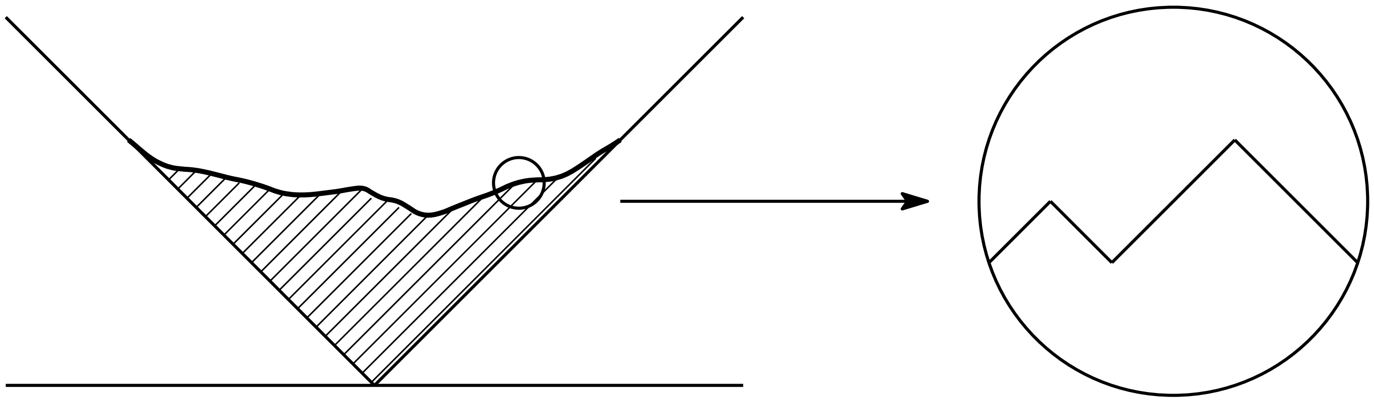
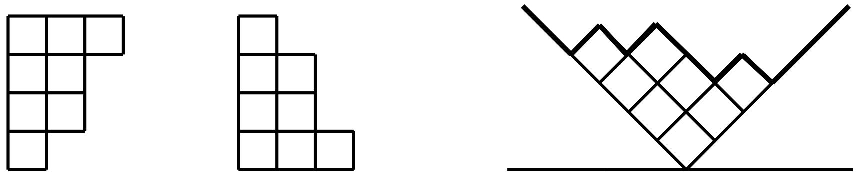


I met Obata san for the first time in the summer of 1983 ......



Ensemble of Young diagrams $\mathbb{Y}_n$ $\quad$ (n : number of boxes)

$$|\mathbb{Y}_n| = \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}n}\Big(1 + O(\frac{1}{\sqrt{n}})\Big)$$

Pick up a diagram of $\mathbb{Y}_n$ at random according to probability $\mathbb{P}_n$.

Derivation of a macroscopic shape (if any) and its time evolution under appropriate scaling limit as $n \to \infty$

$$\lambda \in \mathbb{Y}_n \quad \longleftrightarrow \quad \text{profile } \lambda(x) \quad \longrightarrow \quad \lambda^{\sqrt{n}}(x) = \frac{1}{\sqrt{n}}\lambda(\sqrt{n}x)$$

Probability $\mathbb{P}_n$ determines nature of the model.

▶ $\mathbb{Y}_n$ labels the equivalence classes of irreducible representations of $\mathfrak{S}_n$ (symmetric group of degree n).

$\Longrightarrow$ representation-theoretical ensemble of Young diagrams

▶ Random structure originating from irreducible decomposition or branching rule

Plancherel measure $\quad \mathbb{M}_{\mathrm{Pl}}^{(n)}(\lambda) = \dfrac{(\dim \lambda)^2}{n!}, \qquad \lambda \in \mathbb{Y}_n$

• irreducible decomposition of the bi-regular representation of $\mathfrak{S}_n$:

$$L_2(\mathfrak{S}_n) \cong \bigoplus_{\lambda \in \mathbb{Y}_n} V^\lambda \otimes V^{\overline{\lambda}}$$

