LECTURE NOTE ON INTRODUCTION TO ASYMPTOTIC THEORY FOR REPRESENTATIONS AND CHARACTERS OF SYMMETRIC GROUPS

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Dedicated to Professor Takeshi Hirai for his 70th birthday

Abstract. This is a lecture note for the course I gave at Wrocław University in Poland in April – June of 2007.

1. Overview (Lecture 1)

The first lecture is an overview of the course. The presentation document will be available in a separate form.

Throughout the course we treat growing random Young diagrams and investigate how they describe asymptotic behaviour of corresponding representations and characters of symmetric groups as the sizes of Young diagrams and symmetric groups tend to infinity. Let $\mathfrak{S}_n$ and $\mathbb{Y}_n$ denote the symmetric group of degree n and the set of Young diagrams of size n respectively. The transposed diagram of $\lambda \in \mathbb{Y}_n$ is denoted by λ'.

We focus on two different regimes of scales:

- Vershik–Kerov condition

$$\lambda \in \mathbb{Y}_n; \quad \lambda_i,\ \lambda'_i \ \sim\ (\text{constant})\ \times n$$

- balanced Young diagram

$$\lambda \in \mathbb{Y}_n; \quad \lambda_1,\ \lambda'_1 \ \sim\ (\text{constant})\ \times \sqrt{n}.$$

The course is divided into three parts. In the first part, we introduce basic concepts and tools concerning Young diagrams and the Young graph, including irreducible representations of symmetric groups, spectral properties of the Jucys–Murphy element, Kerov's transition measure for a Young diagram, harmonic functions on the Young graph, the Plancherel measure, characters of the infinite symmetric group, and characterizations for extremality (or ergodicity). In the second part, we treat growing balanced Young diagrams. Main topics are on the limit shape of Young diagrams and fluctuations around it. Some aspects related to quantum probability theory are also interwoven. The third part is devoted to growing Young diagrams with the Vershik–Kerov condition. As corresponding representation-theoretic objects, we deal with characters and finite factorial representations of the infinite symmetric group $\mathfrak{S}_\infty$ and a wreath product of a compact group with $\mathfrak{S}_\infty$.

2. Young diagrams and the Young graph I (Lecture 2)

2.1. **Young diagrams.** A Young diagram λ of size n, i.e. $\lambda \in \mathbb{Y}_n$, is specified by non-increasing integers: $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ such that $\sum_j \lambda_j = n$, where λ_j is the length of the jth row. The size and the number of rows [resp. columns] of $\lambda \in \mathbb{Y}_n$ are denoted by $|\lambda|$ and $r(\lambda)$ [resp. $c(\lambda)$] in this note. Alternatively, $\lambda \in \mathbb{Y}_n$ is expressed as $(1^{m_1(\lambda)}2^{m_2(\lambda)}\cdots)$, $m_j(\lambda)$ denoting the number of rows of length j of λ. In Figure 2.1, $\lambda = (3,2,2,1) = (1^1 2^2 3^1)$ with $|\lambda| = 8$, $r(\lambda) = 4$ and $c(\lambda) = 3$.

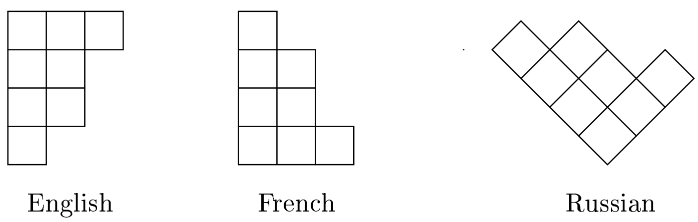

Fig. 2.1. Young diagram $(1^1 2^2 3^1)$

$\mathbb{Y}_n$ parametrizes both the conjugacy classes and the equivalence classes of irreducible representations (IRs for short) of the symmetric group $\mathfrak{S}_n$ of degree n. Let C_ρ denote the conjugacy class of $\mathfrak{S}_n$ corresponding to $\rho \in \mathbb{Y}_n$. An element of C_ρ has by definition the cycle decomposition of type ρ. It is easy to see that

$$|C_\rho| = \frac{n!}{z_\rho}, \qquad z_\rho = \prod_{j=1}^{n} j^{m_j(\rho)} m_j(\rho)!\,.$$

Given $\lambda \in \mathbb{Y}_n$, a Young tableau of shape λ is an array of $\{1,2,\dots,n\}$ put into the boxes of λ one by one. A Young tableau is said to be standard if the arrays are increasing along both rows and columns. The set of tableaux [resp. standard tableaux] of shape λ is denoted by $\mathrm{Tab}(\lambda)$ [resp. $\mathrm{STab}(\lambda)$]. The number of standard tableaux of shape λ, denoted by f^λ, is given by the following hook formula.

Proposition 2.1.

$$f^\lambda = \frac{n!}{\prod_{b\in\lambda} h_\lambda(b)}, \qquad \lambda \in \mathbb{Y}_n\,.$$

Here $h_\lambda(b)$ is the hook length of the box b in λ. See Figure 2.2.

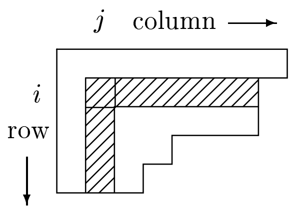

Fig. 2.2. Hook of the (i,j)-box

2.2. **Irreducible representations of $\mathfrak{S}_n$ and the branching rule.** There are several manners to construct an IR of $\mathfrak{S}_n$ associated with $\lambda \in \mathbb{Y}_n$. Here we mention the actions on the Specht polynomials. Set

$$\Delta(x_1, \dots, x_n) = \prod_{1 \le i < j \le n} (x_i - x_j) = \det \begin{pmatrix} x_1^{n-1} & \dots & x_1^2 & x_1 & 1 \\ x_2^{n-1} & \dots & x_2^2 & x_2 & 1 \\ \vdots & & \vdots & \vdots & \vdots \\ x_n^{n-1} & \dots & x_n^2 & x_n & 1 \end{pmatrix}.$$

First let $\lambda \in \mathbb{Y}_n$ be a one-column diagram. For a tableau $T \in \mathrm{Tab}(\lambda)$ filled with $i_1, i_2, \cdots$ from the top, set

$$\Delta(T) = \Delta(x_{i_1}, x_{i_2}, \cdots).$$

If $\lambda \in \mathbb{Y}_n$ is a general shape, for $T \in \mathrm{Tab}(\lambda)$ with T_j as the jth column, set

$$\Delta(T) = \prod_{j=1}^{c(\lambda)} \Delta(T_j).$$

The actions of $g \in \mathfrak{S}_n$ on tableau T and polynomial $F(x_1, \cdots, x_n)$ are given respectively by

$$(gT)(i,j) = g(T(i,j)), \qquad (gF)(x_1, \dots, x_n) = F(x_{g(1)}, \dots, x_{g(n)}),$$

where $T(i,j)$ denotes the letter in the (i,j)-box of T. Since $\Delta(gT) = g\Delta(T)$ holds, $\{\Delta(T) \,|\, T \in \mathrm{Tab}(\lambda)\}$ spans an $\mathfrak{S}_n$-invariant linear space S_λ, called a Specht module. U_λ denoting this action, we have a representation (U_λ, S_λ) of $\mathfrak{S}_n$.

Proposition 2.2. *The set $\{\Delta(T) \,|\, T \in \mathrm{STab}(\lambda)\}$ forms a basis of S_λ. In particular, $f^\lambda = \dim S_\lambda (= \dim \lambda)$.*

Let us see $\mathfrak{S}_{n-1}$-invariant subspaces of S_λ for $\lambda \in \mathbb{Y}_n$. We enumerate the corners of λ as $\square_1, \square_2, \cdots, \square_q$. For each $i = 1, \dots, q$, set

$$\lambda^{(i)} = \lambda \setminus \square_i \in \mathbb{Y}_{n-1},$$

and let V_i be the linear space spanned by the Specht polynomials $\Delta(T)$ with $T \in \mathrm{STab}(\lambda)$ such that the ith corner $\square_i$ of T is filled with the letter n. Setting

$$\tilde{V}_i = V_i + \cdots + V_q ,$$

we have a descending sequence of subspaces

$$S_\lambda = \tilde{V}_1 \supsetneq \tilde{V}_2 \supsetneq \cdots \supsetneq \tilde{V}_q = V_q .$$

Each vector in $\tilde{V}_i$ is a sum of those in V_i and $\tilde{V}_{i+1}$: $\sum \alpha_T \Delta(T) + v$ where $\alpha_T \in \mathbb{C}$, $T \in \mathrm{STab}(\lambda)$ containing n at $\square_i$, and $v \in \tilde{V}_{i+1}$. Setting $\bar{T} = T \setminus \square_i$ (filled with n) $\in \mathrm{STab}(\lambda^{(i)})$, we define linear map $\phi_i : \tilde{V}_i \longrightarrow S_{\lambda^{(i)}}$ by assigning $\sum \alpha_T \Delta(\bar{T})$ to the above vector.

Lemma 2.3. (1) *$\tilde{V}_i$ is $\mathfrak{S}_{n-1}$-invariant.*
(2) *ϕ_i intertwines the actions of $\mathfrak{S}_{n-1}$.*

We hence have

$$\tilde{V}_i / \tilde{V}_{i+1} \cong_{\mathfrak{S}_{n-1}} S_{\lambda^{(i)}} , \quad i = 1, \dots, q-1, \qquad \tilde{V}_q \cong_{\mathfrak{S}_{n-1}} S_{\lambda^{(q)}} ,$$

since $\ker\phi_i = \tilde{V}_{i+1}$. Applying complete reducibility succesively, we obtain $\mathfrak{S}_{n-1}$-invariant subspaces $W_q, W_{q-1}, \cdots, W_1$:

$$\tilde{V}_{q-1} = W_{q-1} \oplus W_q, \qquad W_q = \tilde{V}_q = V_q, \quad W_{q-1} \cong_{\mathfrak{S}_{n-1}} S_{\lambda^{(q-1)}},$$
$$\tilde{V}_{q-2} = W_{q-2} \oplus W_{q-1} \oplus W_q, \qquad W_{q-2} \cong_{\mathfrak{S}_{n-1}} S_{\lambda^{(q-2)}},$$
$$\cdots$$

to reach the following.

Theorem 2.4. *We have a sequence of subspaces $W_1, \cdots, W_q$ of S_λ such that $S_\lambda = W_1 \oplus \cdots \oplus W_q$ and*

$$W_q = V_q \cong_{\mathfrak{S}_{n-1}} S_{\lambda^{(q)}}, \quad W_i \cong_{\mathfrak{S}_{n-1}} S_{\lambda^{(i)}}, \quad i = 1, \ldots, q-1. \tag{2.1}$$

Namely

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} U_\lambda \cong \bigoplus_{\mu:\mu\nearrow\lambda} U_\mu, \qquad \lambda \in \mathbb{Y}_n. \tag{2.2}$$

Theorem 2.5. *For $\lambda \in \mathbb{Y}_n$ let (U_λ, S_λ) be the representation of $\mathfrak{S}_n$ defined above.*
(1) U_λ is an IR of $\mathfrak{S}_n$.
(2) U_λ and U_μ are not equivalent if $\lambda \neq \mu$.
(3) $\{U_\lambda\,;\, \lambda \in \mathbb{Y}_n\}$ is a system of complete representatives of the equivalence classes of IRs of $\mathfrak{S}_n$.

