On Evolution of Macroscopic Profiles (and their Global Fluctuations) for Growing Random Young Diagrams

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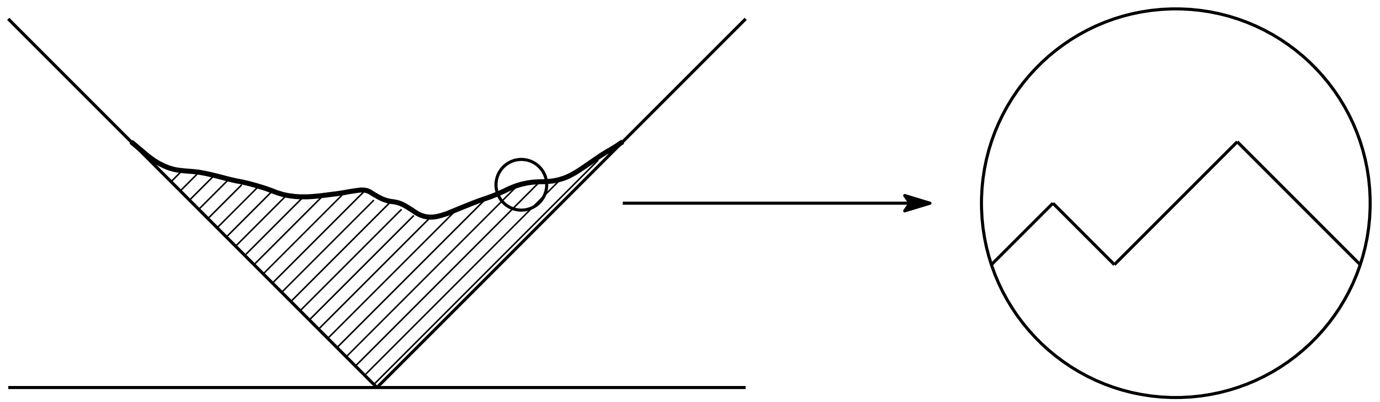


§1 Introduction

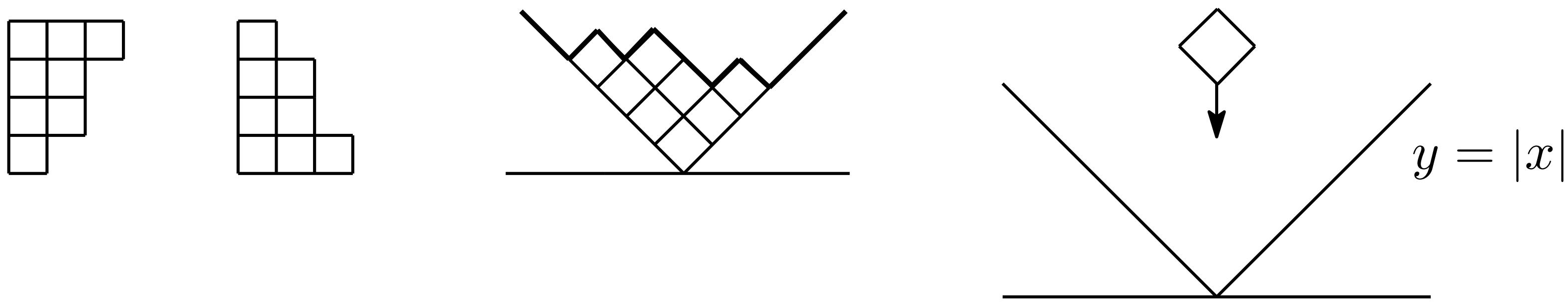


fig. 1 Young diagram $(3 \geqq 2 \geqq 2 \geqq 1) = (1^1 2^2 3^1)$ and its profile

$\mathbb{Y}_n$: the set of Young diagrams of n boxes

$$|\mathbb{Y}_n| = \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}n}\Big(1 + O\big(\frac{1}{\sqrt{n}}\big)\Big)$$

Young diagram λ is characterized by its profile $y = \lambda(x)$

Scheme of the problem

For continuous time Markov chain $(X_s^{(n)})_{s \geqq 0}$ on $\mathbb{Y}_n$,

limiting behavior as $n \to \infty$ and $s \to \infty$ under scaling in space vs time

– macroscopic profile : $1/\sqrt{n}$ both horizontally and vertically

$$\lambda \in \mathbb{Y}_n \quad \longrightarrow \quad \lambda^{\sqrt{n}}(x) = \frac{1}{\sqrt{n}}\lambda(\sqrt{n}x), \quad \lambda^{\sqrt{n}} \in \mathbb{D}$$

– macroscopic time : $t = s/n$ (diffusive scale)

Letting $n \to \infty$, as an effect of LLN($=$ law of large numbers), the distribution of $X_{tn}^{(n)}$ concentrates at a point ω_t, depending on t.

