On a dynamic model for limit profiles and their Gaussian fluctuations in group-theoretical ensembles of Young diagrams

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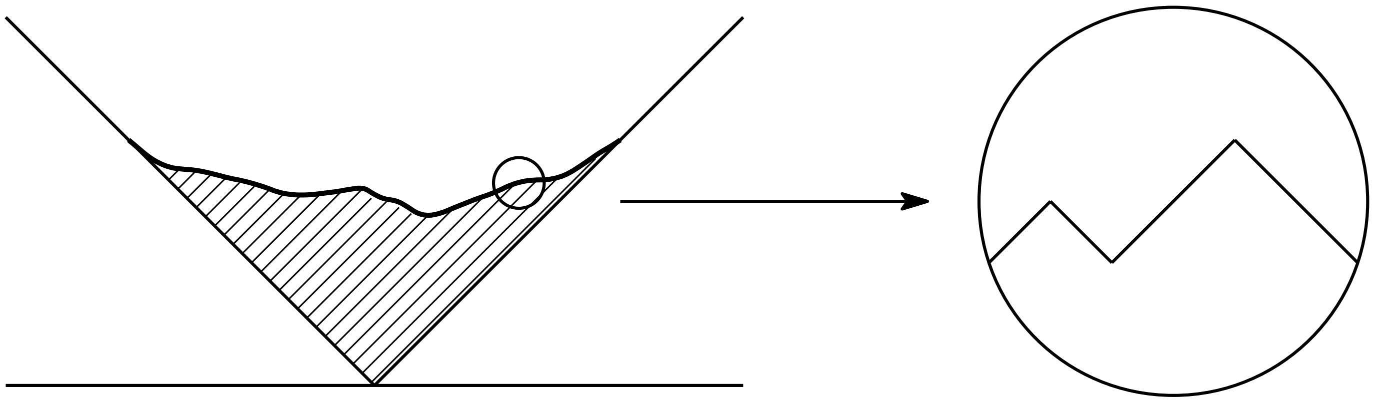
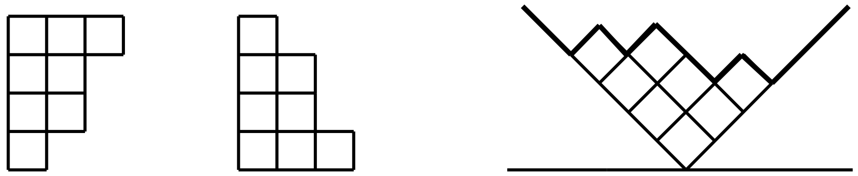




Ensemble of Young diagrams $\mathbb{Y}_n$ $\quad$ (n : number of boxes)

$$|\mathbb{Y}_n| = \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}n}\Big(1 + O\big(\frac{1}{\sqrt{n}}\big)\Big)$$

Pick up a diagram of $\mathbb{Y}_n$ at random according to probability $\mathbb{P}_n$.

Derivation of a macroscopic shape (if any) and its time evolution under appropriate scaling limit as $n \to \infty$

$$\lambda \in \mathbb{Y}_n \;\longleftrightarrow\; \text{profile } \lambda(x) \quad \longrightarrow \quad \lambda^{\sqrt{n}}(x) = \frac{1}{\sqrt{n}}\lambda(\sqrt{n}x)$$

Probability $\mathbb{P}_n$ determines nature of the model.

▶ $\mathbb{Y}_n$ labels the equivalence classes of irreducible representations of $\mathfrak{S}_n$ (symmetric group of degree n).

$\Longrightarrow$ representation-theoretical ensemble of Young diagrams

▶ Random structure originating from irreducible decomposition or branching rule

Plancherel measure $\quad \mathbb{M}_{\mathrm{Pl}}^{(n)}(\lambda) = \dfrac{(\dim \lambda)^2}{n!}, \qquad \lambda \in \mathbb{Y}_n$

• irreducible decomposition of the bi-regular representation of $\mathfrak{S}_n$:

$$L_2(\mathfrak{S}_n) \cong \bigoplus_{\lambda \in \mathbb{Y}_n} V^\lambda \otimes V^{\overline{\lambda}}$$