Proof. Assertions (1) and (2) are simultaneously proved by induction on n. Then (3) follows from a general fact that the cardinality of the set of the equivalence classes of IRs coincides with that of the conjugacy classes. □

The branching graph for $\mathfrak{S}_n$'s is called the Young graph, whose vertex set consists of all Young diagrams:

$$\mathbb{Y} = \bigsqcup_{n=0}^{\infty} \mathbb{Y}_n$$

where $\mathbb{Y}_0 = \{\varnothing\}$. An edge is given by the relation $\lambda \nearrow \mu$ where $|\lambda| + 1 = |\mu|$. The number of paths connecting $\varnothing$ to λ on the Young graph coincides with f^λ. Repeating the decomposition of (2.1), we specify a unique one-dimensional subspace of S_λ to each path from λ back to $\varnothing$. This yields the Young basis $\{v_u\}$ of S_λ:

$$S_\lambda = \bigoplus_{u:\ \text{path}\ \varnothing\nearrow\cdots\nearrow\lambda} \mathbb{C}v_u.$$

2.3. **The Jucys–Murphy elements.** We call

$$J_{n-1} = (1\ n) + (2\ n) + \cdots + (n-1\ n) \quad \in \mathbb{C}[\mathfrak{S}_n]$$

a Jucys–Murphy element. J_{n-1} commutes with $\mathfrak{S}_{n-1}$ embedded in $\mathfrak{S}_n$ as before. For $\lambda \in \mathbb{Y}_n$, $U_\lambda(J_{n-1})$ is called a Jucys–Murphy operator on S_λ.

Proposition 2.6. *Let $\lambda \in \mathbb{Y}_n$ has the (i_k, j_k)-box as the kth corner $\square_k$. If $S \in STab(\lambda)$ contains n at $\square_k$, then*

$$U_\lambda(J_{n-1})\Delta(S) = (j_k - i_k)\Delta(S) + \ (\textit{vector in}\ \tilde{V}_{k+1}).$$

This yields an upper triangularization of the Jucys–Murphy operator with respect to an appropriate ordering of tableaux.

Corollary 2.7. *The spectrum of* $U_\lambda(J_{n-1})$ *is given by*

$$\begin{array}{cccc} j_1 - i_1 & j_2 - i_2 & \cdots & j_q - i_q \\ \dim \lambda^{(1)} & \dim \lambda^{(2)} & \cdots & \dim \lambda^{(q)} \end{array} .$$

Actually, a further discussion yields the following diagonalization.

Theorem 2.8. *In the decomposition*

$$S_\lambda = W_1 \oplus \cdots \oplus W_q, \quad W_k \cong_{\mathfrak{S}_{n-1}} S_{\lambda^{(k)}},$$

W_k *is the eigenspace of* $U_\lambda(J_{n-1})$ *associated with eigenvalue* $j_k - i_k$*.*

Thus the eigenspace decomposition of Jucys–Murphy operator $U_\lambda(J_{n-1})$ describes the branching $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} U_\lambda$.

Remark 2.9. In this section we skipped cumbersome parts of proofs. However, they are all covered by a relation between Specht polynomials called the Garnir relation.

3. Young diagrams and the Young graph II (Lecture 3)

3.1. Profiles and rectangular diagrams.

Definition 3.1. Function $\omega : \mathbb{R} \longrightarrow \mathbb{R}$ is called a rectangular diagram if
(i) ω is continuous and piecewise linear,
(ii) $\omega'(x) = \pm 1$ except finite $x \in \mathbb{R}$,
(iii) $\omega(x) = |x|$ if $|x|$ is large enough.
The set of rectangular diagrams is denoted by $\mathbb{D}_0$.

We assign the interlacing sequence $x_1 < y_1 < \cdots < x_{r-1} < y_{r-1} < y_r$ to $\omega \in \mathbb{D}_0$ where x_i and y_i indicate a valley (local minimum) and a peak (local maximum) respectively as in Figure 3.1. In the present note, this interlacing sequence is called the min-max coordinates of ω.

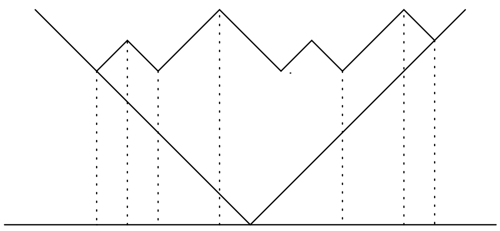

FIG. 3.1. Min-max coordinates

A Young diagram is regarded as a rectangular diagram through its profile as shown by thick lines in Figure 3.2. Here each box is a $\sqrt{2} \times \sqrt{2}$ square. A Young diagram is thus a rectangular diagram whose min-max coordinates are all integers.

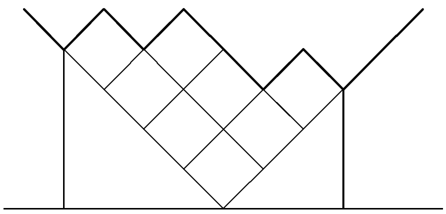

FIG. 3.2. The profile of $\lambda = (1^1 2^2 3^1)$, where $c(\lambda) = 3$ and $r(\lambda) = 4$

Lemma 3.2. (1) *The min-max coordinates of* $\omega \in \mathbb{D}_0$ *satisfy*

$$\sum_{i=1}^{r} x_i = \sum_{i=1}^{r-1} y_i. \tag{3.1}$$

(2) *An interlacing sequence* $x_1 < y_1 < \cdots < y_{r-1} < x_r$ *satisfying* (3.1) *recovers a rectangular diagram.*

Proof. Assertion (1) is proved by induction on the number of valleys r. For (2) observe first that $x_1 < 0$ and $x_r > 0$. Starting at the point $(x_1, -x_1)$, count gain and loss in the vertical direction to reach exactly (x_r, x_r). □

Definition 3.3. For $\omega \in \mathbb{D}_0$ with min-max coordinates $x_1 < y_1 < \cdots < x_r$ set

$$\tau_\omega = \sum_{i=1}^{r} \delta_{x_i} - \sum_{i=1}^{r-1} \delta_{y_i} = \Big(\frac{\omega(x) - |x|}{2}\Big)'' + \delta_0$$

and call it the Rayleigh measure of ω.

The Rayleigh measure of a rectangular diagram is an atomic signed probability on $\mathbb{R}$. Its moment is given by

$$M_k(\tau_\omega) = \sum_{i=1}^{r} x_i^k - \sum_{i=1}^{r-1} y_i^k = \int_{-\infty}^{\infty} x^k \Big(\frac{\omega(x) - |x|}{2}\Big)'' dx, \tag{3.2}$$

e.g. $M_0(\tau_\omega) = 1$, $M_1(\tau_\omega) = 0$, and

$$M_2(\tau_\omega) = \int_{-\infty}^{\infty} (\omega(x) - |x|) dx$$

which is $2n$ if $\omega \in \mathbb{Y}_n$.

3.2. **Kerov's transition measure.** For $\omega \in \mathbb{D}_0$ with min-max coordinates $x_1 < y_1 < \cdots < x_r$, consider partial fraction expansion of a rational function on $\mathbb{C}$:

$$\frac{(z - y_1) \cdots (z - y_{r-1})}{(z - x_1) \cdots (z - x_r)} = \frac{\mu_1}{z - x_1} + \cdots + \frac{\mu_r}{z - x_r}, \tag{3.3}$$

explicitly

$$\mu_i = \frac{x_i - y_1}{x_i - x_1} \cdots \frac{x_i - y_{i-1}}{x_i - x_{i-1}} \frac{x_i - y_i}{x_i - x_{i+1}} \cdots \frac{x_i - y_{r-1}}{x_i - x_r} > 0.$$

Taking $\lim_{z \to \infty} z \times (3.3)$, we have $\sum_{i=1}^{r} \mu_i = 1$.

Definition 3.4. We call the probability

$$\mathfrak{m}_\omega = \sum_{i=1}^{r} \mu_i \delta_{x_i}$$

on $\mathbb{R}$ the transition measure (due to Kerov) of $\omega \in \mathbb{D}_0$.

Equation (3.3) is the Cauchy–Stieltjes (simply, Cauchy or Stieltjes) transform of $\mathfrak{m}_\omega$. As seen later, $\mathfrak{m}_\lambda$ of $\lambda \in \mathbb{Y}$ gives the transition probability of the Plancherel growth process.

Proposition 3.5. *The moments of* $\mathfrak{m}_\omega$ *and* τ_ω *of* $\omega \in \mathbb{D}_0$ *satisfy*

$$\sum_{n=0}^{\infty} \frac{M_n(\mathfrak{m}_\omega)}{z^n} = \exp \sum_{k=1}^{\infty} \frac{M_k(\tau_\omega)}{k} \frac{1}{z^k}.$$

Proof. Expand both sides of (3.3) into series in z^{-1}. □

Proposition 3.5 gives mutual polynomial relations between $M_n(\mathfrak{m}_\omega)$'s and $M_k(\tau_\omega)$'s. For example, $M_1(\mathfrak{m}_\omega) = M_1(\tau_\omega) = 0$ and

$$M_2(\mathfrak{m}_\omega) = \frac{1}{2}(M_1(\tau_\omega)^2 + M_2(\tau_\omega)) = \frac{1}{2} M_2(\tau_\omega)$$

which is n if $\omega \in \mathbb{Y}_n$.

Proposition 3.6. *The map $\omega \mapsto \mathfrak{m}_\omega$ gives a bijection between $\mathbb{D}_0$ and the set of probabilities μ on $\mathbb{R}$ such that suppμ is a finite set and $M_1(\mu) = 0$.*

Proof. We construct a map $\mu \mapsto \omega$ which is inverse to $\omega \mapsto \mu$.