ω_t : macroscopic profile at macroscopic time t

Describe evotuion of ω_t along t !

§2 Markov chain

– transition probability $p(x,y)$ ($x,y \in S$) : $p(x,y) \geqq 0$, $\sum_{y\in S} p(x,y) = 1$

– initial distribution $\nu(x) \geqq 0$, $\sum_{x\in S} \nu(x) = 1$

Then, $\exists (X_n)_{n=0,1,2,\cdots}$ where $X_n : \Omega \longrightarrow S$

$$\mathbb{P}(X_{n+1} = y \,|\, X_n = x) = p(x,y), \quad \mathbb{P}(X_0 = x) = \nu(x)$$

(temporally homogeneous Markov chain on S)

$P = \big(p(x,y)\big)_{x,y\in S}$: transition matrix, $\nu = \big(\nu(x)\big)_{x\in S}$: initial row vector

$$p_n(x,y) = \mathbb{P}(X_n = y \,|\, X_0 = x) = (P^n)_{x,y}, \qquad \mathbb{P}(X_n = x) = (\nu P^n)_x$$

Continuous time Markov chain $(X_{N_s})_{s\geqq 0}$ where

$(N_s)_{s\geqq 0}$: Poisson process on $\{0,1,\cdots\}$, $N_0 = 0$ a.s., independent of (X_n)

$$\tilde{\mathbb{P}}(X_{N_s} = x) = \big(\nu e^{s(P-I)}\big)_x, \quad x \in S$$

§3 Plancherel growth process

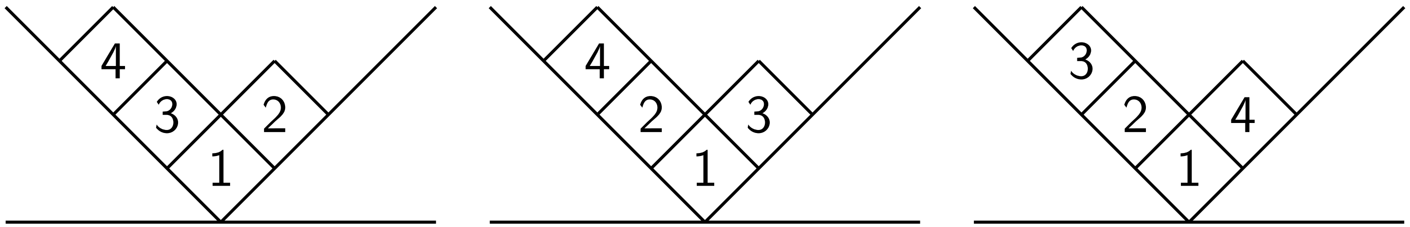


fig. 2 standard tableaux for $(2 \geqq 1 \geqq 1) = (1^2 2^1) \in \mathbb{Y}_4$

To Young diagram λ, dimension of λ is assigned:

$$\begin{aligned} \dim \lambda &= \text{number of standard tableaux for } \lambda \\ &= \text{number of paths from } \varnothing \text{ to } \lambda \text{ on the Young graph} \end{aligned}$$

Ex. $\dim(1^2 2^1) = 3$

Young graph vertices: $\mathbb{Y} = \bigsqcup_{n=0}^{\infty} \mathbb{Y}_n, \quad \mathbb{Y}_0 = \{\varnothing\}$