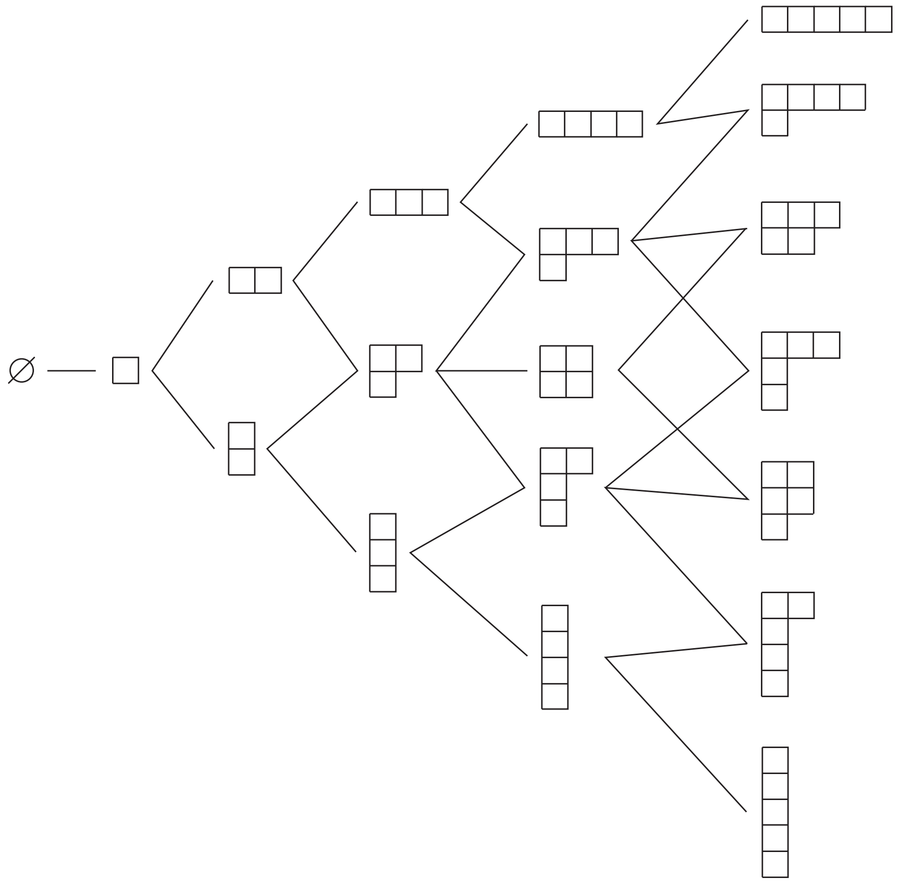
• Robinson–Schensted correspondence

$\mathfrak{S}_n \;\cong\; \{(P,Q) \,|\, P,Q : \lambda\text{-type standard tableaux}, \; \lambda \in \mathbb{Y}_n\}$

length of the longest increasing subsequence in x = length of the first row of λ

Young graph

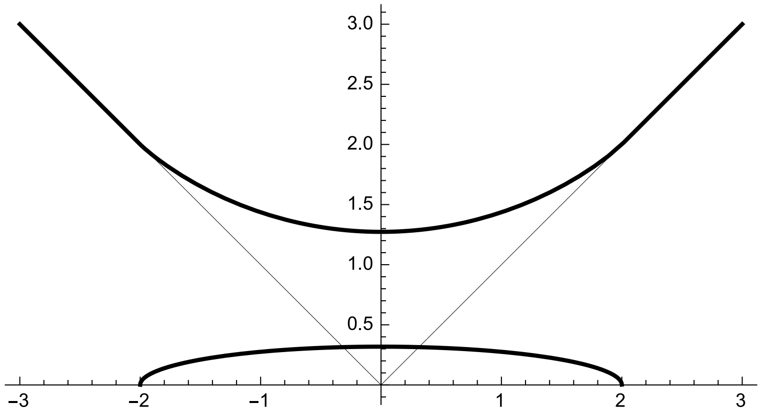
(number of paths from $\varnothing$ = dimension of the irreducible representation)



Static model for the Plancherel ensemble $(\mathbb{Y}_n, \mathbb{M}_{\mathrm{Pl}}^{(n)})$

Vershik – Kerov (1977), Logan – Shepp (1977)

$$\Omega(x) = \begin{cases} \frac{2}{\pi}\Big(x \arcsin\frac{x}{2} + \sqrt{4-x^2}\Big), & |x| \leqq 2 \\ |x|, & |x| > 2 \end{cases}$$



$$\lim_{n\to\infty} \mathbb{M}_{\mathrm{Pl}}^{(n)}\Big(\Big\{\lambda \in \mathbb{Y}_n \;\Big|\; \sup_{x\in\mathbb{R}} |\lambda^{\sqrt{n}}(x) - \Omega(x)| \geqq \epsilon\Big\}\Big) = 0 \qquad (\forall \epsilon > 0)$$

Other representation-theoretical ensembles of Young diagrams

— Kerov, Biane, ...