Given $\mu = \sum_{i=1}^r \mu_i \delta_{x_i}$ such that $x_1 < \cdots < x_r$, $\mu_i > 0$, $\sum_{i=1}^r \mu_i = 1$ and $\sum_{i=1}^r x_i \mu_i = 0$, set

$$\frac{\mu_1}{z - x_1} + \cdots + \frac{\mu_r}{z - x_r} = \frac{f(z)}{(z - x_1)\cdots(z - x_r)} \tag{3.4}$$

where $f(z)$ is a polynomial with $\deg f = r - 1$. Taking $\lim_{z\to\infty} z\times(3.4)$, we see that $f(z)$ is monic. Furthermore,

$$f(x_1) = \mu_1(x_1 - x_2)\cdots(x_1 - x_r)$$
$$f(x_2) = \mu_2(x_2 - x_1)(x_2 - x_3)\cdots(x_2 - x_r)$$
$$\cdots$$
$$f(x_r) = \mu_r(x_r - x_1)\cdots(x_r - x_{r-1}).$$

Since these have alternative signs, there exist zeros of $f(z)$: $y_1, \cdots, y_{r-1}$ such that $x_1 < y_1 < x_2 < y_2 < \cdots < x_{r-1} < y_{r-1} < x_r$. Then $f(z) = (z - y_1)\cdots(z - y_{r-1})$. Equation (3.4) yields

$$\sum_{i=1}^r x_i - \sum_{i=1}^{r-1} y_i = M_1(\mu) = 0.$$

Hence the interlacing sequence produces $\omega \in \mathbb{D}_0$. □

3.3. **A trace formula.** The irreducible character of $\mathfrak{S}_n$ corresponding to $\lambda \in \mathbb{Y}_n$ and the normalized one are denoted by $\chi^\lambda = \mathrm{tr}U_\lambda$ and $\tilde{\chi}^\lambda = \chi^\lambda / \dim \lambda$ respectively. Let linear map $\mathbb{E}_n : \mathbb{C}[\mathfrak{S}_{n+1}] \longrightarrow \mathbb{C}[\mathfrak{S}_n]$ be defined by $\mathbb{E}_n(g) = g$ for $g \in \mathfrak{S}_n$ and $\mathbb{E}_n(g) = 0$ for $g \notin \mathfrak{S}_n$. More generally for finite group G and its subgroup H, the map $\mathbb{E} : \mathbb{C}[G] \longrightarrow \mathbb{C}[H]$ is similarly defined. It has the property of a conditional expectation that $\mathbb{E}(ba) = (\mathbb{E}b)a$ and $\mathbb{E}(ab) = a(\mathbb{E}b)$ for $a \in \mathbb{C}[H]$, $b \in \mathbb{C}[G]$. Since the Jucys–Murphy element J_n commutes with $\mathfrak{S}_n$, $\mathbb{E}_n J_n^k$ belongs to the center of $\mathbb{C}[\mathfrak{S}_n]$.

Theorem 3.7. *For $\lambda \in \mathbb{Y}_n$, we have*

$$\tilde{\chi}^\lambda(\mathbb{E}_n J_n^k) = M_k(\mathfrak{m}_\lambda), \qquad k \in \mathbb{N}.$$

The proof of Theorem 3.7 is divided into two parts. Let $\lambda \in \mathbb{Y}_n$ have min-max coordinates $x_1 < y_1 < \cdots < x_r$. The Young diagram formed by putting one box at the ith valley (coordinate x_i) of λ is denoted by $\Lambda^{(i)} \in \mathbb{Y}_{n+1}$.

Lemma 3.8.

$$\tilde{\chi}^\lambda(\mathbb{E}_n J_n^k) = \sum_{i=1}^r x_i^k \frac{\dim \Lambda^{(i)}}{(n+1)\dim\lambda}$$

Lemma 3.9.

$$\frac{\dim \Lambda^{(i)}}{(n+1)\dim\lambda} = \mathfrak{m}_\lambda(\{x_i\})$$

Proof of Lemma 3.8. From general theory of representations of finite groups, we have

$$\tilde{\chi}^\lambda(\mathbb{E}_n J_n^k) = \frac{1}{(n+1)(\dim\lambda)^2} \sum_{\mu\in\mathbb{Y}_{n+1}} \dim\mu \, \mathrm{tr}(U_\mu(J_n^k)U_\mu(e_\lambda)) \tag{3.5}$$

where e_λ is the central projection in $\mathbb{C}[\mathfrak{S}_n]$ corresponding to $\lambda \in \mathbb{Y}_n$. $U_\mu(e_\lambda)$ is nontrivial if and only if $\lambda \nearrow \mu$. Then $\mu = \Lambda^{(i)}$ for some i. Furthermore $U_\mu(e_\lambda)$ is the projector onto the λ-component in the Specht module S_μ. The content of the box $\Lambda^{(i)} \setminus \lambda$ is x_i. Theorem 2.8 yields that the Jucys–Murphy operator $U_{\Lambda^{(i)}}(J_n)$ acts as multiplication by x_i on the λ-component in $S_{\Lambda^{(i)}}$. Hence we have

$$\begin{aligned}\tilde{\chi}^\lambda(\mathbb{E}_n J_n^k) &= \frac{1}{(n+1)(\dim\lambda)^2}\sum_{i=1}^r \dim\Lambda^{(i)} \mathrm{tr}(x_i^k U_{\Lambda^{(i)}}(e_\lambda)) \\ &= \frac{1}{(n+1)(\dim\lambda)^2}\sum_{i=1}^r x_i^k \dim\Lambda^{(i)} \dim\lambda.\end{aligned}$$

This completes the proof. □

Proof of Lemma 3.9. From the hook formula (Proposition 2.1), we have

$$\frac{\dim\Lambda^{(i)}}{(n+1)\dim\lambda} = \frac{\prod_{b\in\lambda} h_\lambda(b)}{\prod_{b\in\Lambda^{(i)}} h_{\Lambda^{(i)}}(b)}.$$

For most boxes b, namely unless box b have the box $\Lambda^{(i)} \setminus \lambda$ in its hook, $h_\lambda(b) = h_{\Lambda^{(i)}}(b)$ holds. For the other involved hooks, we rewrite the ratio in the right side by using the min-max coordinates of λ. □

Remark 3.10. Taking the dimension of an induced representation

$$\mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} U_\lambda \cong \bigoplus_{i=1}^r U_{\Lambda^{(i)}},$$

we recognize from Lemma 3.9 that the transition measure of a Young diagram is a probability.

4. Young diagrams and the Young graph III (Lecture 4)

4.1. **Harmonic functions on $\mathbb{Y}$.** Let $\mathfrak{S}_\infty = \bigcup_{n=1}^\infty \mathfrak{S}_n$ be the infinite symmetric group. Its element is by definition a finite permutation of $\mathbb{N}$. The identiy element is denoted by e (until Section 9).

Definition 4.1. $\mathbb{C}$-valued function f on $\mathfrak{S}_\infty$ is said to be:
(i) positive-definite if $\sum_{i,j} \overline{\alpha_i}\alpha_j f(g_i^{-1}g_j) \geq 0$ for any $\alpha_i \in \mathbb{C}$ and $g_i \in \mathfrak{S}_\infty$,
(ii) central (or invariant, class) if $f(ghg^{-1}) = f(h)$ for any $g, h \in \mathfrak{S}_\infty$,
(iii) normalized if $f(e) = 1$.
The set of positive-definite and central functions on $\mathfrak{S}_\infty$ is denoted by $\mathcal{K}(\mathfrak{S}_\infty)$ and the set of normalized elements in it by $\mathcal{K}_1(\mathfrak{S}_\infty)$.

Definition 4.2. $\mathbb{C}$-valued function φ on $\mathbb{Y}$ is said to be:
(i) harmonic if $\varphi(\lambda) = \sum_{\mu:\lambda\nearrow\mu} \varphi(\mu)$ for any $\lambda \in \mathbb{Y}$,
(ii) normalized if $\varphi(\varnothing) = 1$.
The set of nonnegative harmonic functions on $\mathbb{Y}$ is denoted by $\mathcal{H}(\mathbb{Y})$ and the set of normalized elements in it by $\mathcal{H}_1(\mathbb{Y})$.

Proposition 4.3. *There exists a bijection between $\mathcal{K}_1(\mathfrak{S}_\infty)$ and $\mathcal{H}_1(\mathbb{Y})$ through*

$$f\big|_{\mathfrak{S}_n} = \sum_{\lambda\in\mathbb{Y}_n} \varphi(\lambda)\chi^\lambda, \qquad n \in \mathbb{N}. \tag{4.1}$$

Proof. ($f \mapsto \varphi$) Given $f \in \mathcal{K}_1(\mathfrak{S}_\infty)$, expand $f\big|_{\mathfrak{S}_n}$ as (4.1) to define φ and set $\varphi(\varnothing) = 1$. Then $\varphi(\lambda) \geq 0$ for any $\lambda \in \mathbb{Y}_n$. Harmonicity of φ follows from the consistency condition $f\big|_{\mathfrak{S}_{n+1}}\big|_{\mathfrak{S}_n} = f\big|_{\mathfrak{S}_n}$. Note $1 = f(e) = \varphi(\square)$.

($\varphi \mapsto f$) Harmonicity of φ ensures well-definedness of f, while positivity of φ implies positive-definiteness of f. □

Example 4.4. Orthogonality of irreducible characters yields

$$\delta_e = \sum_{\lambda\in\mathbb{Y}_n} \frac{\dim\lambda}{n!}\chi^\lambda, \qquad n \in \mathbb{N}.$$

Thus $\dim\lambda/|\lambda|!$ is the harmonic function corresponding to the delta function $\delta_e \in \mathcal{K}_1(\mathfrak{S}_\infty)$.

4.2. **Probabilities on the path space of the Young graph.** A path on the Young graph starting at $\varnothing$ is expressed as $t = (t(0) \nearrow t(1) \nearrow \cdots \nearrow t(n) \nearrow \cdots)$ where $t(n) \in \mathbb{Y}_n$ is the nth level diagram. The first and second level diagrams are always $\varnothing$ and $\square$ respectively. The set of all paths starting at $\varnothing$ is denoted by $\mathfrak{T}$ (standing for "Tableaux"). An element of $\mathfrak{T}$ is regarded also as an infinite standard tableau. For $\lambda \in \mathbb{Y}$ let $\mathfrak{T}(\lambda)$ denote the set of finite paths starting at $\varnothing$ and ending at λ. For finite path $u = (u(0) \nearrow u(1) \nearrow \cdots \nearrow u(n))$ of length n, set $C_u = \{t \in \mathfrak{T} \,|\, t(0) = u(0), \cdots, t(n) = u(n)\}$ (a cylindrical subset). The topology on $\mathfrak{T}$ generated by the cylindrical subsets coincides with the relative topology of $\prod_{n=0}^\infty \mathbb{Y}_n$ which makes $\mathfrak{T}$ compact and totally disconnected. The σ-field generated by the cylindrical subsets is the topological σ-field. A finitely additive probability measure on the finite field generated by the cylindrical subsets of $\mathfrak{T}$ is always extendable to a countably additive probability measure on the σ-field.

Let $\mathfrak{S}_{\mathfrak{T}(\lambda)}$ be the set of permutations of elements of $\mathfrak{T}(\lambda)$. We consider $g \in \mathfrak{S}_{\mathfrak{T}(\lambda)}$ to act on $\mathfrak{T}$. If $t \in \mathfrak{T}$ passes through λ, t_n denoting the truncated finite path up to $t(n) = \lambda$, set $g(t) = (g(t_n) \nearrow t(n+1) \nearrow \cdots)$. Otherwise set $g(t) = t$.