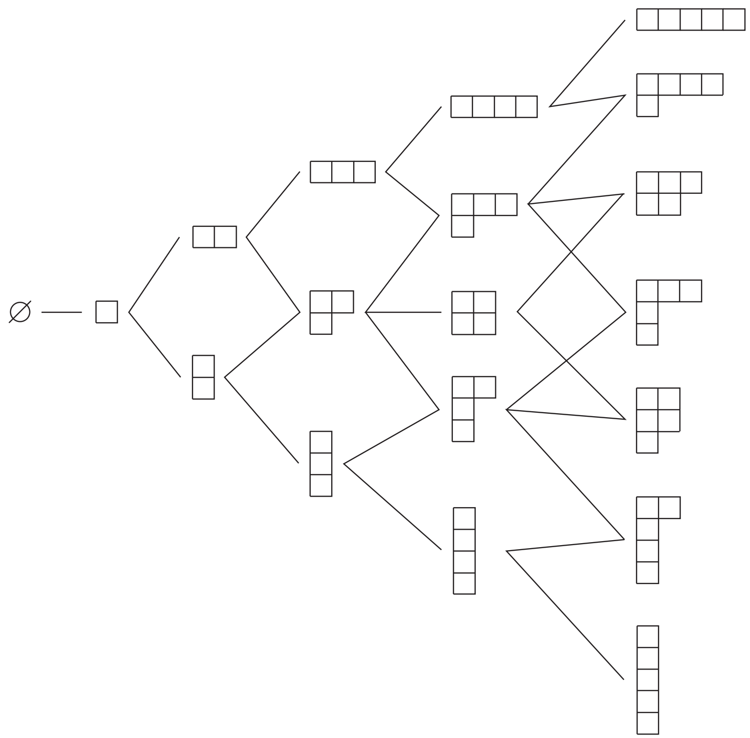


fig. 3 Young graph: dimension in 5th strara — 1, 4, 5, 6, 5, 4, 1 :
$1^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 1^2 = 5!$

Plancherel growth process is Markov chain (Z_n) on Young diagrams s.t.

$$\nu = \delta_{\varnothing}$$

$$p(\lambda,\mu) = p^{\uparrow}(\lambda,\mu) : \text{ proportional to } \dim\mu$$

$$= \frac{\dim\mu}{(|\lambda|+1)\dim\lambda}, \qquad \lambda,\mu\in\mathbb{Y}, \quad \lambda\nearrow\mu$$

Then, the distribution after n step is

$$p_n(\varnothing,\lambda) = \mathbb{P}(Z_n=\lambda) = \frac{(\dim\lambda)^2}{n!} = \mathbb{M}_{\mathrm{Pl}}^{(n)}(\lambda)$$

called Plancherel measure on $\mathbb{Y}_n$ ($\leftarrow$ Fourier transform on $\mathfrak{S}_n$)

- rectangular diagram

$$\mathbb{D}_0 = \Big\{\lambda : \mathbb{R}\longrightarrow\mathbb{R} \,\Big|\, \text{continuous, piecewise linear,}$$
$$\lambda'(x) = \pm 1,\ \lambda(x) = |x| \ (|x| \text{ large enough})\Big\}$$

• continuous diagram

$$\mathbb{D} = \left\{\omega : \mathbb{R} \longrightarrow \mathbb{R} \,\middle|\, |\omega(x) - \omega(y)| \leqq |x - y|,\ \omega(x) = |x|\ (|x| \text{ large enough})\right\}$$

$\lambda^{\sqrt{n}} \in \mathbb{D}_0 \subset \mathbb{D}$ (recall $\lambda^{\sqrt{n}}(x) = \frac{1}{\sqrt{n}}\lambda(\sqrt{n}x)$)

$$\Omega(x) = \begin{cases} \frac{2}{\pi}\left(x \arcsin\frac{x}{2} + \sqrt{4 - x^2}\right), & |x| \leqq 2 \\ |x|, & |x| > 2 \end{cases}$$ limit shape

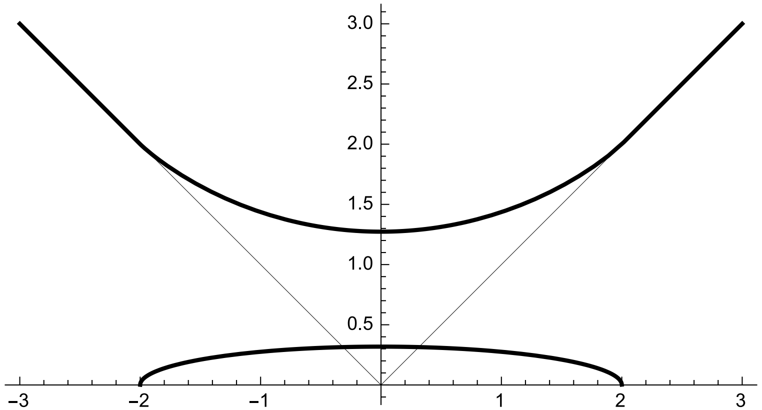


fig. 4 limit shape Ω and its transition measure

The following LLN holds (static scaling limit for the Plancherel measure)
Vershik – Kerov (1977), Logan – Shepp (1977)