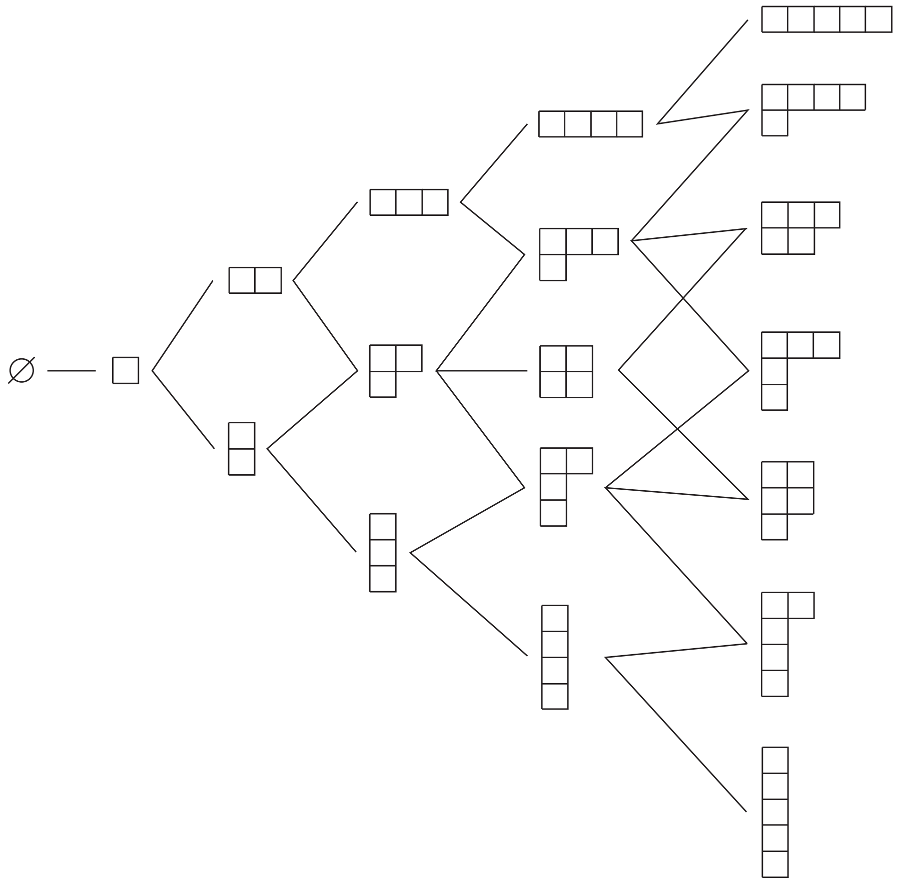
• Robinson–Schensted correspondence

$\mathfrak{S}_n \;\cong\; \{(P,Q) \,|\, P, Q : \lambda\text{-type standard tableaux}, \; \lambda \in \mathbb{Y}_n\}$

length of the longest increasing subsequence in x = length of the first row of λ

Young graph

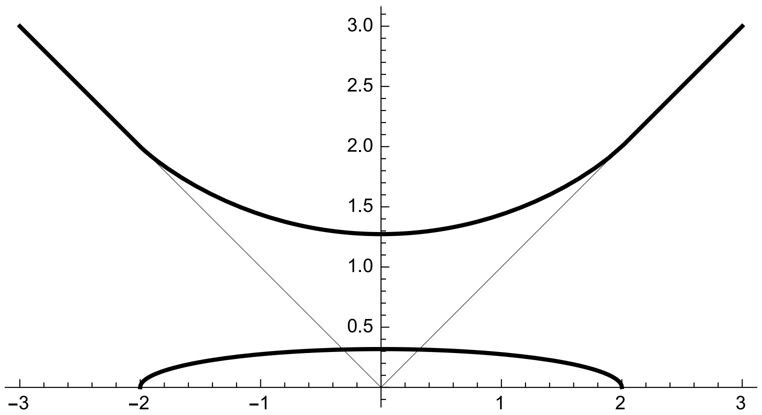
(number of paths from $\varnothing$ = dimension of the irreducible representation)



Static model for the Plancherel ensemble $(\mathbb{Y}_n, \mathbb{M}_{\mathrm{Pl}}^{(n)})$

Vershik – Kerov (1977), Logan – Shepp (1977)

$$\Omega(x) = \begin{cases} \frac{2}{\pi}\left(x \arcsin\frac{x}{2} + \sqrt{4-x^2}\right), & |x| \leqq 2 \\ |x|, & |x| > 2 \end{cases}$$



$$\lim_{n\to\infty} \mathbb{M}_{\mathrm{Pl}}^{(n)}\left(\left\{\lambda \in \mathbb{Y}_n \;\middle|\; \sup_{x\in\mathbb{R}} |\lambda^{\sqrt{n}}(x) - \Omega(x)| \geqq \epsilon\right\}\right) = 0 \qquad (\forall \epsilon > 0)$$

Other representation-theoretical ensembles of Young diagrams

— Kerov, Biane, ...

Approximate factorization property for ensemble of Young diagrams

— a weak ergodicity

- positive-definite function on $\mathfrak{S}_n$ corresponding to probability $\mathbb{M}^{(n)}$ on $\mathbb{Y}_n$

$$f^{(n)}(x) = \sum_{\lambda \in \mathbb{Y}_n} \mathbb{M}^{(n)}(\lambda) \widetilde{\chi}^\lambda(x), \qquad x \in \mathfrak{S}_n$$

- For $x, y \in \mathfrak{S}_n$ such that $\operatorname{supp} x \cap \operatorname{supp} y = \varnothing$

$$f^{(n)}(xy) - f^{(n)}(x) f^{(n)}(y) = o\big(n^{-\frac{|x|+|y|}{2}}\big)$$

where $|x|$ is the minimal number of transpositions needed to present x

Case of the Plancherel measure : $f^{(n)} = \delta_e$

Gaussian fluctuation

Fundamental fact :