Definition 4.5. Probability M on the path space $\mathfrak{T}$ is said to be central if it is invariant under the action of $\bigcup_{\lambda\in\mathbb{Y}}\mathfrak{S}_{\mathfrak{T}(\lambda)}$. The set of central probabilities on $\mathfrak{T}$ is denoted by $\mathcal{M}_1(\mathfrak{T})$.

M is central if and only if $M(C_u) = M(C_v)$ for any $\lambda \in \mathbb{Y}$ and $u, v \in \mathfrak{T}(\lambda)$.

Proposition 4.6. *There exists a bijection between* $\mathcal{H}_1(\mathbb{Y})$ *and* $\mathcal{M}_1(\mathfrak{T})$ *through*

$$\varphi(\lambda) = M(C_u), \qquad u \in \mathfrak{T}(\lambda), \quad \lambda \in \mathbb{Y}. \tag{4.2}$$

Proof. ($\varphi \mapsto M$) Harmonicity of φ ensures consistency of M, namely M is well-defined on the finite field generated by the cylindrical subsets. Then it is extendable to a true measure.

($M \mapsto \varphi$) φ is well-defined by virtue of centrality of M. □

4.3. **The Plancherel measure.** Example 4.4 and Proposition 4.6 determine the following.

Definition 4.7. The Plancherel measure $\mathfrak{P}$ on $\mathfrak{T}$ is defined by

$$\mathfrak{P}(C_u) = \frac{\dim\lambda}{n!}, \qquad u \in \mathfrak{T}(\lambda), \quad \lambda \in \mathbb{Y}_n.$$

Its marginal $\mathfrak{P}_n$ defined by

$$\mathfrak{P}_n(\lambda) = \mathfrak{P}(\{t \in \mathfrak{T} \,|\, t(n) = \lambda\}) = \frac{(\dim\lambda)^2}{n!}, \qquad \lambda \in \mathbb{Y}_n$$

is also called the Plancherel measure on $\mathbb{Y}_n$

The Fourier transform of $f : \mathfrak{S}_n \longrightarrow \mathbb{C}$ is defined by $\widehat{f}(\lambda) = \sum_{x\in\mathfrak{S}_n} f(x)U_\lambda(x) \in \mathcal{L}(S_\lambda)$ where $\mathcal{L}(S_\lambda)$ denotes the set of linear operators on the Specht module S_λ. Then we have the Plancherel formula

$$\sum_{x\in\mathfrak{S}_n} \overline{f_1(x)} f_2(x) = \sum_{\lambda\in\mathbb{Y}_n} \mathrm{Tr}(\widehat{f}_1(\lambda)^* \widehat{f}_2(\lambda))\mathfrak{P}_n(\lambda)$$

where Tr is the normalized trace on $\mathcal{L}(S_\lambda)$ so that $\mathrm{Tr}(I) = 1$.

In general, central probability M on $\mathfrak{T}$ gives rise to a Markov chain on $\mathbb{Y}$ since

$$M\big(t(n+1) = \mu \,\big|\, t(0) = \varnothing, t(1) = \square, \cdots, t(n) = \lambda\big) = \begin{cases} \frac{\varphi(\mu)}{\varphi(\lambda)}, & \lambda \nearrow \mu, \\ 0, & \text{otherwise} \end{cases}$$

holds. Especially for the Plancherel measure $\mathfrak{P}$, we have

$$\frac{\varphi(\mu)}{\varphi(\lambda)} = \frac{\dim\mu}{(n+1)\dim\lambda} = \mathfrak{m}_\lambda(\{x_i\}), \qquad \lambda \nearrow \mu$$

by Lemma 3.9, x_i denoting the valley of λ at which the box $\mu \setminus \lambda$ is put. Hence the transition probability of the Plancherel growth process, the induced Markov chain above, from λ is given by Kerov's transition measure $\mathfrak{m}_\lambda$.

4.4. **Extremality and ergodicity.** We consider the extremal points of convex sets

$$\mathcal{K}_1(\mathfrak{S}_\infty) \;\longleftrightarrow\; \mathcal{H}_1(\mathbb{Y}) \;\longleftrightarrow\; \mathcal{M}_1(\mathfrak{T}).$$

An extremal measure is often said to be ergodic. In general, let M be a G-invariant probability with respect to some group G. M is said to be G-ergodic if the following mutually equivalent conditions hold.

(1) M is an extremal point: if $M = cM_1 + (1-c)M_2$ holds for G-invariant probabilities M_1, M_2 and $0 \le c \le 1$, it holds that $c = 0$ or 1.
(2) An orbital mean and an ensemble mean coincide: G-invariant function F is constant M-a.s. The constant is $\int F(t)M(dt)$. Note that an appropriate limit of a G-orbital mean is a G-invariant function.
(3) Each orbit passes through every point (with exaggeration): if A is a G-invariant set w.r.t. M i.e. $M(A \triangle gA) = 0$ for any $g \in G$, it holds that $M(A) = 0$ or 1. Here $\triangle$ means symmetric difference of sets. Hence $M(B) > 0$ implies $M((\bigcup_{g\in G} gB)^c) = 0$.

Definition 4.8. Set
$E(\mathfrak{S}_\infty) = \{f \in \mathcal{K}_1(\mathfrak{S}_\infty) \mid f \text{ is extremal}\}$,
$F(\mathfrak{S}_\infty) = \{f \in \mathcal{K}_1(\mathfrak{S}_\infty) \mid f \text{ is factorizable}\}$.
Here f is said to be factorizable (or multiplicative) if $f(xy) = f(x)f(y)$ holds for any $x, y \in \mathfrak{S}_\infty$ such that $\mathrm{supp}x \cap \mathrm{supp}y = \varnothing$.

Theorem 4.9 (Thoma)**.**

$$E(\mathfrak{S}_\infty) = F(\mathfrak{S}_\infty)$$

Proof. (1) $E(\mathfrak{S}_\infty) \subset F(\mathfrak{S}_\infty)$: Given $f \in E(\mathfrak{S}_\infty)$ and $g_1, g_2 \in \mathfrak{S}_\infty$ with disjoint supports, we show $f(g_1g_2) = f(g_1)f(g_2)$. Let G_1 and G_2 be the sets of permutations on $\mathrm{supp}g_1$ and $\mathbb{N} \setminus \mathrm{supp}g_1$ respectively. G_2 is isomorphic to $\mathfrak{S}_\infty$. $f(gh)$ regarded as a function in g is central on G_1 since G_1 commutes with G_2. Expand this with respect to irreducible characters of G_1 to have

$$f(gh) = \sum_{\alpha \in \widehat{G_1}} c_\alpha(h)\chi^\alpha(g). \tag{4.3}$$

$c_\alpha(h)$ is a central and positive-definite function on G_2. In fact, we use again G_2 commuting with G_1 and see that

$$\sum_{\alpha\in\widehat{G_1}} \sum_{i,j} \overline{\beta_i}\beta_j c_\alpha(h_i^{-1}h_j)\chi^\alpha(g) = \sum_{i,j} \overline{\beta_i}\beta_j f(gh_i^{-1}h_j)$$

is positive-definite on G_1 as a function in g. Putting $g = e$ in (4.3), we have decomposition

$$f(h) = \sum_{\alpha\in\widehat{G_1}} (\dim\alpha)c_\alpha(h).$$

However, $f\big|_{G_2} \in E(G_2)$ because f is central. Hence we conclude $(\dim\alpha)c_\alpha(h) = k_\alpha f(h)$ for some nonnegative constant k_α. Putting this into (4.3), we have

$$f(gh) = \Big(\sum_{\alpha\in\widehat{G_1}} k_\alpha \tilde{\chi}^\alpha(g)\Big) f(h). \tag{4.4}$$

The sum in (4.4) is $f(g)$, which is seen by putting $g = e$.

(2) $E(\mathfrak{S}_\infty) \supset F(\mathfrak{S}_\infty)$: Central function f on $\mathfrak{S}_\infty$ induces a function on $(\mathbb{Z}_{\ge 0})^{\{2,3,\dots\}}$ by

$$\tilde{f}(m_2, m_3, \cdots) = f((2^{m_2}3^{m_3}\cdots)).$$

Given $f \in F(\mathfrak{S}_\infty)$, let s_j denote the value of f at a j-cycle. Positive-definiteness, centrality and normality of f imply $f(g) = \overline{f(g^{-1})} = \overline{f(g)} \in [-1, 1]$. Applying factorizability, we have

$$\tilde{f}(m_2, m_3, \cdots) = s_2^{m_2}s_3^{m_3}\cdots, \qquad m_2, m_3, \cdots \in \mathbb{Z}_{\ge 0}.$$

Choquet's theorem ensures an integral expression

$$f((2^{m_2}3^{m_3}\cdots)) = s_2^{m_2}s_3^{m_3}\cdots = \int_B \psi\nu(d\psi) = \int_{\tilde{B}} t_2^{m_2}t_3^{m_3}\cdots\tilde{\nu}(dt), \tag{4.5}$$

where $B \subset E(\mathfrak{S}_\infty) \subset F(\mathfrak{S}_\infty)$ (by virtue of (1)) and $\tilde{B} \subset [-1,1]^\infty$. However, this is impossible unless $\tilde{\nu} = \delta_s$, $s = (s_2, s_3, \cdots)$. In fact, it is a sort of moment problem. First let $m_3 = m_4 = \cdots = 0$ in (4.5) to have

$$s_2^{m_2} = \int t_2^{m_2}\nu_2(dt_2), \qquad m_2 \in \mathbb{Z}_{\geq 0},$$

where ν_2 is a marginal of $\tilde{\nu}$ such that $\mathrm{supp}\nu_2 \subset [-1,1]$. This yields $\nu_2 = \delta_{s_2}$. Similarly all marginals of $\tilde{\nu}$ are Dirac measures, which concludes that $\tilde{\nu}$ itself and hence ν is of Dirac. In (4.5), B is thus taken as a singleton subset of $E(\mathfrak{S}_\infty)$. □

5. Young diagrams and the Young graph IV (Lecture 5)

5.1. **Symmetric functions.** We briefly recall some properties of symmetric functions, especially Schur functions. Let $\mathbf{\Lambda}_n^k$ be the homogeneous symmetric polynomials of degree k in n variables, equipped also with 0. For $\lambda \in \mathbb{Y}$, $n \geq r(\lambda)$,

$$m_\lambda(x_1, \cdots, x_n) = \sum_{(\alpha_1, \cdots, \alpha_n)} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

denotes a monomial, where $(\alpha_1, \cdots, \alpha_n)$ runs over all distinct permutations of $(\lambda_1, \cdots, \lambda_n)$. The set of monomials $\{m_\lambda(x_1, \cdots, x_n) \mid \lambda \in \mathbb{Y}_k,\ r(\lambda) \leq n\}$ is a linear basis of $\mathbf{\Lambda}_n^k$. Along the projective system given by $p_{nm} : \mathbf{\Lambda}_m^k \longrightarrow \mathbf{\Lambda}_n^k$ sending $f(x_1, \cdots, x_m)$ to $f(x_1, \cdots, x_n, 0, \cdots, 0)$, $m > n$, let $\mathbf{\Lambda}^k$ be the projective limit of $\mathbf{\Lambda}_n^k$ as $n \to \infty$. $m_\lambda \in \mathbf{\Lambda}^k$ denotes the monomial symmetric function of degree k. Then $\{m_\lambda \mid \lambda \in \mathbb{Y}_k\}$ is a linear basis of $\mathbf{\Lambda}^k$. Finally

$$\mathbf{\Lambda} = \bigoplus_{k=0}^{\infty} \mathbf{\Lambda}^k = \mathrm{span}\{m_\lambda \mid \lambda \in \mathbb{Y}\}$$

is the algebra of symmetric functions. Power sums are defined by

$$\begin{aligned} &p_k(x_1, x_2, \cdots) = m_{(k)}(x_1, x_2, \cdots) = x_1^k + x_2^k + \cdots, \qquad p_0 = 1, \\ &p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots, \qquad \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \in \mathbb{Y}. \end{aligned}$$

The set of power sums $\{p_\lambda \mid \lambda \in \mathbb{Y}\}$ is a linear basis of $\mathbf{\Lambda}$.