$$\mathbb{M}_{\mathrm{Pl}}^{(n)}\Big(\Big\{\lambda \in \mathbb{Y}_n \;\Big|\; \sup_{x\in\mathbb{R}} |\lambda^{\sqrt{n}}(x) - \Omega(x)| \geqq \epsilon\Big\}\Big) = \mathbb{P}\big(\|Z_n^{\sqrt{n}} - \Omega\|_{\sup} \geqq \epsilon\big) \xrightarrow[n\to\infty]{} 0 \qquad (\forall \epsilon > 0)$$

Namely, distribution of $Z_n^{\sqrt{n}}$ converges to δ_Ω as $n \to \infty$. ■

Continuous time Plancherel growth process $\tilde{Z}_s = Z_{N_s}$
with initial distribution $\delta_\varnothing$, transition matrix $e^{s(P-I)}$

$$\tilde{\mathbb{P}}(\tilde{Z}_s = \lambda) = \sum_{n=0}^{\infty} \frac{e^{-s}s^n}{n!} \mathbb{M}_{\mathrm{Pl}}^{(n)}(\lambda), \qquad \lambda \in \mathbb{Y}$$

(Poissonization of the Plancherel measures)

Dynamical scaling limit

s: microscopic time, t: macroscopic time $\quad s = tn$

Then $\quad \tilde{Z}_{tn}^{\sqrt{n}} \xrightarrow[n\to\infty]{} ?$

$$\tilde{\mathbb{P}}\big(\|\tilde{Z}_{tn}^{\sqrt{n}} - \Omega_t\|_{\sup} \geqq \epsilon\big) = \tilde{\mathbb{P}}^{\tilde{Z}_{tn}}\big(\|\lambda^{\sqrt{n}} - \Omega_t\|_{\sup} \geqq \epsilon\big)$$
$$= \sum_{k=0}^{\infty} \frac{e^{-tn}(tn)^k}{k!} \mathbb{M}_{\mathrm{Pl}}^{(k)}\big(\|\lambda^{\sqrt{n}} - \Omega_t\|_{\sup} \geqq \epsilon\big)$$

The above Poisson distribution has mean tn and standard deviation $\sqrt{tn}$

Under $\mathbb{M}_{\mathrm{Pl}}^{(\lfloor tn \rfloor)}$, $\quad \lambda^{\sqrt{tn}} \to \Omega \iff \lambda^{\sqrt{n}} \to \Omega_t \quad$ where

$$\Omega_t(x) = \begin{cases} \frac{2}{\pi}\big(x \arcsin\frac{x}{2\sqrt{t}} + \sqrt{4t - x^2}\big), & |x| \leqq 2\sqrt{t} \\ |x|, & |x| > 2\sqrt{t} \end{cases}$$

Proposition $\quad \tilde{Z}_{tn}^{\sqrt{n}} \xrightarrow[n\to\infty]{} \Omega_t$ in probability ■

(Ω_t is "similar" to $\Omega_1 = \Omega$ (static limit), so this is not very remarkable as a dynamic model.)

§4 Restriction-induction chain

restriction $\leftrightarrow$ removing 1 box,   induction $\leftrightarrow$ adding 1 box

$p^{\uparrow}(\lambda, \mu)$ as before

$$p^{\downarrow}(\lambda, \mu) \quad \text{(proportional to } \dim \mu) \; = \begin{cases} \frac{\dim \mu}{\dim \lambda}, & \mu \nearrow \lambda \\ 0, & \text{otherwise} \end{cases}$$