- Kerov's central limit theorem for the Plancherel measure (Kerov 1993)

$$\left\{n^{\frac{k}{2}}\,\widetilde{\chi}^{\lambda}_{(k,1^{n-k})}\right\}_{k=2,3,\cdots} \quad \text{on} \quad (\mathbb{Y}_n, \mathbb{M}^{(n)}_{\mathrm{Pl}})$$

$$\xrightarrow[n\to\infty]{} \quad \{X_k\}_{k=2,3,\cdots} \quad : \text{ independent}, \quad X_k \sim N(0,k)$$

- Equivalently,

conjugacy class $C \subset \mathfrak{S}_n \;\leftrightarrow\;$ adjacency operator $A = \sum_{x\in C} x \;\curvearrowright \ell^2(\mathfrak{S}_n)$,

$$\lim_{n\to\infty}\left\langle \delta_e, \left(\frac{A_{(2,1^{n-2})}}{\sqrt{|C_{(2,1^{n-2})}|}}\right)^{p_2}\cdots\left(\frac{A_{(k,1^{n-k})}}{\sqrt{|C_{(k,1^{n-k})}|}}\right)^{p_k}\delta_e\right\rangle = \prod_{j=2}^{k}\int_{\mathbb{R}} x^{p_j}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx$$

for $\forall k \in \{2,3,\cdots\}$, $\forall p_2,\cdots,p_k \in \mathbb{N}\cup\{0\}$

Kerov's CLT is extended to many directions

quantum decomposition $A_{(j,1^{n-j})} = A^{+}_{(j,1^{n-j})} + A^{-}_{(j,1^{n-j})} + A^{o}_{(j,1^{n-j})}$

$$A^{+}_{(j,1^{n-j})} = \sum_{x \in C_{(j,1^{n-j})}} x^{+}, \qquad x^{+}\delta_y = \begin{cases} \delta_{xy}, & \text{if } c(xy) < c(y) \\ 0, & \text{otherwise} \end{cases} \qquad \text{etc.}$$

$c(y) = \sharp$ of cycles in decomposing y

▶ In the sense of convergence of any matrix element of any mixed product of operators (quantum CLT in HO book 2007):

$$\Big\{ \Big(\frac{A^{+}_{(j,1^{n-j})}}{\sqrt{|C_{(j,1^{n-j})}|}}, \frac{A^{-}_{(j,1^{n-j})}}{\sqrt{|C_{(j,1^{n-j})}|}}, \frac{A^{o}_{(j,1^{n-j})}}{\sqrt{|C_{(j,1^{n-j})}|}} \Big) \Big\}_j \xrightarrow{n\to\infty} \{ (B_j^{+}, B_j^{-}, 0) \}_j$$

$B_j^{+} = a^{+}(v_j)$ (creation), $B_j^{-} = a^{-}(v_j)$ (annihilation), $\{v_j\}$: ONB in H, $B_j^{\pm}$ act on the Boson Fock space $\Gamma(H) = \bigoplus_n H^{\hat{\otimes} n}$

To describe fluctuation of the profile :

profile ω vs its transition measure $\mathfrak{m}_\omega$ via the Markov transform

$$\frac{1}{z}\exp\Big\{\int_{\mathbb{R}}\frac{1}{x-z}\Big(\frac{\omega(x)-|x|}{2}\Big)' dx\Big\} = \int_{\mathbb{R}}\frac{1}{z-x}\mathfrak{m}_\omega(dx)$$

$$= \sum_{n=0}^{\infty}\frac{M_n(\mathfrak{m}_\omega)}{z^{n+1}}, \qquad z \in \mathbb{C}^+$$

- $\mathfrak{m}_\Omega$ is the standard Wigner (semicircle) distribution

$$\mathfrak{m}_\Omega(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\, 1_{[-2,2]}(x)\, dx$$

- For Young diagram $\lambda = (x_1 < y_1 < x_2 < \cdots < y_{r-1} < x_{r-1} < x_r)$

(x_i: valley, y_i: peak),

$$\frac{(z-y_1)\cdots(z-y_{r-1})}{(z-x_1)\cdots(z-x_r)} = \int_{\mathbb{R}}\frac{1}{z-x}\mathfrak{m}_\lambda(dx) = \frac{\mu_1}{z-x_1}+\cdots+\frac{\mu_r}{z-x_r}$$

By Vershik–Kerov and Logan–Shepp, we know

$$\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_{\Omega} \quad \xrightarrow[n\to\infty]{} \quad 0$$

Interpretation of Kerov's CLT by Ivanov – Olshanski (2002)

Consider

$$\sqrt{n}\left(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_{\Omega}\right) \quad \xrightarrow[n\to\infty]{} \quad ?$$

- Bulk and edge of $\lambda^{\sqrt{n}}$ have different scales of fluctuation

$\Longrightarrow$ distribution-valued r.v.