Definition 5.1. For $\lambda \in \mathbb{Y}$ and $n \geq r(\lambda)$, the Schur polynomial is defined by

$$s_\lambda(x_1, \cdots, x_n) = \frac{\det(x_i^{\lambda_j + n - j})}{\det(x_i^{n-j})} \tag{5.1}$$

to be a homogeneous symmetric polynomial of degree $|\lambda|$. Further set $s_\varnothing = 1$.

Equation (5.1) is the Weyl character formula for the unitary group $U(n)$, in which x_i's are taken to be spectral parameters (eigenvalues) of $g \in U(n)$. For $\lambda \in \mathbb{Y}_k$ and $m > n \geq k$, we have $s_\lambda(x_1, \cdots, x_n, 0, \cdots, 0) = s_\lambda(x_1, \cdots, x_n)$ and hence $s_\lambda \in \mathbf{\Lambda}^k$, which is called a Schur function. We recall two celebrated formulas.

Theorem 5.2 (Pieri's formula)**.**

$$s_{(1)} s_\lambda = \sum_{\mu : \lambda \nearrow \mu} s_\mu.$$

Theorem 5.3 (Frobenius character formula)**.**

$$p_\rho = \sum_{\lambda \in \mathbb{Y}_n} \chi_\rho^\lambda s_\lambda, \qquad \rho \in \mathbb{Y}_n.$$

For later use, we mention here the diagonalization of adjacency operators on the symmetric group in terms of Schur functions. Set

$$A_\rho = \sum_{x \in C_\rho} x \ \in \mathbb{C}[\mathfrak{S}_n], \qquad \rho \in \mathbb{Y}_n$$

which acts on $\ell^2(\mathfrak{S}_n)$ under the left regular representation and is called an adjacency operator. Consider

$$\Gamma(\mathfrak{S}_n) = \mathrm{span}\Big\{\xi_\sigma = \sum_{x \in C_\sigma} \delta_x \;\Big|\; \sigma \in \mathbb{Y}_n\Big\} \ \subset\ \ell^2(\mathfrak{S}_n).$$

$\Gamma(\mathfrak{S}_n)$ is an invariant subspace for adjacency operators. Setting $\Phi(\sigma) = \xi_\sigma/\sqrt{|C_\sigma|}$, we have an orthonormal basis $\{\Phi(\sigma) \,|\, \sigma \in \mathbb{Y}_n\}$ of $\Gamma(\mathfrak{S}_n)$. Define a unitary operator

$$I : \sqrt{z_\rho}\Phi(\rho) \in \Gamma(\mathfrak{S}_n) \longmapsto p_\rho \in \mathbf{\Lambda}^n, \qquad \rho \in \mathbb{Y}_n, \tag{5.2}$$

where the scalar product in $\mathbf{\Lambda}$ is defined as usual by $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$.

Proposition 5.4.

$$I\Big(\frac{A_\rho}{|C_\rho|}\Big)I^{-1}s_\lambda = \tilde{\chi}^\lambda_\rho s_\lambda, \qquad \rho, \lambda \in \mathbb{Y}_n.$$

Proof. We see from general theory for representations of finite groups

$$A_\rho = \sum_{\lambda \in \mathbb{Y}_n} |C_\rho| \tilde{\chi}^\lambda_\rho E_\lambda, \qquad \rho \in \mathbb{Y}_n,$$

where $\{E_\lambda \,|\, \lambda \in \mathbb{Y}_n\}$ forms a total system of idempotents in the center of $\mathbb{C}[\mathfrak{S}_n]$. Setting $\zeta_\lambda = E_\lambda \delta_e$, we get from this

$$\xi_\rho = \sum_{\lambda \in \mathbb{Y}_n} |C_\rho| \tilde{\chi}^\lambda_\rho \zeta_\lambda, \tag{5.3}$$

$$A_\rho \zeta_\lambda = |C_\rho| \tilde{\chi}^\lambda_\rho \zeta_\lambda. \tag{5.4}$$

Invert (5.3) by using orthogonality of irreducible characters to have

$$\frac{\zeta_\lambda}{\dim \lambda} = \sum_{\rho \in \mathbb{Y}_n} \frac{\chi^\lambda_\rho}{n!} \xi_\rho.$$

Applying operator I and using Frobenius character formula (inversion of Theorem 5.3), we have $I(\zeta_\lambda/\dim\lambda) = s_\lambda/\sqrt{n!}$. Combined with (5.4), this yields the desired equality. □

5.2. **Extremality via symmetric functions.** We give a characterization of $\mathcal{H}_1(\mathbb{Y})$ (hence $\mathcal{K}_1(\mathfrak{S}_\infty)$ and $\mathcal{M}_1(\mathfrak{T})$ also) and its extremal points in terms of symmetric functions.

Proposition 5.5. $\mathcal{H}_1(\mathbb{Y})$*, the set of normalized nonnegative harmonic functions on* $\mathbb{Y}$*, has a bijective correspondence with*

$$\{\psi : \mathbf{\Lambda} \longrightarrow \mathbb{C} \,|\, \psi \textit{ is linear},\ \psi(1) = 1,\ \psi(s_\lambda) \geq 0,\ \ker\psi \supset (s_1 - 1)\mathbf{\Lambda}\} \tag{5.5}$$

through

$$\varphi(\lambda) = \psi(s_\lambda). \tag{5.6}$$

Proof. We see the following mutually converse map.

($\varphi \mapsto \psi$) Since $\{s_\lambda \,|\, \lambda \in \mathbb{Y}\}$ is a basis of $\mathbf{\Lambda}$, ψ is determined by (5.6). The last property follows from Pieri's formula for Schur functions:

$$\psi((s_1 - 1)s_\lambda) = \sum_{\mu:\lambda\nearrow\mu} \psi(s_\mu) - \psi(s_\lambda) = \sum_{\mu:\lambda\nearrow\mu} \varphi(\mu) - \varphi(\lambda).$$

($\psi \mapsto \varphi$) Harmonicity of φ follows again from Pieri's formula and the kernel property of ψ. □

Set

$$\overline{\mathbb{Y}} = \{\rho \in \mathbb{Y} \mid m_1(\rho) = 0\}.$$

$\overline{\mathbb{Y}}$ parametrizes the conjugacy classes of $\mathfrak{S}_\infty$. The corresponding conjugacy class of $\mathfrak{S}_\infty$ to $\rho \in \overline{\mathbb{Y}}$ is denoted by C_ρ. Compared with the notation of a conjugacy class of $\mathfrak{S}_n$, it holds that $C_\rho \cap \mathfrak{S}_n = C_{(\rho,1^{n-|\rho|})}$ for $\rho \in \overline{\mathbb{Y}}$ and $n \geq |\rho|$. Note that $C_\varnothing = \{e\}$, $\varnothing \in \overline{\mathbb{Y}}$.

Theorem 5.6. *Under the correspondence between $\mathcal{H}_1(\mathbb{Y})$ and* (5.5) *through* (5.6)*, φ is extremal if and only if ψ is an algebra homomorphism.*

Proof. This proof is based on Thoma's theorem for characters of $\mathfrak{S}_\infty$. Let φ and ψ be given as (5.6). Set $\chi : \mathfrak{S}_\infty \longrightarrow \mathbb{C}$ by

$$\chi(g) = \psi(p_\rho), \qquad g \in C_\rho,\ \rho \in \overline{\mathbb{Y}}. \tag{5.7}$$

Since $\psi((s_1 - 1)p_\rho) = 0$ holds from (5.5), we have $\psi(p_1^k p_\rho) = \psi(p_\rho)$ for any k. Then, for $g \in C_{(\rho,1^{n-|\rho|})} \subset \mathfrak{S}_n$, $\rho \in \overline{\mathbb{Y}}$ and $n \geq |\rho|$, Frobenius character formula yields

$$\chi\big|_{\mathfrak{S}_n}(g) = \psi(p_\rho) = \psi(p_{(\rho,1^{n-|\rho|})}) = \psi\Big(\sum_{\lambda \in \mathbb{Y}_n} \chi^\lambda_{(\rho,1^{n-|\rho|})} s_\lambda\Big) = \sum_{\lambda \in \mathbb{Y}_n} \varphi(\lambda)\chi^\lambda(g).$$

Hence χ is related to φ as in Proposition 4.3, in particular $\chi \in \mathcal{K}_1(\mathfrak{S}_\infty)$. Theorem 4.9 tells that φ is extremal if and only if $\chi \in F(\mathfrak{S}_\infty)$. This factorizability means that

$$\chi(c_1 \cdots c_q) = \chi(c_1) \cdots \chi(c_q) \qquad \text{for any disjoint cycles } c_1, \cdots, c_q, \tag{5.8}$$

where $\chi(c_i) = \psi(p_{r_i})$ if c_i is an r_i-cycle ($r_i \geq 2$). We have

$$\begin{aligned}
(5.8) &\iff \psi(p_\rho) = \psi(p_{\rho_1})\psi(p_{\rho_2}) \cdots, && \rho \in \overline{\mathbb{Y}}, \\
&\iff \psi(p_\rho) = \psi(p_{\rho_1})\psi(p_{\rho_2}) \cdots, && \rho \in \mathbb{Y} \quad (\text{since } \psi(p_1) = 1), \\
&\iff \psi(p_\rho p_\sigma) = \psi(p_\rho)\psi(p_\sigma), && \rho,\ \sigma \in \mathbb{Y}.
\end{aligned}$$

Since $\{p_\rho \mid \rho \in \mathbb{Y}\}$ is a basis of $\mathbf{\Lambda}$, this means that ψ is an algebra homomorphism. □

5.3. Martin kernel on the Young graph.

Definition 5.7. For λ, $\mu \in \mathbb{Y}$, the number of paths connecting λ with μ on the Young graph is denoted by $d(\lambda, \mu)$ and called the (combinatorial) dimension function. If there are no such paths, we set $d(\lambda, \mu) = 0$ by definition.