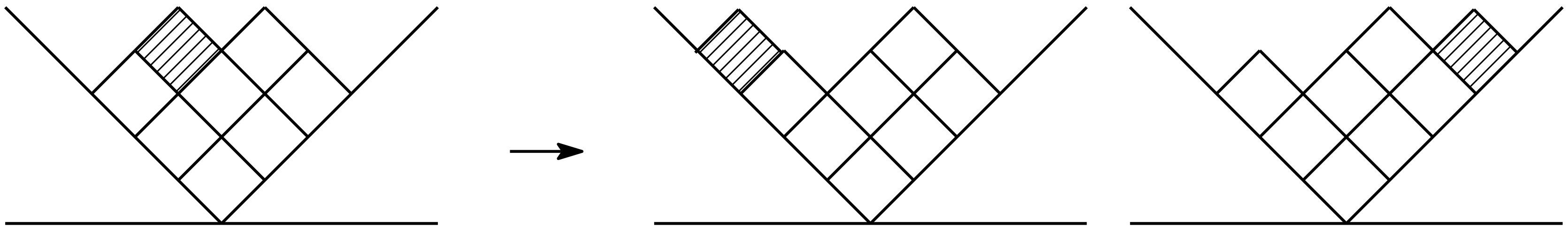


fig. 5   Res-Ind chain: transition from $\lambda = (3, 3, 2)$

Res-Ind chain $(X_m^{(n)})_{m=0,1,2,\cdots}$ on $\mathbb{Y}_n$ has transition matrix
$P^{(n)} = P^{\downarrow}P^{\uparrow} = \big(p^{(n)}(\lambda,\mu)\big)_{\lambda,\mu\in\mathbb{Y}_n}$

$$p^{(n)}(\lambda,\mu) = \sum_{\nu\in\mathbb{Y}_{n-1}:\nu\nearrow\lambda,\nu\nearrow\mu} p^{\downarrow}(\lambda,\nu)p^{\uparrow}(\nu,\mu), \qquad \lambda,\mu\in\mathbb{Y}_n$$

Lemma Res-Ind chain is symmetric w.r.t. the Plancherel measure:

$$\mathbb{M}_{\mathrm{Pl}}^{(n)}(\lambda)p^{(n)}(\lambda,\mu) = \mathbb{M}_{\mathrm{Pl}}^{(n)}(\mu)p^{(n)}(\mu,\lambda), \quad \lambda,\mu\in\mathbb{Y}_n,$$

hence the Plancherel measure is invariant distribution for Res-Ind chain ■

Continuous time Res-Ind chain $\tilde{X}_s^{(n)} = X_{N_s}^{(n)}$ on $\mathbb{Y}_n$ with

transition matrix $e^{s(P^{(n)}-I)}$,

initial distribution $\mathbb{M}_0^{(n)}$ (see Remark),

invariant distribution $\mathbb{M}_{\mathrm{Pl}}^{(n)}$

Remark For a sequence of probability spaces $(\mathbb{Y}_n, \mathbb{M}^{(n)})$, we know some sufficient condition for LLN

$$\mathbb{M}^{(n)}\Big(\Big\{\lambda \in \mathbb{Y}_n \;\Big|\; \|\lambda^{\sqrt{n}} - \psi\|_{\sup} \geqq \epsilon\Big\}\Big) \xrightarrow[n\to\infty]{} 0 \quad (\forall \epsilon > 0)$$

to hold with some continuous diagram $\psi \in \mathbb{D}$, which we call a concentration property at ψ (a certain approximate factorization property).

Ex. Plancherel measures $(\mathbb{Y}_n, \mathbb{M}^{(n)}_{\mathrm{Pl}})$ satisfy this concentration property.

Dynamic scaling limit

s: microscopic time, t: macroscopic time $\quad s = tn$

Then $\tilde{X}_{tn}{}^{(n)\,\sqrt{n}} \xrightarrow[n\to\infty]{} ?$ (macroscopic profile depending on t)

Let $\mathbb{M}^{(n)}_t = \tilde{\mathbb{P}}^{\tilde{X}^{(n)}_{tn}}$: distribution of $\tilde{X}^{(n)}_{tn}$ on $\mathbb{Y}_n$