Approximate factorization property for ensemble of Young diagrams

— a weak ergodicity

- positive-definite function on $\mathfrak{S}_n$ corresponding to probability $\mathbb{M}^{(n)}$ on $\mathbb{Y}_n$

$$f^{(n)}(x) = \sum_{\lambda \in \mathbb{Y}_n} \mathbb{M}^{(n)}(\lambda) \widetilde{\chi}^{\lambda}(x), \qquad x \in \mathfrak{S}_n$$

- For $x, y \in \mathfrak{S}_n$ such that $\operatorname{supp} x \cap \operatorname{supp} y = \varnothing$

$$f^{(n)}(xy) - f^{(n)}(x) f^{(n)}(y) = o\big(n^{-\frac{|x|+|y|}{2}}\big)$$

where $|x|$ is the minimal number of transpositions needed to present x

Case of the Plancherel measure : $f^{(n)} = \delta_e$

Gaussian fluctuation

Fundamental fact :

▶ Kerov's central limit theorem for the Plancherel measure (Kerov 1993)

$$\left\{n^{\frac{k}{2}}\,\widetilde{\chi}^{\lambda}_{(k,1^{n-k})}\right\}_{k=2,3,\cdots} \quad \text{on} \quad (\mathbb{Y}_n, \mathbb{M}^{(n)}_{\mathrm{Pl}})$$

$$\xrightarrow[n\to\infty]{} \quad \{X_k\}_{k=2,3,\cdots} \quad : \text{independent}, \quad X_k \sim N(0,k)$$

Extended to many directions

• profile ω vs its transition measure $\mathfrak{m}_\omega$

e.g. $\mathfrak{m}_\Omega$ is the standard Wigner (semicircle) distribution :

$$\mathfrak{m}_\Omega(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\,1_{[-2,2]}(x)\,dx$$

Markov transform :

$$\frac{1}{z}\exp\Big\{\int_{\mathbb{R}}\frac{1}{x-z}\Big(\frac{\omega(x)-|x|}{2}\Big)'dx\Big\} = \int_{\mathbb{R}}\frac{1}{z-x}\mathfrak{m}_\omega(dx)$$

$$= \sum_{n=0}^{\infty}\frac{M_n(\mathfrak{m}_\omega)}{z^{n+1}}, \qquad z\in\mathbb{C}^+$$

For Young diagram $\lambda = (x_1 < y_1 < x_2 < \cdots < y_{r-1} < x_{r-1} < x_r)$

(x_i: valley, y_i: peak),

$$\frac{(z-y_1)\cdots(z-y_{r-1})}{(z-x_1)\cdots(z-x_r)} = \int_{\mathbb{R}}\frac{1}{z-x}\mathfrak{m}_\lambda(dx) = \frac{\mu_1}{z-x_1}+\cdots+\frac{\mu_r}{z-x_r}$$

We know $\quad \mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_\Omega \quad \xrightarrow[n\to\infty]{} \quad 0,$

Then $\quad \sqrt{n}\big(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_\Omega\big) \quad \xrightarrow[n\to\infty]{} \quad ?$ Ivanov – Olshanski (2002)

Interpretation of Kerov's CLT

$$M_n(\mu) = \int_{\mathbb{R}} x^n \mu(dx) = \langle x^n, \mu \rangle$$

$$\Sigma_k(\lambda) = n(n-1)\cdots(n-k+1)\,\widetilde{\chi}^{\lambda}_{(k,1^{n-k})} \qquad (\lambda \in \mathbb{Y}_n)$$

- Algebra of functions on $\mathbb{Y}$: $\langle M_n(\mathfrak{m}_\lambda)\rangle_n = \langle R_n(\mathfrak{m}_\lambda)\rangle_n = \langle \Sigma_k(\lambda)\rangle_k$

free cumulant-moment formula, Kerov polynomial

$$\langle x^{2p}, \sqrt{n}(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_\Omega)\rangle = \sum_{j=1}^{p-1} \binom{2p}{p-j-1} n^{-\frac{2j+1}{2}} \Sigma_{2j+1}(\lambda) + (\text{remainder}),$$

$$\langle x^{2p-1}, \sqrt{n}(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_\Omega)\rangle = \sum_{j=1}^{p-1} \binom{2p-1}{p-j-1} n^{-j} \Sigma_{2j}(\lambda) + (\text{remainder})$$

on $\mathbb{Y}_n$ (remainder : w.r.t. $\mathbb{M}^{(n)}_{\mathrm{Pl}}$)

$\longrightarrow$ Inversion!

$$n^{-\frac{k}{2}}\Sigma_k(\lambda) = \langle(\text{Chebyshev polynomial}), \sqrt{n}(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_{\Omega})\rangle + (\text{remainder})$$