$$M_n(\mu) = \int_{\mathbb{R}} x^n \mu(dx) = \langle x^n, \mu \rangle$$

$$\Sigma_k(\lambda) = n(n-1)\cdots(n-k+1)\,\widetilde{\chi}^{\lambda}_{(k,1^{n-k})} \qquad (\lambda \in \mathbb{Y}_n)$$

- Algebra of functions on $\mathbb{Y}$: $\langle M_n(\mathfrak{m}_\lambda)\rangle_n = \langle R_n(\mathfrak{m}_\lambda)\rangle_n = \langle \Sigma_k(\lambda)\rangle_k$

free cumulant-moment formula, Kerov polynomial

$$\langle x^{2p}, \sqrt{n}(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_\Omega)\rangle = \sum_{j=1}^{p-1} \binom{2p}{p-j-1} n^{-\frac{2j+1}{2}} \Sigma_{2j+1}(\lambda) + \text{(remainder)},$$

$$\langle x^{2p-1}, \sqrt{n}(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_\Omega)\rangle = \sum_{j=1}^{p-1} \binom{2p-1}{p-j-1} n^{-j} \Sigma_{2j}(\lambda) + \text{(remainder)}$$

on $\mathbb{Y}_n$ (remainder : w.r.t. $\mathbb{M}^{(n)}_{\mathrm{Pl}}$)

$\longrightarrow$ Inversion!

$$n^{-\frac{k}{2}}\Sigma_k(\lambda) = \langle (\text{Chebyshev polynomial}), \sqrt{n}(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_{\Omega})\rangle + (\text{remainder})$$

LHS $\xrightarrow[n\to\infty]{}$ (independent) Gaussian r.v. by Kerov's CLT

Chebyshev polynomials form ONB in $L^2\Big((-2,2), \frac{1}{\pi\sqrt{4-x^2}}dx\Big)$

RHS : Coefficient of a random Fourier series

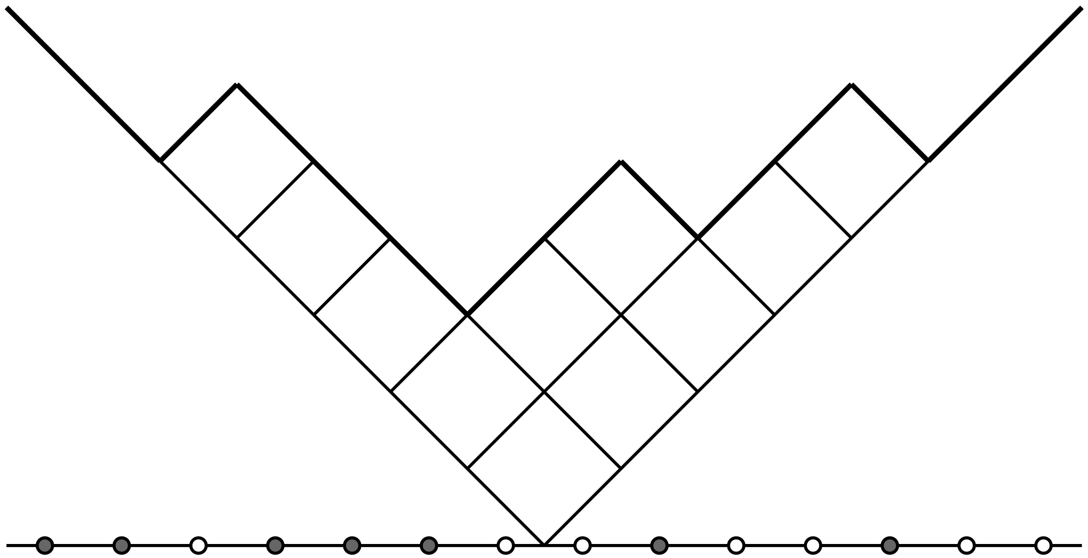
- $\sqrt{n}(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_{\Omega})$ converges as $n \to \infty$ to

$$\sum (\text{Gaussian r.v.}) \times (\text{Chebyshev polynomials}) \frac{1}{\pi\sqrt{4-x^2}}$$

$$= \sum_{k=3}^{\infty} \sqrt{k-1}\,\xi_{k-1}\,T_k\big(\frac{x}{2}\big)\,\frac{1}{\pi\sqrt{4-x^2}}$$

$\{\xi_{k-1}\}$: independent standard Gaussian, $\qquad T_k(\cos\theta) = \cos k\theta$

Dynamic model — scaling limit under not only growth of Young diagrams but also transition between diagrams (Markov chain)



1 step transition :

Move one box (peak $\to$ valley)

$\Longleftrightarrow$ twice adjacent flips $\bullet \leftrightarrow \circ$ in Maya diagram