Obviously $\dim \lambda = d(\varnothing, \lambda)$. If φ is harmonic, we have

$$\varphi(\lambda) = \sum_{\mu \in \mathbb{Y}_m} d(\lambda, \mu)\varphi(\mu), \qquad \lambda \in \mathbb{Y}_n,\ n < m.$$

The ratio of dimension functions

$$K(\lambda, \mu) = \frac{d(\lambda, \mu)}{d(\varnothing, \mu)}$$

is (an analogous object of) the Martin kernel on the Young graph.

Lemma 5.8. *If*

$$\varphi(\lambda) = \lim_{m \to \infty} K(\lambda, \mu_m), \qquad \lambda \in \mathbb{Y}$$

exists, then $\varphi \in \mathcal{H}_1(\mathbb{Y})$.

Proof. Harmonicity follows immediately from

$$\varphi(\lambda) = \lim_{m\to\infty} \frac{d(\lambda,\mu_m)}{d(\varnothing,\mu_m)} = \lim_{m\to\infty} \frac{\sum_{\mu:\lambda\nearrow\mu} d(\mu,\mu_m)}{d(\varnothing,\mu_m)} = \sum_{\mu:\lambda\nearrow\mu} \varphi(\mu).$$

□

Recalling the correspondence between χ, M, φ and ψ, we anticipate the following two results which we will prove in later sections (the third part of the course) in more general contexts of wreath product groups.

Theorem 5.9 (Thoma). *The extremal points in $\mathcal{K}_1(\mathfrak{S}_\infty)$ are given as the supersymmetric power sums. Namely, $\chi \in E(\mathfrak{S}_\infty)$ satisfies*

$$\chi(C_{(k)}) = p_k(\alpha,\beta) = \sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} \sum_{i=1}^{\infty} \beta_i^k, \qquad k \geq 2.$$

Here the Thoma parameter runs over

$$\Big\{ \big(\alpha = (\alpha_i),\, \beta = (\beta_i)\big) \;\big|\; \alpha_1 \geq \alpha_2 \geq \cdots \geq 0,\ \beta_1 \geq \beta_2 \geq \cdots \geq 0,\ \sum_{i=1}^{\infty} (\alpha_i + \beta_i) \leq 1 \Big\}.$$

Since χ is factorizable, we have the value at an element of C_ρ as

$$\chi(C_\rho) = p_{\rho_1}(\alpha,\beta) p_{\rho_2}(\alpha,\beta) \cdots = p_\rho(\alpha,\beta), \qquad \rho \in \overline{\mathbb{Y}}.$$

Remark 5.10. By definition we set $p_1(\alpha,\beta) = 1$, not necessarily coinciding with $\sum_{i=1}^{\infty} (\alpha_i + \beta_i)$.

Theorem 5.11 (Vershik–Kerov). *Let $M \in \mathcal{M}_1(\mathfrak{T})$ be ergodic and φ the corresponding extremal element of $\mathcal{H}_1(\mathbb{Y})$. We have*

$$\lim_{n\to\infty} K(\lambda, t(n)) = \varphi(\lambda), \qquad \lambda \in \mathbb{Y}$$

for M-a.s. path $t \in \mathfrak{T}$.

Let $\chi_{\alpha,\beta}$ and $\varphi_{\alpha,\beta}$ correspond to the Thoma parameter (α,β). Taking Remark 5.10 into account, we have

$$\chi_{\alpha,\beta}\big|_{\mathfrak{S}_n} (C_{(\rho, 1^{n-|\rho|})}) = p_{(\rho,1^{n-|\rho|})}(\alpha,\beta), \qquad \rho \in \overline{\mathbb{Y}},\ n \geq |\rho|.$$

Hence we have

$$p_\rho(\alpha,\beta) = \sum_{\lambda\in\mathbb{Y}_n} \chi_\rho^\lambda \varphi_{\alpha,\beta}(\lambda), \qquad \rho \in \mathbb{Y}_n,\ n \in \mathbb{N}. \tag{5.9}$$

Equation (5.9) suggests us to write

$$\varphi_{\alpha,\beta}(\lambda) = s_\lambda(\alpha,\beta)$$

and call it a supersymmetric Schur function.

6. Limit shape of Young diagrams and fluctuations I (Lecture 6)

6.1. **Continuous diagrams and transition measures.** We prepare a framework for limiting objects of growing Young diagrams (in the balanced regime).

Definition 6.1. A continuous diagram is a function $\omega : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying
(i) $|\omega(x_1) - \omega(x_2)| \leq |x_1 - x_2|, \qquad x_1, x_2 \in \mathbb{R}$,
(ii) $\omega(x) = |x| \quad$ if $|x|$ is large enough.
The set of continuous diagrams is denoted by $\mathbb{D}$.

It follows that $\mathbb{D} \supset \mathbb{D}_0 \supset \mathbb{Y}$. Let us define Kerov's transition measure of a continuous diagram through approximation by rectangular diagrams. Given $\omega \in \mathbb{D}$, take a sequence of $\omega_n \in \mathbb{D}_0$ which converges to ω in the uniform norm. Then $\{M_k(\tau_{\omega_n})\}$ is convergent for any $k \in \mathbb{N}$ by (3.2). Proposition 3.5 ensures that $\{M_k(\mathfrak{m}_{\omega_n})\}$ is also convergent to have the limit M_k as $n \to \infty$. Note that $\mathrm{supp}\mathfrak{m}_{\omega_n}$'s are included in a common compact set of $\mathbb{R}$. By virtue of the determinate moment problem, we have a unique probability μ whose kth moment agrees with M_k. It is obvious that the resulting probability μ does not depend on the choice of an approximating sequence of $\omega_n \in \mathbb{D}_0$.

Definition 6.2. The probability μ constructed above for $\omega \in \mathbb{D}$ is called the transition measure of ω.

It is seen from the construction that $\mathrm{supp}\mathfrak{m}_\omega$ coincides with the support of the function $\omega(x) - |x|$. Proposition 3.5 gives us the following characterization of the Stieltjes transform of $\mathfrak{m}_\omega$.

Proposition 6.3.

$$\int_{-\infty}^{\infty} \frac{1}{z-x} \mathfrak{m}_\omega(dx) = \frac{1}{z} \exp\Big\{ \int_{-\infty}^{\infty} \frac{1}{x-z} \Big(\frac{\omega(x) - |x|}{2} \Big)' dx \Big\}, \qquad \omega \in \mathbb{D}.$$

Further we set

$$\tau_\omega = \Big(\frac{\omega(x) - |x|}{2} \Big)'' + \delta_0, \qquad \omega \in \mathbb{D}. \tag{6.1}$$

τ_ω is called the Rayleigh measure of $\omega \in \mathbb{D}$ if it becomes a (signed) measure.

Definition 6.4 (limit shape of Young diagrams)**.**

$$\Omega(x) = \begin{cases} \frac{2}{\pi}\big(x \arcsin\frac{x}{2} + \sqrt{4-x^2}\big), & |x| \leq 2, \\ |x|, & |x| > 2. \end{cases}$$

Proposition 6.5.

$$\mathfrak{m}_\Omega(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\ 1_{[-2,2]}(x)dx \qquad \textit{(semicircle law)},$$
$$\tau_\Omega(dx) = \frac{1}{\pi\sqrt{4-x^2}}\ 1_{(-2,2)}(x)dx \qquad \textit{(arcsine law)}.$$

6.2. **Limit shape of Young diagrams.** The limit shape of Young diagrams describes a law of large numbers for the Plancherel measure $\mathfrak{P}$. In other words, it is a concentration phenomenon in the irreducible decomposition of the regular representation of $\mathfrak{S}_n$ as $n \to \infty$:

$$L_n \cong \bigoplus_{\lambda \in \mathbb{Y}_n} [\dim \lambda]\, U_\lambda,$$

$$\delta_e = \sum_{\lambda \in \mathbb{Y}_n} \frac{\dim \lambda}{n!} \chi^\lambda = \sum_{\lambda \in \mathbb{Y}_n} \mathfrak{P}_n(\lambda) \tilde{\chi}^\lambda.$$

For $\lambda \in \mathbb{Y} \subset \mathbb{D}_0$, we consider rescaled $\lambda^{\sqrt{n}} \in \mathbb{D}_0$ determined by

$$\lambda^{\sqrt{n}}(x) = \frac{1}{\sqrt{n}} \lambda(\sqrt{n} x). \tag{6.2}$$

Theorem 6.6. *For $\mathfrak{P}$-a.s. path $t \in \mathfrak{T}$, we have*

(1) $$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |t(n)^{\sqrt{n}}(x) - \Omega(x)| = 0,$$

(2) $$\lim_{n \to \infty} M_k(\mathfrak{m}_{t(n)^{\sqrt{n}}}) = M_k(\mathfrak{m}_\Omega), \qquad k \in \mathbb{N}.$$

Statements (1) and (2) of Theorem 6.6 show concentration to the limit shape Ω in the uniform and the moment topologies on $\mathbb{D}$ respectively. $\mathbb{D}$ is equipped with the latter topology by the family of semi-distances

$$d_k(\omega_1, \omega_2) = |M_k(\mathfrak{m}_{\omega_1}) - M_k(\mathfrak{m}_{\omega_2})|, \qquad k \in \mathbb{N}.$$

Both topologies are equivalent if they are restricted on $\mathbb{D}_K = \{\omega \in \mathbb{D} \mid \operatorname{supp}(\omega(x) - |x|) \subset [-K, K]\}$ $(K > 0)$, while they are not comparable on the whole $\mathbb{D}$. We intend to approach this result via moment analysis of the Jucys–Murphy element. Proof of Theorem 6.6 is given in Section 7.2.

6.3. A variation of the Young graph.

We introduce a variation of the Young graph for combinatorics of moments of the Jucys–Murphy element and also for later use in quantum central limit theorems.

Definition 6.7. Recall $\overline{\mathbb{Y}} = \{\rho \in \mathbb{Y} \mid m_1(\rho) = 0\}$.
(1) For $g \in C_\rho$, the conjugacy class of $\mathfrak{S}_\infty$ corresponding to $\rho \in \overline{\mathbb{Y}}$, set $\operatorname{type}(g) = \rho$.
(2) Set $l(\rho) = |\rho| - r(\rho)$ for $\rho \in \overline{\mathbb{Y}}$.