LHS $\xrightarrow[n\to\infty]{}$ (independent) Gaussian r.v. by Kerov's CLT

Chebyshev polynomials form ONB in $L^2\Big((-2,2), \frac{1}{\pi\sqrt{4-x^2}}dx\Big)$

RHS : Coefficient of a random Fourier series

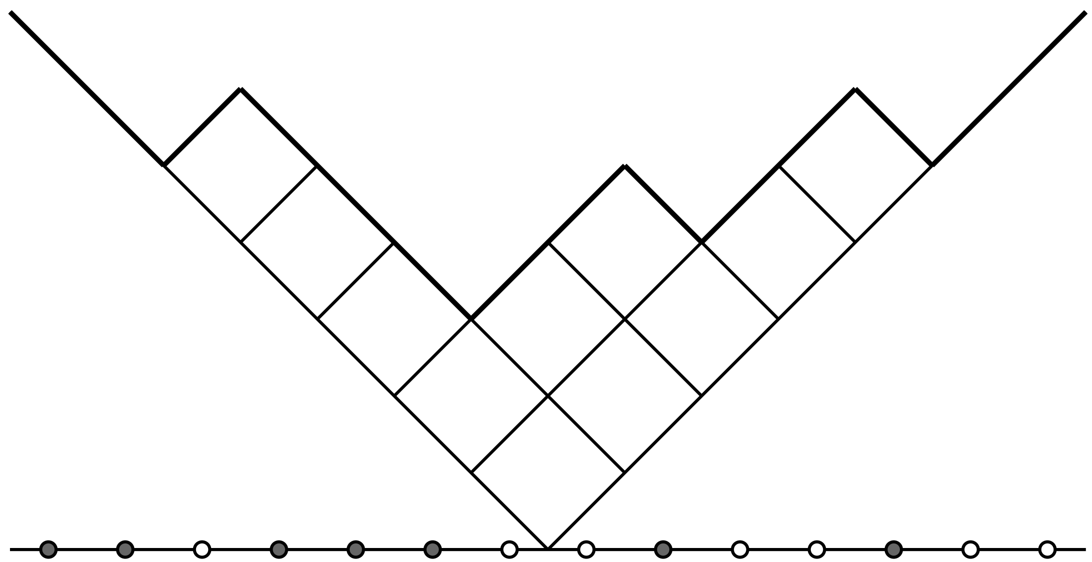
- $\sqrt{n}(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_{\Omega})$ converges to

$$\sum(\text{Gaussian r.v.}) \times (\text{Chebyshev polynomials})\,\frac{1}{\pi\sqrt{4-x^2}}$$

$$= \sum_{k=3}^{\infty} \sqrt{k-1}\,\xi_{k-1}\,T_k\Big(\frac{x}{2}\Big)\,\frac{1}{\pi\sqrt{4-x^2}}$$

$\{\xi_{k-1}\}$: independent standard Gaussian, $\qquad T_k(\cos\theta) = \cos k\theta$

Dynamic model — scaling limit under not only growth of Young diagrams but also transition between diagrams (Markov chain)



1 step transition :

Move one box (peak $\to$ valley)

$\Longleftrightarrow$ twice adjacent flips $\bullet \leftrightarrow \circ$ in Maya diagram

Canonical setting

$$P^{\downarrow}_{\lambda,\nu} = \begin{cases} \frac{\dim \nu}{\dim \lambda}, & \nu \nearrow \lambda, \\ 0, & \text{otherwise} \end{cases}, \qquad P^{\uparrow}_{\nu,\mu} = \begin{cases} \frac{\dim \mu}{(|\nu|+1) \dim \nu}, & \nu \nearrow \mu, \\ 0, & \text{otherwise} \end{cases}$$

Irreducible decomposition of restriction/induction of representations

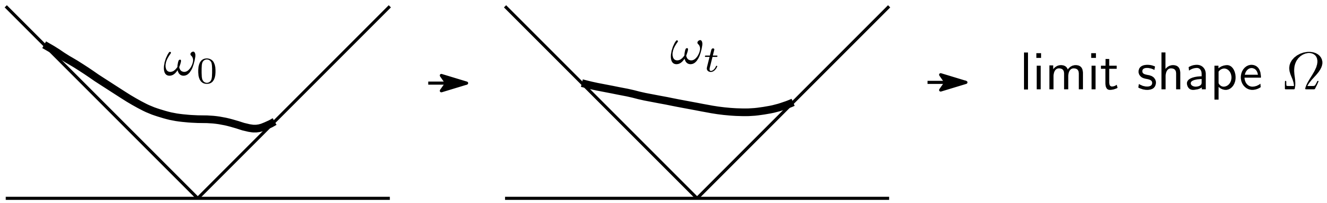
$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \pi^{\lambda} \cong \bigoplus_{\nu \in \mathbb{Y}_{n-1} : \nu \nearrow \lambda} \pi^{\nu}, \qquad \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \pi^{\nu} \cong \bigoplus_{\mu \in \mathbb{Y}_{n} : \nu \nearrow \mu} \pi^{\mu}$$