Canonical setting

$$P^{\downarrow}_{\lambda,\nu} = \begin{cases} \frac{\dim \nu}{\dim \lambda}, & \nu \nearrow \lambda, \\ 0, & \text{otherwise} \end{cases}, \qquad P^{\uparrow}_{\nu,\mu} = \begin{cases} \frac{\dim \mu}{(|\nu|+1)\dim \nu}, & \nu \nearrow \mu, \\ 0, & \text{otherwise} \end{cases}$$

Irreducible decomposition of restriction/induction of representations

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \pi^{\lambda} \cong \bigoplus_{\nu \in \mathbb{Y}_{n-1} : \nu \nearrow \lambda} \pi^{\nu}, \qquad \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \pi^{\nu} \cong \bigoplus_{\mu \in \mathbb{Y}_n : \nu \nearrow \mu} \pi^{\mu}$$

The Plancherel measure $\mathbb{M}^{(n)}_{\mathrm{Pl}}$ on $\mathbb{Y}_n$ is kept invariant by transition probability $P^{(n)} = P^{\downarrow} P^{\uparrow}$.

Continuous time Markov chain $(X^{(n)}_s)_{s \geqq 0}$ on $\mathbb{Y}_n$

- macroscpoic time t, microscopic time $s = tn$, spatial rescale by $\sqrt{n}$

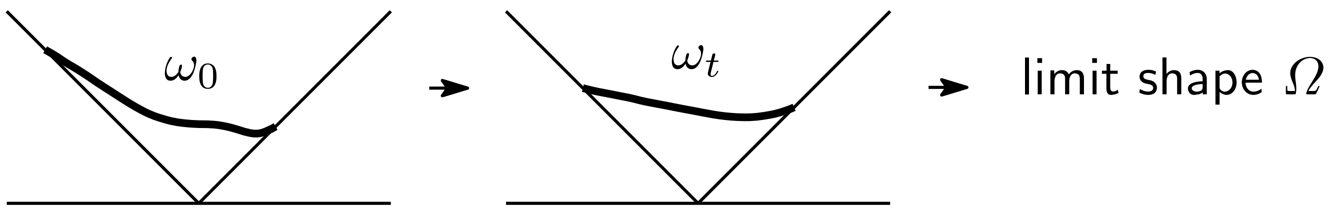
initial distribution $\mathbb{M}_0^{(n)}$ $\longrightarrow$ ω_0

$\downarrow$ $1/\sqrt{n}$, $n \to \infty$

distribution at time tn $\mathbb{M}_t^{(n)}$ $\longrightarrow$ ω_t

$\downarrow$

stationary distribution $\mathbb{M}_{\mathrm{Pl}}^{(n)}$ $\longrightarrow$ Ω



Funaki – Sasada : [CMP 2010]

Grand canonical

- probability on $\mathbb{Y}$ $\mu^{\epsilon}(\lambda) = Z^{-1}\epsilon^{|\lambda|}$, $\lambda \in \mathbb{Y}$ $(0 < \epsilon < 1)$

  such that $\mathbb{E}_{\mu^{\epsilon}}\big[|\lambda|\big] = N^2$ $(\Longrightarrow \lim_{N\to\infty} \epsilon = 1)$
- number of boxes ± 1 at 1 step transition (at random for peaks or valleys)

  — μ^{ϵ} kept invariant

Continuous time Markov chain $\longrightarrow$ distribution ν_s^N at time s

▶ Rescale for time $s = tN^2$, for space $\lambda^N(x) = \frac{1}{N}\lambda(Nx)$

Assumption : weak LLN under initial ensemble ν^N : $\lim_{N\to\infty} \lambda^N = \psi_0$

Result : $\forall t > 0$, weak LLN under $\nu_{tN^2}^N$: $\lim_{N\to\infty} \lambda^N = \psi_t$

PDE satisfied by $\psi_t(x)$ is obtained.

Theorem [Publ. RIMS 2015] (canonical setting)

Assumption : Initial ensemble $\{(\mathbb{Y}_n, \mathbb{M}_0^{(n)})\}_n$ satisfies approximate factorization property.

Result : $\forall t > 0$, $\mathbb{M}_t^{(n)}$ denoting distribution of $X_{tn}^{(n)}$, the ensemble $\{(\mathbb{Y}_n, \mathbb{M}_t^{(n)})\}_n$ at time t also satisfies approximate factorization property, and weak LLN : $\forall \epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{M}_t^{(n)}\Big(\Big\{\lambda \in \mathbb{Y}_n \,\Big|\, \sup_{x\in\mathbb{R}} |\lambda^{\sqrt{n}}(x) - \omega_t(x)| \geqq \epsilon\Big\}\Big) = 0$$

holds. Macroscopic shape ω_t at t is characterized by

$$\mathfrak{m}_{\omega_t} = (\mathfrak{m}_{\omega_0})_{e^{-t}} \boxplus (\mathfrak{m}_{\Omega})_{1-e^{-t}}$$

(free convolution of free compressions of Kerov transition measures).