Theorem The concentration property is propagated as time goes by, i.e. if initial distributions $\mathbb{M}_0^{(n)}$ satisfy the concentration property at $\omega_0 \in \mathbb{D}$, then for $\forall t > 0$ $\mathbb{M}_t^{(n)}$ also satisfy the concentration property, hence there exists $\omega_t \in \mathbb{D}$ s.t. LLN

$$\mathbb{M}_t^{(n)}\left(\left\{\lambda \in \mathbb{Y}_n \;\middle|\; \|\lambda^{\sqrt{n}} - \omega_t\|_{\sup} \geqq \epsilon\right\}\right) \xrightarrow[n\to\infty]{} 0 \quad (\forall \epsilon > 0)$$

holds. ■

– ω_0 can be taken arbitrarily in $\mathbb{D}$

– ω_t converges to Ω (limit shape) in $\mathbb{D}$ as $t \to \infty$

– The area is kept invariant: $\int_{\mathbb{R}}(\omega_t(x) - |x|)dx = 2$ for $\forall t$

– ω_t is described precisely by using free probability

(see the following sections)

initial distribution $\mathbb{M}_0^{(n)}$	$\longrightarrow$	ω_0
$\downarrow$	$1/\sqrt{n}, \quad n \to \infty$	
distribution at time tn $\mathbb{M}_t^{(n)}$	$\longrightarrow$	ω_t
$\downarrow$		
invariant distribution $\mathbb{M}_{\mathrm{Pl}}^{(n)}$	$\longrightarrow$	Ω

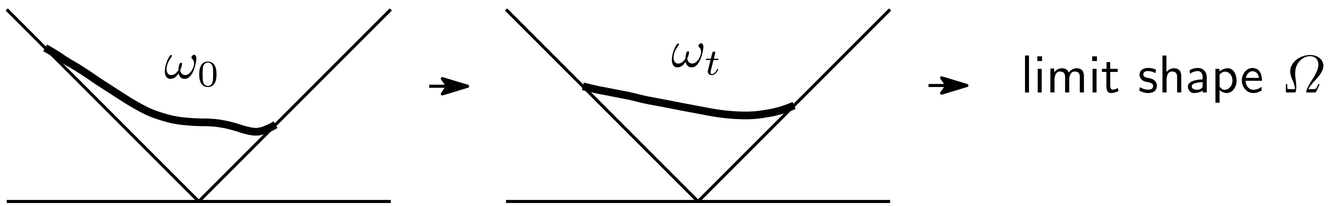


fig. 6 evolution of macroscopic profile

§5 **Technical digressions** — Markov transform and free probability

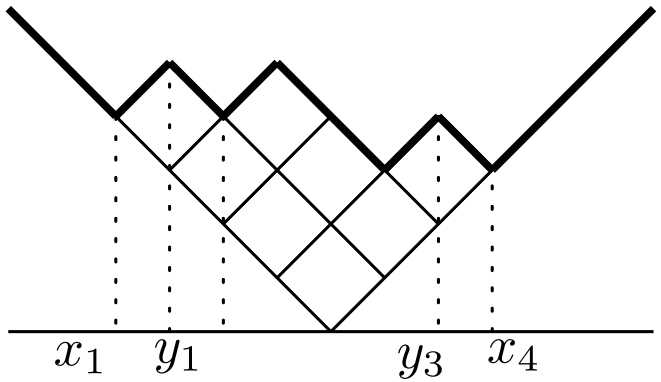


fig. 7 peak-valley coordinates of a Young diagram

peak-valley coordinates of $\lambda \in \mathbb{D}_0$: $(x_1 < y_1 < x_2 < \cdots < y_{r-1} < x_r)$

$$G_\lambda(z) = \frac{(z-y_1)\cdots(z-y_{r-1})}{(z-x_1)\cdots(z-x_r)} = \frac{\mu_1}{z-x_1} + \cdots + \frac{\mu_r}{z-x_r}$$

Then, $\mu_i > 0$ and $\sum_{i=1}^{r} \mu_i = 1$, so $\mathfrak{m}_\lambda = \sum_{i=1}^{r} \mu_i \delta_{x_i} \in \mathcal{P}(\mathbb{R})$

Lemma $\mathbb{D}_0 \cong \{\mu \in \mathcal{P}(\mathbb{R}) \,|\, \text{mean } 0, \mathrm{supp}\mu \text{ is a finite set}\}$ by $\lambda \leftrightarrow \mathfrak{m}_\lambda$

Lemma Extended to embedding $\mathbb{D} \longrightarrow \mathcal{P}(\mathbb{R})$: Markov(-Krein) transform