The Plancherel measure $\mathbb{M}^{(n)}_{\mathrm{Pl}}$ on $\mathbb{Y}_n$ is kept invariant by transition probability $P^{(n)} = P^{\downarrow} P^{\uparrow}$.

Continuous time Markov chain $(X^{(n)}_s)_{s \geqq 0}$ on $\mathbb{Y}_n$

- macroscpoic time t, microscopic time $s = tn$, spatial rescale by $\sqrt{n}$

initial distribution $\mathbb{M}_0^{(n)}$	$\longrightarrow$	ω_0
$\downarrow$	$1/\sqrt{n}$, $n \to \infty$	
distribution at time tn $\mathbb{M}_t^{(n)}$	$\longrightarrow$	ω_t
$\downarrow$		
stationary distribution $\mathbb{M}_{\mathrm{Pl}}^{(n)}$	$\longrightarrow$	Ω



Funaki – Sasada : [CMP 2010]

Grand canonical

- probability on $\mathbb{Y}$ $\mu^{\epsilon}(\lambda) = Z^{-1}\epsilon^{|\lambda|}$, $\lambda \in \mathbb{Y}$ $(0 < \epsilon < 1)$

  such that $\mathbb{E}_{\mu^{\epsilon}}\big[|\lambda|\big] = N^2$ $(\Longrightarrow \lim_{N\to\infty} \epsilon = 1)$
- number of boxes ± 1 at 1 step transition (at random for peaks or valleys)

  — μ^{ϵ} kept invariant

Continuous time Markov chain $\longrightarrow$ distribution ν_s^N at time s

▶ Rescale for time $s = tN^2$, for space $\lambda^N(x) = \frac{1}{N}\lambda(Nx)$

Assumption : weak LLN under initial ensemble ν^N : $\lim_{N\to\infty} \lambda^N = \psi_0$

Result : $\forall t > 0$, weak LLN under $\nu_{tN^2}^N$: $\lim_{N\to\infty} \lambda^N = \psi_t$

PDE satisfied by $\psi_t(x)$ is obtained.

Theorem [Publ. RIMS 2015] (canonical setting)

Assumption : Initial ensemble $\{(\mathbb{Y}_n, \mathbb{M}_0^{(n)})\}_n$ satisfies approximate factorization property.

Result : $\forall t > 0$, $\mathbb{M}_t^{(n)}$ denoting distribution of $X_{tn}^{(n)}$, the ensemble $\{(\mathbb{Y}_n, \mathbb{M}_t^{(n)})\}_n$ at time t also satisfies approximate factorization property, and weak LLN : $\forall \epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{M}_t^{(n)}\Big(\Big\{\lambda \in \mathbb{Y}_n \,\Big|\, \sup_{x\in\mathbb{R}} |\lambda^{\sqrt{n}}(x) - \omega_t(x)| \geqq \epsilon\Big\}\Big) = 0$$

holds. Macroscopic shape ω_t at t is characterized by

$$\mathfrak{m}_{\omega_t} = (\mathfrak{m}_{\omega_0})_{e^{-t}} \boxplus (\mathfrak{m}_{\Omega})_{1-e^{-t}}$$

(free convolution of free compressions of Kerov transition measures).

Time evolution of the distributions could be described through their Stieltjes transforms

- Stieltjes transform of semicircle distribution μ_t of mean 0 and variance t

$$g(t,z) = \int_{\mathbb{R}} \frac{1}{z-x}\mu_t(dx) = \int_{-2\sqrt{t}}^{2\sqrt{t}} \frac{1}{z-x}\frac{\sqrt{4t-x^2}}{2\pi t}dx$$
$$= \frac{z-\sqrt{z^2-4t}}{2t}$$

  satisfies PDE :

$$\frac{\partial g}{\partial t} = -g\,\frac{\partial g}{\partial z}$$

▶ PDE describing time evolution of transition measure $\mathfrak{m}_{\omega_t}$

$$G(t,z) = \int_{\mathbb{R}} \frac{1}{z-x} \mathfrak{m}_{\omega_t}(dx),$$

$$\frac{\partial G}{\partial t} = \frac{1}{G} \frac{\partial G}{\partial z} + G - G \frac{\partial G}{\partial z}, \qquad t > 0, \ z \in \mathbb{C}^+$$

▷ PDE for $\omega(t,x) = \omega_t(x) \cdots$

$$c \frac{\partial \omega}{\partial t} = \frac{\partial^2 \omega}{\partial x^2} - \frac{4}{\pi^2} \Big(\omega - x \frac{\partial \omega}{\partial x} \Big)^{-1} \quad ??$$

(Constraint of constant area ?)