Time evolution of the distributions could be described through their Stieltjes transforms

- Stieltjes transform of semicircle distribution μ_t of mean 0 and variance t

$$g(t,z) = \int_{\mathbb{R}} \frac{1}{z-x}\mu_t(dx) = \int_{-2\sqrt{t}}^{2\sqrt{t}} \frac{1}{z-x}\frac{\sqrt{4t-x^2}}{2\pi t}dx$$
$$= \frac{z-\sqrt{z^2-4t}}{2t}$$

satisfies PDE :

$$\frac{\partial g}{\partial t} = -g\,\frac{\partial g}{\partial z}$$

▶ PDE describing time evolution of transition measure $\mathfrak{m}_{\omega_t}$

$$G(t,z) = \int_{\mathbb{R}} \frac{1}{z-x} \mathfrak{m}_{\omega_t}(dx),$$

$$\frac{\partial G}{\partial t} = \frac{1}{G}\frac{\partial G}{\partial z} + G - G\frac{\partial G}{\partial z}, \qquad t>0,\ z \in \mathbb{C}^+$$

▷ PDE for $\omega(t,x) = \omega_t(x) \cdots$

$$c\,\frac{\partial \omega}{\partial t} = \frac{\partial^2 \omega}{\partial x^2} - \frac{4}{\pi^2}\Big(\omega - x\,\frac{\partial \omega}{\partial x}\Big)^{-1} \quad ??$$

(Constraint of constant area ?)

▶ ODE for Ω :

$$\frac{\partial^2 \Omega}{\partial x^2} - \frac{4}{\pi^2}\Big(\Omega - x\,\frac{\partial \Omega}{\partial x}\Big)^{-1} = 0$$

Case of Funaki – Sasada

In French style

time evolution $$\partial_t \psi = \partial_u \Big(\frac{\partial_u \psi}{1 - \partial_u \psi} \Big) + \frac{\pi}{\sqrt{6}} \frac{\partial_u \psi}{1 - \partial_u \psi}$$

$t \to \infty$ $$e^{-(\pi/\sqrt{6})u} + e^{-(\pi/\sqrt{6})\psi(u)} = 1$$

In Russian style

time evolution $$\frac{\partial \phi}{\partial t} = \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2} - \frac{\pi}{4\sqrt{3}} \Big(1 - \big(\frac{\partial \phi}{\partial x} \big)^2 \Big)$$

$t \to \infty$ $$\phi(x) = \frac{2\sqrt{3}}{\pi} \log \Big(e^{(\pi/2\sqrt{3})x} + e^{-(\pi/2\sqrt{3})x} \Big)$$

Although PDE of time evolution for ω_t is unknown, we have a solution in some sense (through Markov transform) ......

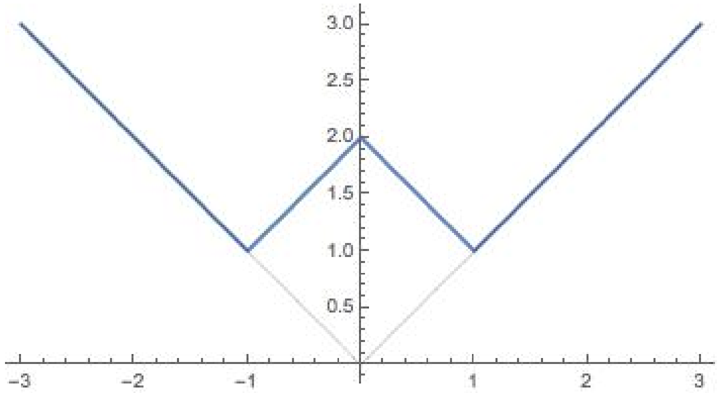
▶ Relation between profile ω_t and its Kerov transition measure $\mathfrak{m}_{\omega_t}$ (recall the Markov transform)

$$\int_{\mathbb{R}} \frac{1}{z-x} \mathfrak{m}_{\omega_t}(dx) = \frac{1}{z} \exp\Big\{ \int_{\mathbb{R}} \frac{1}{x-z} \Big(\frac{\omega_t(x) - |x|}{2} \Big)' dx \Big\}, \quad z \in \mathbb{C}^+$$

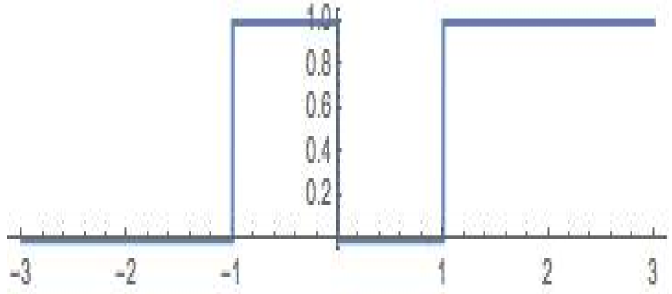
Procedure of computing ω_t from $\mathfrak{m}_{\omega_t}$:

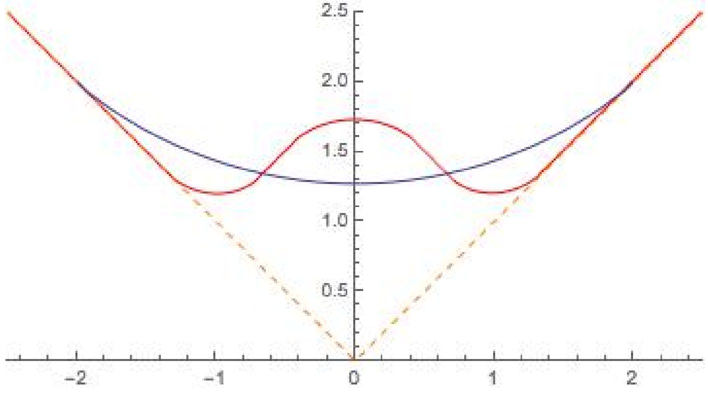
free convolution, free compression $\longrightarrow$ free cumulants
$\longrightarrow$ Voiculescu R-transform $\longrightarrow$ Stieltjes transform
$\longrightarrow$ taking $\log$ $\longrightarrow$ inversion of Stieltjes transform $\longrightarrow$ integration!

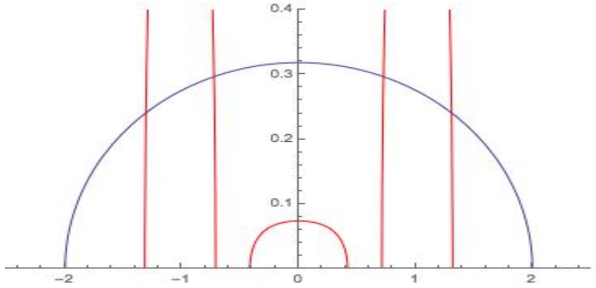
profile	transition measure	Rayleigh function

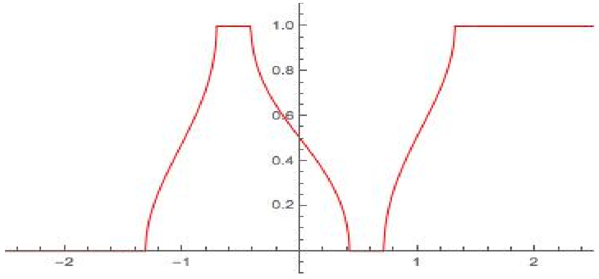


$$\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$$









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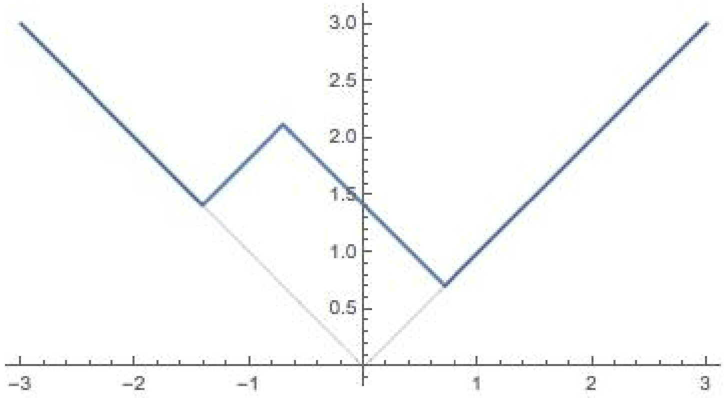
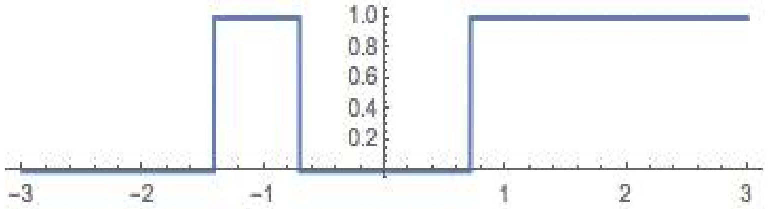
profile