Lemma 6.8. *Consider the length function $\partial(e, g)$, denoting the minimal distance between e and g, on the Cayley graph $(\mathfrak{S}_\infty, \{\text{transpositions}\})$. Then,*

$$\partial(e, g) = |\operatorname{type}(g)| - r(\operatorname{type}(g)) = l(\operatorname{type}(g)), \qquad g \in \mathfrak{S}_\infty.$$

Lemma 6.9. *For $g \in \mathfrak{S}_\infty$ and transposition $(i\ j)$, set $\rho = \operatorname{type}(g)$ and $\sigma = \operatorname{type}((i\ j)g)$. Then, either $l(\sigma) = l(\rho) \pm 1$ holds.*

Remark 6.10. In Lemma 6.9, expressing g as a tableau of ρ-type with the infinitely long leg (i.e. one-box rows), we have

$$+1 \iff i, j \text{ in distinct rows of } \rho \quad \text{(merge)}$$
$$-1 \iff i, j \text{ in the same row of } \rho \quad \text{(split)}.$$

Definition 6.11 (variation of the Young graph)**.** Adopt $\overline{\mathbb{Y}}$ as the vertex set. Two vertices $\rho, \sigma \in \overline{\mathbb{Y}}$ are defined to be adjacent if there exit $x, y \in \mathfrak{S}_\infty$ and $(i\ j)$ such that $\rho = \operatorname{type}(x)$, $\sigma = \operatorname{type}(y)$ and $(i\ j)x = y$. For simplicity of notations, this graph is denoted also by $\overline{\mathbb{Y}}$.

Lemma 6.9 tells that adjacent ρ and σ satisfy $l(\sigma) = l(\rho) \pm 1$. Thus the vertices of $\overline{\mathbb{Y}}$ should be stratified according to l. To draw the graph $\overline{\mathbb{Y}}$, glue a copy of the first column (shaded in Figure 6.1) to the leftmost side for each diagram of the Young graph. The edges of the Young graph are all inherited, which are referred to as old edges in Figure 6.1, while some edges are added (as new edges) according to the adjacency relation in Definition 6.11.

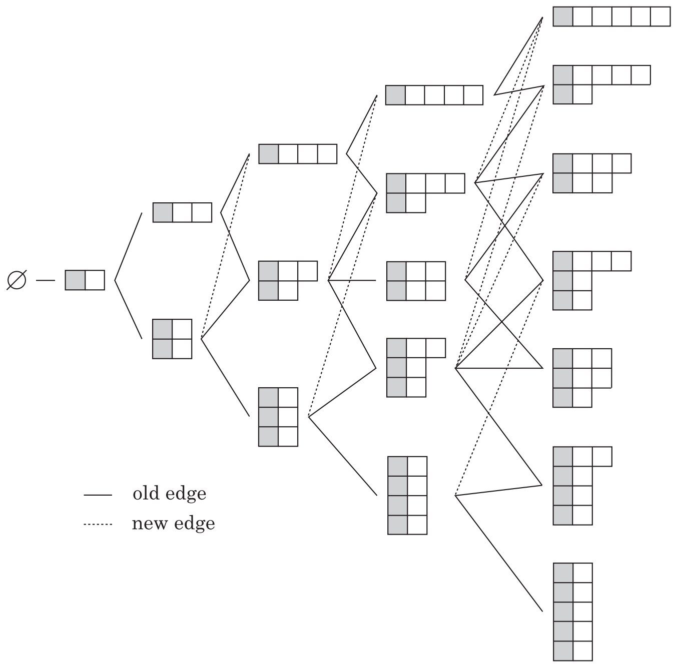

FIG. 6.1. A variation of the Young graph

6.4. **Admissible walks in $\overline{\mathbb{Y}}$.** We consider the Jucys–Murphy element $J_n = (1\; n+1) + (2\; n+1) + \cdots + (n\; n+1)$ in $\mathbb{C}[\mathfrak{S}_{n+1}]$, or J_n in $\mathfrak{S}_{\{1,\dots,n,*\}}$, replacing $n+1$ by extra letter $*$. Expand its kth power as a big sum:

$$J_n^k = \sum_{i_1,\cdots,i_k \in \{1,\dots,n\}} (i_k\; *) \cdots (i_2\; *)(i_1\; *). \tag{6.3}$$

To a sequence $i_1, i_2, \cdots, i_k$ in (6.3), we assign a k-walk on the Cayley graph of $\mathfrak{S}_{n+1}$

$$e \to (i_1\; *) \to (i_2\; *)(i_1\; *) \to \cdots \to (i_k\; *) \cdots (i_2\; *)(i_1\; *) \tag{6.4}$$

and, setting $\rho_j = \mathrm{type}((i_j\; *) \cdots (i_1\; *))$, a k-walk in $\overline{\mathbb{Y}}$

$$\varnothing \to \rho_1 \to \rho_2 \to \cdots \to \rho_k. \tag{6.5}$$

Definition 6.12. (1) $i_1, \cdots, i_k$ in (6.3) is called an admissible sequence if $(i_k\; *) \cdots (i_1\; *) \in \mathfrak{S}_n$ in the k-walk (6.4).
(2) The walk (6.5) is called the projection of (6.4).
(3) A k-walk $\varnothing \to \rho_1 \to \rho_2 \to \cdots \to \rho_k$ in $\overline{\mathbb{Y}}$ is said to be admissible if it is the projection of such a k-walk as (6.4) induced from some admissible sequence

$i_1, \cdots, i_k \in \{1, \ldots, n\}$ for some $n \in \mathbb{N}$. If $l(\rho_{j+1}) = l(\rho_j) + 1$ [resp. -1], the step $\rho_j \to \rho_{j+1}$ is said to be up [resp. down].

Remark 6.13. If $i_1, \cdots, i_k \in \{1, \ldots, n\}$ is admissible, so is as a sequence in $\{1, \ldots, m\}$ for any $m > n$. Hence Definition 6.12 (3) actually holds for any $m > n$.

Set $\overline{\mathbb{Y}}_k = \mathbb{Y}_k \cap \overline{\mathbb{Y}} = \{\rho \in \mathbb{Y}_k \mid m_1(\rho) = 0\}$. We define an operation on $\overline{\mathbb{Y}}$ by

$$\sigma \longmapsto \sigma^\circ = (2^{m_3(\sigma)} 3^{m_4(\sigma)} \cdots).$$

Namely, the shaded boxes are removed to obtain σ° in Figure 6.2. This operation is injective when restricted on $\overline{\mathbb{Y}}_k$.

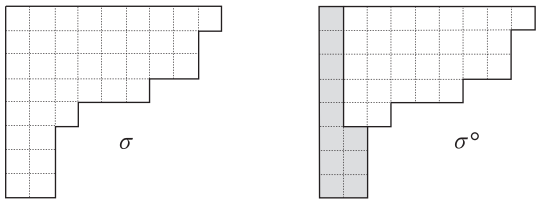

FIG. 6.2. The operation $\sigma \mapsto \sigma^\circ$

We characterize the ending vertices of the admissble walks in $\overline{\mathbb{Y}}$.

Proposition 6.14. *The following are equivalent for $\rho \in \overline{\mathbb{Y}}$ and $k \in \mathbb{N}$.*
(1) *ρ is the ending vertex of some admissible k-walk in $\overline{\mathbb{Y}}$.*
(2) *$l(\rho) \equiv k \pmod 2$ and $|\rho| + r(\rho) \leq k$.*
(3) *$\rho = \sigma^\circ$ holds for some $\sigma \in \overline{\mathbb{Y}}_k$.*

Proof. (1) $\Longrightarrow$ (2). Given admissible walk $\varnothing \to \rho_1 \to \cdots \to \rho_k = \rho$, let u and d be the numbers of up and down steps respectively. Since $u + d = k$ and $u - d = l(\rho)$ hold, we have

$$u = \frac{k + l(\rho)}{2}, \qquad d = \frac{k - l(\rho)}{2}. \tag{6.6}$$

In particular, $k \equiv l(\rho) \pmod 2$. Take an admissible sequence $i_1, \cdots, i_k \in \{1, \ldots, n\}$ for the given k-walk. $(i_j\ *) \cdots (i_1\ *)$ cannot contain a cycle without $*$ until a down step appears. Hence $r(\rho) \leq d$. Combined with (6.6), it holds $k \geq l(\rho) + 2r(\rho) = |\rho| + r(\rho)$.

(2) $\Longrightarrow$ (1). Given $\rho = (\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{r(\rho)})$ and k as (2), take $n > k$ and distinct $i_1, \cdots, i_{\rho_1} \in \{1, \ldots, n\}$ with $i_{\rho_1+1} = i_1$. Then we have a $\rho_1 + 1$-walk ending at

$$(i_{\rho_1+1}\ *)(i_{\rho_1}\ *) \cdots (i_1\ *) = (i_1\ i_2\ \cdots\ i_{\rho_1}) \in \mathfrak{S}_n.$$

Next take distinct $j_1, \cdots, j_{\rho_2} \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_{\rho_1}\}$ with $j_{\rho_2+1} = j_1$ to have a similar $\rho_2 + 1$-walk. Continuing this procedure, we get an admissible walk of length $(\rho_1 + 1) + \cdots + (\rho_{r(\rho)} + 1) = |\rho| + r(\rho)$. Since there remain $k - (|\rho| + r(\rho))$ steps, which is even from the assumption, we repeat multiplying $(s\ *)$ with letter s which never appeared.

(2) $\Longleftrightarrow$ (3). Straightforward by the definition of σ°. $\square$

7. Limit shape of Young diagrams and fluctuations II (Lecture 7)

7.1. Moments of the Jucys–Murphy element. Since $\mathbb{E}_n J_n^k$ is in the center of $\mathbb{C}[\mathfrak{S}_n]$, Proposition 6.14 implies

$$\mathbb{E}_n J_n^k = \sum_{\sigma \in \overline{\mathbb{Y}}_k} K_{\sigma,n} A_{(\sigma^\circ, 1^{n-|\sigma^\circ|})}, \qquad K_{\sigma,n} \in \mathbb{Z}_{\geq 0}. \tag{7.1}$$

$K_{\sigma,n}$ counts the number of those terms in $\mathbb{E}_n J_n^k$ which result in an arbitrarily given element of σ°-type.

Proposition 7.1. *In* (7.1)*, we have an asymptotic order as*

$$K_{\sigma,n} \asymp n^{m_2(\sigma)}, \qquad \sigma \in \overline{\mathbb{Y}}_k.$$

Proof. Recall (6.6) about the numbers of ups and downs along a k-walk from $\varnothing$ to σ°:

$$u = \frac{k + l(\sigma^\circ)}{2}, \qquad d = \frac{k - l(\sigma^\circ)}{2}. \tag{7.2}$$

An up step $\rho_j \to \rho_{j+1}$ can be realized by taking $(i\ *)x$ such that $\mathrm{type}(x) = \rho_j$ and $\mathrm{type}((i\ *)x) = \rho_{j+1}$ when (i) $i \notin \mathrm{supp}\, x$, $* \notin \mathrm{supp}\, x$, (ii) $i \notin \mathrm{supp}\, x$, $* \in \mathrm{supp}\, x$, (iii) $i \in \mathrm{supp}\, x$, $* \notin \mathrm{supp}\, x$, and (iv) $i, *$ are contained in distinct cycles of x.