▶ ODE for Ω :

$$\frac{\partial^2 \Omega}{\partial x^2} - \frac{4}{\pi^2} \Big(\Omega - x \frac{\partial \Omega}{\partial x} \Big)^{-1} = 0$$

Case of Funaki – Sasada

In French style

time evolution $$\partial_t \psi = \partial_u \Big(\frac{\partial_u \psi}{1 - \partial_u \psi} \Big) + \frac{\pi}{\sqrt{6}} \frac{\partial_u \psi}{1 - \partial_u \psi}$$

$t \to \infty$ $$e^{-(\pi/\sqrt{6})u} + e^{-(\pi/\sqrt{6})\psi(u)} = 1$$

In Russian style

time evolution $$\frac{\partial \phi}{\partial t} = \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2} - \frac{\pi}{4\sqrt{3}} \Big(1 - (\frac{\partial \phi}{\partial x})^2 \Big)$$

$t \to \infty$ $$\phi(x) = \frac{2\sqrt{3}}{\pi} \log\Big(e^{(\pi/2\sqrt{3})x} + e^{-(\pi/2\sqrt{3})x} \Big)$$

Although PDE of time evolution for ω_t is unknown, we have a solution in some sense (through Markov transform) ......

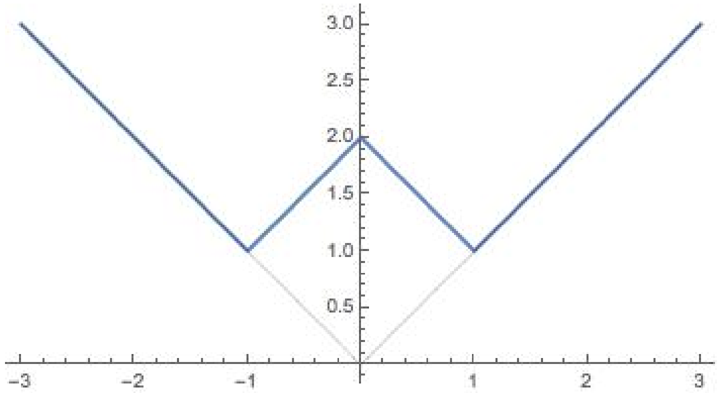
▶ Relation between profile ω_t and its Kerov transition measure $\mathfrak{m}_{\omega_t}$ (recall the Markov transform)

$$\int_{\mathbb{R}} \frac{1}{z-x} \mathfrak{m}_{\omega_t}(dx) = \frac{1}{z} \exp\Big\{ \int_{\mathbb{R}} \frac{1}{x-z} \Big(\frac{\omega_t(x) - |x|}{2} \Big)' dx \Big\}, \quad z \in \mathbb{C}^+$$

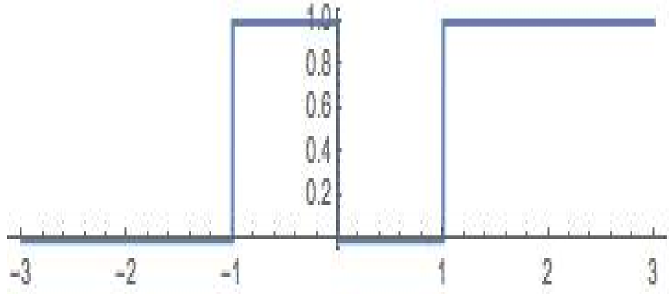
Procedure of computing ω_t from $\mathfrak{m}_{\omega_t}$:

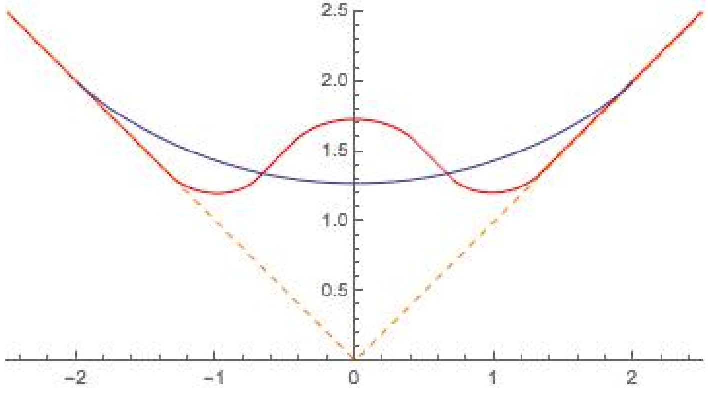
free convolution, free compression $\longrightarrow$ free cumulants
$\longrightarrow$ Voiculescu R-transform $\longrightarrow$ Stieltjes transform
$\longrightarrow$ taking $\log$ $\longrightarrow$ inversion of Stieltjes transform $\longrightarrow$ integration!