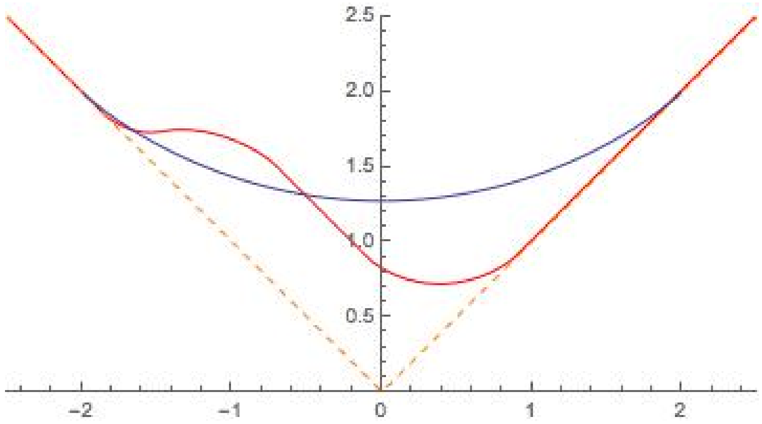
transition measure

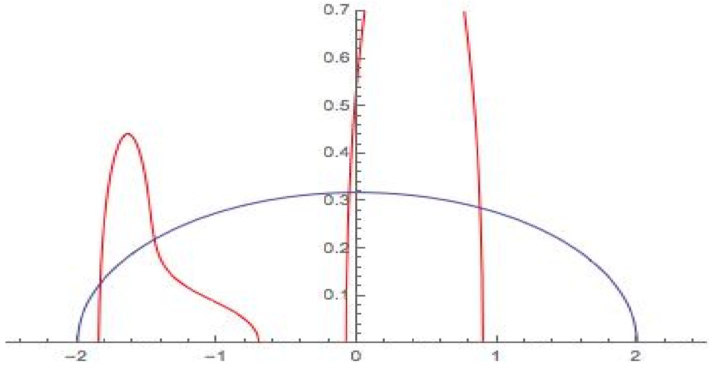
Rayleigh function

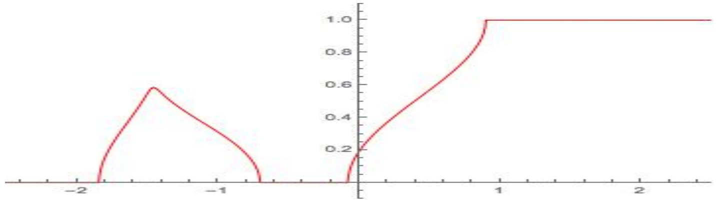


$$\frac{1}{3}\delta_{-\sqrt{2}} + \frac{2}{3}\delta_{1/\sqrt{2}}$$









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Dynamic model : initial $\to\to\to$ Plancherel

Fluctuation for other (non-Plancherel) ensembles

Śniady (2005) "character factorization property"

- $(\mathbb{Y}_n, \mathbb{M}^{(n)})$ or $(Z(\mathbb{C}[\mathfrak{S}_n]), \phi^{(n)})$,

  C : cumulant functional w.r.t. $E_{\mathbb{M}^{(n)}}$ or $\phi^{(n)}$

Assume

$$C\big[\Sigma_{j_1}, \cdots, \Sigma_{j_k}\big] = O\big(n^{\frac{j_1+\cdots+j_k-k+2}{2}}\big).$$

Then

$$\Big\{\sqrt{n}\Big(n^{-\frac{j+1}{2}}\Sigma_j - E_{\mathbb{M}^{(n)}}\big[n^{-\frac{j+1}{2}}\Sigma_j\big]\Big)\Big\}_{j\geqq 2} \xrightarrow[n\to\infty]{} \{X_j\} \text{ : Gaussian, mean } 0.$$

Theorem (not satisfactory)

In our model, character factorization property is propagated at any macroscopic time t.

Hence $\sqrt{n}\big(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_{\omega_t}\big)$ on $(\mathbb{Y}_n, \mathbb{M}_t^{(n)})$ converges as $n \to \infty$ to the fluctuation at t, i.e.

- $\big\{\langle x^j, \sqrt{n}(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_{\omega_t})\rangle\big\}_j \xrightarrow[n\to\infty]{}$ Gaussian system with mean 0
- Covariance has complicated t-dependence, vanishes as $t \to \infty$.

Grand canonical setting $\quad \mathbb{Y} = \bigsqcup_{n=0}^{\infty} \mathbb{Y}_n$

Poissonization of the Plancherel measure

$$\mathbb{M}_{\mathrm{PP}}^{(\xi)} = \sum_{n=0}^{\infty} \frac{e^{-\xi}\xi^n}{n!} \mathbb{M}_{\mathrm{Pl}}^{(n)}, \qquad \xi > 0$$

is kept invariant under transition probability $P^{(\xi)}$ on $\mathbb{Y}$:

$$P^{(\xi)} = \alpha_\xi(n) P^{\uparrow(n)} + (1 - \alpha_\xi(n)) P^{\downarrow(n)},$$

$$\alpha_\xi(n) = \int_0^1 \xi e^{-\xi x}(1-x)^n dx$$

Continuous time Markov chain $(X_s^{(\xi)})_{s \geqq 0}$

- Rescale for time $t\xi$, $\qquad$ for space $\frac{1}{\sqrt{\xi}}\lambda(\sqrt{\xi}x) \quad (\lambda \in \mathbb{Y})$

Behavior as $\xi \to \infty$ ......

Method of proofs

- profile of Young diagram $\lambda \longleftrightarrow$ transition measure $\mathfrak{m}_\lambda$
- method of symmetric functions (generators, generating functions)
- free cumulant $R_k(\mathfrak{m}_\lambda)$ vs irreducible character value χ^λ at cycle

$\Longleftarrow$ Kerov polynomial

- diagonalizing transition probability by using irreducible characters
- estimate of ensemble expectation by using approximate (character)

factorization property

END