$\omega \mapsto \mathfrak{m}_\omega$: transition measure of continuous diagram

$$\frac{1}{z} \exp\Big\{ \int_{\mathbb{R}} \frac{1}{x-z} \Big(\frac{\omega(x) - |x|}{2} \Big)' dx \Big\} = \int_{\mathbb{R}} \frac{1}{z-x} \mathfrak{m}_\omega(dx), \quad z \in \mathbb{C}^+$$

Remark Add a box at the ith valley x_i of $\lambda \in \mathbb{Y}$ to make $\mu^{(i)}$, then

$$\mathfrak{m}_\lambda(x_i) = \frac{\dim \mu^{(i)}}{(|\lambda| + 1) \dim \lambda} = p^{\uparrow}(\lambda, \mu^{(i)})$$

(transition probability for Plancherel growth process)

Freeness is a notion for describing relation between random variables.

Free structure often appears in large random matrices/permutations.

In several mathematical contexts,

independence vs freeness for random variables

results in/from interesting contrasts such as

- direct product vs free product (as group or algebra structure)
- lattice vs tree (as Laplacian)
- Gauss vs Wigner (as central limit theorem)
- Boson Fock vs full Fock (as creation and annihilation) etc.

Let a, b be real random variables (typically, self-adjoint elements in function or operator algebra) with distributions μ, ν respectively

$$\mathbb{E}[a^n] = \int_{\mathbb{R}} x^n \mu(dx), \; \mathbb{E}[b^n] = \int_{\mathbb{R}} x^n \nu(dx) \implies \mathbb{E}[(a+b)^n] = \int_{\mathbb{R}} x^n \, ? \, (dx)$$

$a + b \longrightarrow \mu * \nu$	convolution	if a, b are independent
$\longrightarrow \mu \boxplus \nu$	free convolution	if a, b are free

p : projection free to $a \longrightarrow pap$: free compression

$c =$ expectation of $p \in (0,1)$ i.e. $\mathbb{E}[p] = \mathbb{E}[p^2] = c$

μ_c : distribution of pap (no commutative analogue)

$$\mathbb{E}[(pap)^n] = \int_{\mathbb{R}} x^n \, \mu_c(dx)$$

§6 Characterization of the macroscopic profile at time t

Theorem (recall) [Publ. RIMS 2015, SpringerBriefs Math-Phys. 2016]

The concentration property is propagated as time goes by.

If initial distributions $\mathbb{M}_0^{(n)}$ satisfy the concentration property with $\omega_0 \in \mathbb{D}$, then for $\forall t > 0$ $\mathbb{M}_t^{(n)}$ also satisfy the concentration property, hence there exists $\omega_t \in \mathbb{D}$ s.t. LLN

$$\mathbb{M}_t^{(n)}\Big(\Big\{\lambda \in \mathbb{Y}_n \,\Big|\, \|\lambda^{\sqrt{n}} - \omega_t\|_{\sup} \geqq \epsilon\Big\}\Big) \xrightarrow[n\to\infty]{} 0 \quad (\forall \epsilon > 0).$$

Here ω_t is determined by

$$\mathfrak{m}_{\omega_t} = (\mathfrak{m}_{\omega_0})_{e^{-t}} \boxplus (\mathfrak{m}_{\Omega})_{1-e^{-t}}$$

(free convolution of free compressions of transition measures).

Furthermore time evolution of the distributions is described through its Stieltjes transform: $G(t,z) = \int_{\mathbb{R}} \frac{1}{z-x} \mathfrak{m}_{\omega_t}(dx)$. ■

▶ PDE describing time evolution of transition measure $\mathfrak{m}_{\omega_t}$

$$\frac{\partial G}{\partial t} = -G\,\frac{\partial G}{\partial z} + \frac{1}{G}\,\frac{\partial G}{\partial z} + G, \qquad t > 0,\ \ z \in \mathbb{C}^+$$

Remark Transition measure of Ω_t (limit shape of Plancherel growth process at time t) is semicircle distribution of mean 0 and variance t,

$$g(t,z) = \int_{\mathbb{R}} \frac{1}{z-x}\mathfrak{m}_{\Omega_t}(dx) = \frac{z - \sqrt{z^2 - 4t}}{2t}$$

satisfying PDE :

$$\frac{\partial g}{\partial t} = -g\,\frac{\partial g}{\partial z}$$

- Equation for $\omega(t,x) = \omega_t(x)$ is still open.
- 1 step transition of Res-Ind chain is non-local.