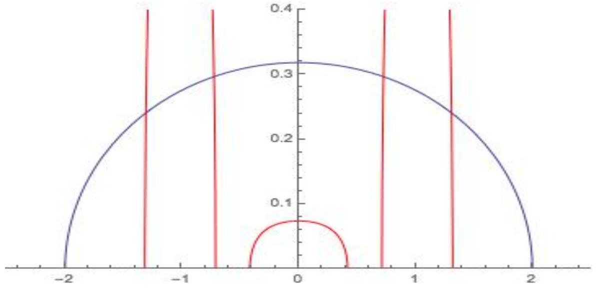
profile	transition measure	Rayleigh function

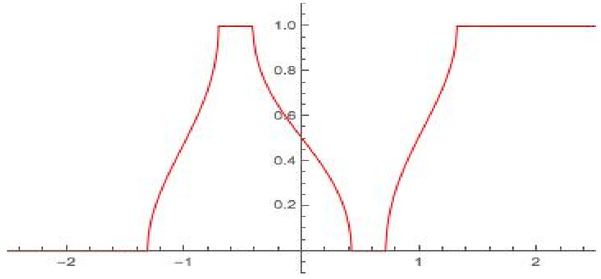


$$\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{1}$$









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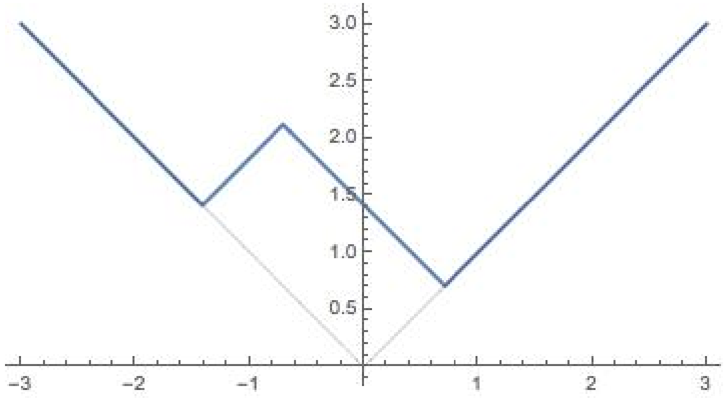
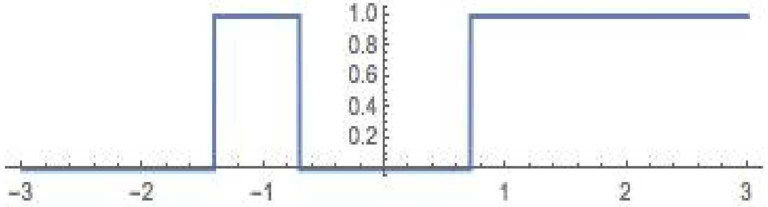
profile

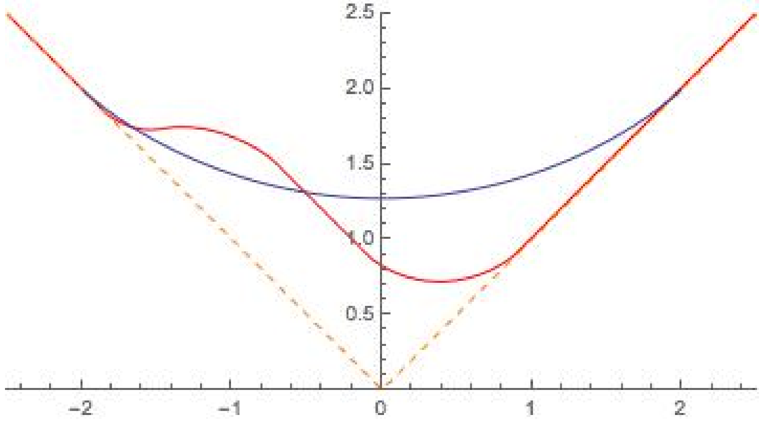
transition measure

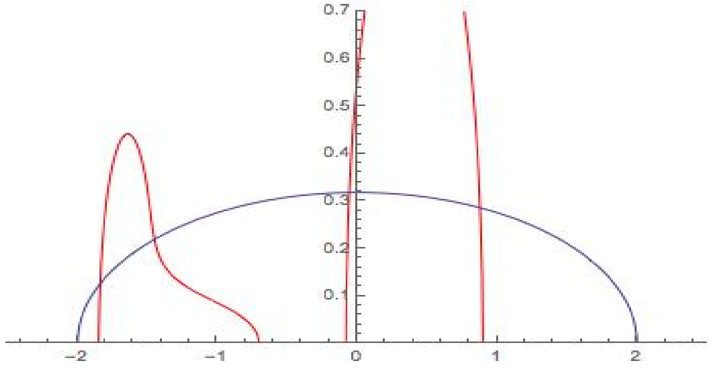
Rayleigh function

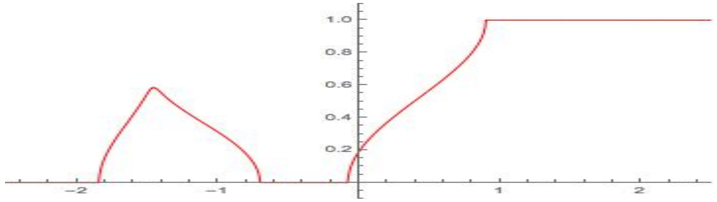


$$\frac{1}{3}\delta_{-\sqrt{2}} + \frac{2}{3}\delta_{1/\sqrt{2}}$$









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Dynamic model : initial $\to\to\to$ Plancherel

Fluctuation for other (non-Plancherel) ensembles

Śniady (2005) "character factorization property"

- $(\mathbb{Y}_n, \mathbb{M}^{(n)})$, C : cumulant functional w.r.t. $E_{\mathbb{M}^{(n)}}$

Assume $C\big[\Sigma_{j_1}, \cdots, \Sigma_{j_k}\big] = O\big(n^{\frac{j_1+\cdots+j_k-k+2}{2}}\big)$. Then

$$\Big\{\sqrt{n}\Big(n^{-\frac{j+1}{2}}\Sigma_j - E_{\mathbb{M}^{(n)}}\big[n^{-\frac{j+1}{2}}\Sigma_j\big]\Big)\Big\}_{j\geqq 2} \xrightarrow[n\to\infty]{} \{X_j\} : \text{Gaussian, mean } 0$$

In our model :

$\sqrt{n}\big(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_{\omega_t}\big)$ on $(\mathbb{Y}_n, \mathbb{M}_t^{(n)})$ $\xrightarrow[n\to\infty]{}$ fluctuation at macroscopic time t

- $\big\{\langle x^j, \sqrt{n}(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_{\omega_t})\rangle\big\}_j \xrightarrow[n\to\infty]{}$ Gaussian system with mean 0
- Covariance has complicated t-dependence, vanishes as $t \to \infty$.

Grand canonical setting $\mathbb{Y} = \bigsqcup_{n=0}^{\infty} \mathbb{Y}_n$

Poissonization of the Plancherel measure

$$\mathbb{M}_{\mathrm{PP}}^{(\xi)} = \sum_{n=0}^{\infty} \frac{e^{-\xi}\xi^n}{n!} \mathbb{M}_{\mathrm{Pl}}^{(n)}, \qquad \xi > 0$$

is kept invariant under transition probability $P^{(\xi)}$ on $\mathbb{Y}$:

$$P^{(\xi)} = \alpha_\xi(n) P^{\uparrow(n)} + (1 - \alpha_\xi(n)) P^{\downarrow(n)},$$

$$\alpha_\xi(n) = \int_0^1 \xi e^{-\xi x} (1-x)^n dx$$

Continuous time Markov chain $(X_s^{(\xi)})_{s \geqq 0}$

- Rescale for time $t\xi$, for space $\frac{1}{\sqrt{\xi}} \lambda(\sqrt{\xi} x) \quad (\lambda \in \mathbb{Y})$

Behavior as $\xi \to \infty$ ......

Method of proofs

- profile of Young diagram $\lambda \longleftrightarrow$ transition measure $\mathfrak{m}_\lambda$
- method of symmetric functions (generators, generating functions)
- free cumulant $R_k(\mathfrak{m}_\lambda)$ vs irreducible character value χ^λ at cycle

$\Longleftarrow$ Kerov polynomial

- diagonalizing transition probability by using irreducible characters
- estimate of ensemble expectation by using approximate factorization property

END