§7 Global fluctuation

Dynamic model : initial $\to\to\to$ Plancherel

Fluctuation for other (non-Plancherel) ensembles

Śniady (2005) "character factorization property"

- ▶ $(\mathbb{Y}_n, \mathbb{M}^{(n)})$ or $(Z(\mathbb{C}[\mathfrak{S}_n]), \phi^{(n)})$,

  C : cumulant functional w.r.t. $E_{\mathbb{M}^{(n)}}$ or $\phi^{(n)}$

Assume

$$C\big[\Sigma_{j_1}, \cdots, \Sigma_{j_k}\big] = O\big(n^{\frac{j_1+\cdots+j_k-k+2}{2}}\big).$$

Then

$$\Big\{\sqrt{n}\Big(n^{-\frac{j+1}{2}}\Sigma_j - E_{\mathbb{M}^{(n)}}\big[n^{-\frac{j+1}{2}}\Sigma_j\big]\Big)\Big\}_{j\geqq 2} \xrightarrow[n\to\infty]{} \{X_j\} \text{ : Gaussian, mean } 0. \tag{$\star$}$$

Theorem (not satisfactory)

In our model, character factorization property is propagated at any macroscopic time t.

Hence $\sqrt{n}\big(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_{\omega_t}\big)$ on $(\mathbb{Y}_n, \mathbb{M}_t^{(n)})$ converges as $n \to \infty$ to the fluctuation at t, i.e.

▶ $\big\{\langle x^j, \sqrt{n}(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_{\omega_t})\rangle\big\}_j \xrightarrow[n\to\infty]{}$ Gaussian system with mean 0

▶ Covariance of (X_j) in $(\star)$ for $\mathbb{M}^{(n)} = \mathbb{M}_t^{(n)}$ has complicated t-dependence, vanishes as $t \to \infty$. ■

Grand canonical setting $\mathbb{Y} = \bigsqcup_{n=0}^{\infty} \mathbb{Y}_n$

Poissonization of the Plancherel measure

$$\mathbb{M}_{\mathrm{PP}}^{(\xi)} = \sum_{n=0}^{\infty} \frac{e^{-\xi}\xi^n}{n!} \mathbb{M}_{\mathrm{Pl}}^{(n)}, \qquad \xi > 0$$

is kept invariant under transition probability $P^{(\xi)}$ on $\mathbb{Y}$:

$$P^{(\xi)} = \alpha_\xi(n) P^{\uparrow(n)} + (1 - \alpha_\xi(n)) P^{\downarrow(n)},$$

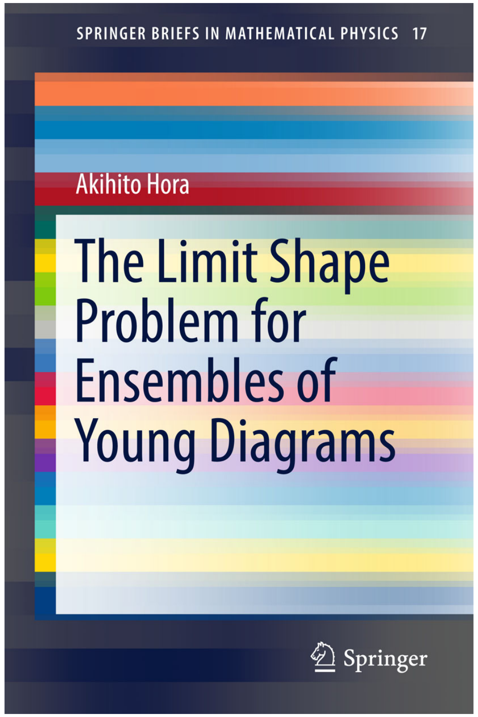
$$\alpha_\xi(n) = \int_0^1 \xi e^{-\xi x}(1-x)^n dx$$

Continuous time Markov chain $(X_s^{(\xi)})_{s \geqq 0}$

▶ Rescale for time $t\xi$, for space $\frac{1}{\sqrt{\xi}}\lambda(\sqrt{\xi}x)$ $(\lambda \in \mathbb{Y})$

Behavior as $\xi \to \infty$ ......

Reference



A. Hora: The Limit Shape Problem for Ensembles of Young Diagrams, SpringerBriefs in Mathematical Physics 17, Springer, 2016

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