The orders of growth are (i) n, (ii) n, (iii) 1 and (iv) 1 respectively. On the other hand, as is seen from the choice of admissible sequences in the proof of Proposition 6.14, we have k-walks of order n^u, projected onto a k-walk from $\varnothing$ to σ°, by taking only (i) and (ii) cases as up steps. The number of k-walks in $\overline{\mathbb{Y}}$ is independent of n. Such k-walks of order n^u are evenly distributed over $C_{(\sigma^\circ, 1^{n-|\sigma^\circ|})}$. Hence $K_{\sigma,n}$, per each element, has order n to the power

$$u - |\sigma^\circ| = \frac{|\sigma| + l(\sigma^\circ)}{2} - |\sigma^\circ| = m_2(\sigma)$$

by using (7.2). □

Remark 7.2. Actually we know that

$$\lim_{n\to\infty} \frac{K_{\sigma,n}}{n^{m_2(\sigma)}} = |\mathrm{NC}(\sigma)| z_{\sigma^\circ} = |\mathrm{NC}(\sigma)| \prod_{j\geq 2} j^{m_{j+1}(\sigma)} m_{j+1}(\sigma)!$$

holds where $\mathrm{NC}(\sigma)$ denotes the set of noncrossing partitions of $\{1, 2, \ldots, k = |\sigma|\}$ of block type σ.

Proposition 7.3. *For even* $2p$, $p \in \mathbb{N}$,

$$\delta_e(J_n^{2p}) = \frac{1}{p+1}\binom{2p}{p} n^p + O(n^{p-1}), \tag{7.3}$$

while $\delta_e(J_n^{2p+1}) = 0$.

Proof. Apply δ_e to (6.3). The surviving terms satisfy that the projected k-walks start at $\varnothing$ and end at $\varnothing$. Such a walk can exist only if k is even. Set $k = 2p$.

Consider a $2p$-walk w on $\mathfrak{S}_{n+1}$ projected onto an admissible $2p$-walk in $\overline{\mathbb{Y}}$, which necessarily has p up steps and p down steps. Once appeared a nontrivial cycle c without $*$ in w, then we need an up step in w later at which some letter i chosen from c is involved, namely $(i\ *)$ acts. However, since these up steps are just of order 1, those walks on $\mathfrak{S}_{n+1}$ containing such a step are of smaller order $O(n^{p-1})$ compared with those which have up steps only of order n.

Let us consider the remaining cases, that is, the $2p$-walks in $\overline{\mathbb{Y}}$ moving within the upper boundary line in Figure 6.1. For each $2p$-walk of this sort starting at $\varnothing$ and ending at $\varnothing$, every up step can have almost n choices. Hence the number of corresponding walks on $\mathfrak{S}_{n+1}$ is $n^p(1+O(1/n))$. Thus we have only to count the $2p$-walks along the upper boundary. This walk is equivalent to the simple random walk on the half line $\mathbb{Z}_{\geq 0}$ returning to 0 at time $2p$, and hence has a bijective correspondence to a Catalan path of length $2p$. □

Proposition 7.4.

$$\delta_e((\mathbb{E}_n J_n^k)^2) - \delta_e(J_n^k)^2 = O(n^{k-1}).$$

Proof. Equation (7.1) and Proposition 7.1 yield

$$\delta_e((\mathbb{E}_n J_n^k)^2) = \sum_{\sigma\in\overline{\mathbb{Y}}_k} K_{\sigma,n}^{\,2}|C_{(\sigma^\circ,1^{n-|\sigma^\circ|})}| = \sum_{\sigma\in\overline{\mathbb{Y}}_k} c_\sigma n^{2m_2(\sigma)}(1+O(1/n))n^{|\sigma^\circ|}$$

with constant $c_\sigma > 0$. We have

$$2m_2(\sigma) + |\sigma^\circ| = l(\sigma) + m_2(\sigma) = k - (m_3(\sigma) + m_4(\sigma) + \cdots),$$

which is equal to the maximum k if and only if k is even and $\sigma = (2^{k/2})$. If $k = 2p$ is even, the leading term is at $\sigma = (2^p)$ (hence $\sigma^\circ = \varnothing$). Then

$$K_{\sigma,n}^{\,2}|C_{(\sigma^\circ,1^{n-|\sigma^\circ|})}| = K_{(2^p),n}^{\,2},$$

which is square of the e-component in $\mathbb{E}_n J_n^k$ i.e. of $\delta_e(J_n^k)$. The remainder contributes as $O(n^{k-1})$. □

7.2. **LLN for the Plancherel ensemble.** $\mathbb{E}_n J_n^k$ being considered as a random variable in $\mathbb{C}[\mathfrak{S}_n]$ (or actually in its center) with respect to δ_e, Proposition 7.3 and Proposition 7.4 imply its mean $\sim\ n^{k/2}$ and its standard deviation of $o(n^{k/2})$. These properties immediately show a weak law of large numbers in a usual manner. In fact, since

$$\delta_e(J_n^k) = n^{k/2}M_k(\mathfrak{m}_\Omega) + O(n^{k/2-1}), \qquad k \geq 2$$

holds from Proposition 6.5 and Proposition 7.3, using Theorem 3.7, Proposition 7.4 and multiplicativity of normalized irreducible character $\tilde{\chi}^\lambda$ on the center, we have

$$\begin{aligned}
&E^{\mathfrak{P}}[(M_k(\mathfrak{m}_{t(n)\sqrt{n}}) - M_k(\mathfrak{m}_\Omega))^2]\\
&= \sum_{\lambda\in\mathbb{Y}_n} \mathfrak{P}_n(\lambda)\big(n^{-k/2}M_k(\mathfrak{m}_\lambda) - M_k(\mathfrak{m}_\Omega)\big)^2 = \sum_{\lambda\in\mathbb{Y}_n} \mathfrak{P}_n(\lambda)\big(n^{-k/2}\tilde{\chi}^\lambda(\mathbb{E}_n J_n^k) - M_k(\mathfrak{m}_\Omega)\big)^2\\
&= \sum_{\lambda\in\mathbb{Y}_n} \mathfrak{P}_n(\lambda)\,\tilde{\chi}^\lambda\big((n^{-k/2}\mathbb{E}_n J_n^k - M_k(\mathfrak{m}_\Omega))^2\big) = \delta_e\big((n^{-k/2}\mathbb{E}_n J_n^k - M_k(\mathfrak{m}_\Omega))^2\big)\\
&= n^{-k}\big(\delta_e(J_n^k)^2 + O(n^{k-1})\big) - 2n^{-k/2}M_k(\mathfrak{m}_\Omega)\big(M_k(\mathfrak{m}_\Omega)n^{k/2} + O(n^{k/2-1})\big) + M_k(\mathfrak{m}_\Omega)^2\\
&= O(1/n).
\end{aligned}$$

Then it holds for any $k \in \mathbb{N}$ and $\epsilon > 0$ that

$$\begin{aligned}
\mathfrak{P}\big(\{t\in\mathfrak{T}\,\big|\,|M_k(\mathfrak{m}_{t(n)\sqrt{n}}) - M_k(\mathfrak{m}_\Omega)| \geq \epsilon\}\big) &\leq \frac{1}{\epsilon^2}E^{\mathfrak{P}}[(M_k(\mathfrak{m}_{t(n)\sqrt{n}}) - M_k(\mathfrak{m}_\Omega))^2]\\
&= O(1/n).
\end{aligned}$$

For strong LLN, we first mention the following estimate.

Theorem 7.5.

$$\delta_e\big((n^{-k/2}\mathbb{E}_n J_n^k - M_k(\mathfrak{m}_\Omega))^4\big) = O(1/n^2).$$

Proof. It needs a bit finer estimates than Proposition 7.1, which involve intersection numbers $p_{\rho\sigma}^{\ \tau}$ for (the association scheme of) the symmetric group $\mathfrak{S}_n$ defined by

$$A_{C_\rho} A_{C_\sigma} = \sum_{\tau \in \mathbb{Y}_n} p_{\rho\sigma}^{\ \tau} A_{C_\tau}, \qquad \rho, \sigma \in \mathbb{Y}_n. \tag{7.4}$$

We omit the details. □

Through a similar argument as before, Theorem 7.5 yields

$$\sum_{n=1}^{\infty} \mathfrak{P}\big(\big\{t \in \mathfrak{T} \,\big|\, |M_k(\mathfrak{m}_{t(n)^{\sqrt{n}}}) - M_k(\mathfrak{m}_\Omega)| \geq \epsilon\big\}\big) < \infty.$$

Then, taking a sequence $\epsilon_j \searrow 0$ and setting

$$\mathfrak{T}_0 = \bigcap_{k=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{l=1}^{\infty} \bigcap_{n=l}^{\infty} \big\{t \in \mathfrak{T} \,\big|\, |M_k(\mathfrak{m}_{t(n)^{\sqrt{n}}}) - M_k(\mathfrak{m}_\Omega)| < \epsilon_j\big\},$$

we have $\mathfrak{P}(\mathfrak{T}_0) = 1$ from Borel–Cantelli's lemma. Since $t \in \mathfrak{T}_0$ is equivalent to

$$\lim_{n \to \infty} |M_k(\mathfrak{m}_{t(n)^{\sqrt{n}}}) - M_k(\mathfrak{m}_\Omega)| = 0, \qquad k \in \mathbb{N},$$

this completes the proof of Theorem 6.6 (2).

For Theorem 6.6 (1), we reduce the argument to a compact interval of $\mathbb{R}$.

Theorem 7.6. *There exists a constant $c > 0$ such that*

$$\sum_{n=1}^{\infty} \mathfrak{P}_n\big(\big\{\lambda \in \mathbb{Y}_n \,\big|\, r(\lambda) > c\sqrt{n} \text{ or } c(\lambda) > c\sqrt{n}\big\}\big) < \infty.$$

We take $c \geq 2$ and set

$$\mathfrak{T}_1 = \bigcup_{l=1}^{\infty} \bigcap_{n=l}^{\infty} \big\{t \in \mathfrak{T} \,\big|\, r(t(n)) \leq c\sqrt{n},\ c(t(n)) \leq c\sqrt{n}\big\}.$$

Theorem 7.6 with Borel–Cantelli's lemma again yields $\mathfrak{P}(\mathfrak{T}_1) = 1$. If $t \in \mathfrak{T}_1$, then $t(n)^{\sqrt{n}}(x) - \Omega(x) = 0$ for any $x \notin [-c,\ c]$ if n is large enough. Hence if $t \in \mathfrak{T}_0 \cap \mathfrak{T}_1$, we conclude that, for any $\epsilon > 0$, sufficiently large n gives

$$\sup_{x \in \mathbb{R}} |t(n)^{\sqrt{n}}(x) - \Omega(x)| \leq \epsilon.$$

We thus showed Theorem 6.6 (1).

Proof of Theorem 7.6[1] The RSK (or Schensted) correspondence gives us

$$P(L_n = l) = \mathfrak{P}_n(\{\lambda \in \mathbb{Y}_n \,|\, c(\lambda) = l\}), \qquad l \in \{1, 2, \ldots, n\}, \tag{7.5}$$

where L_n denotes the length of the longest increasing subsequence of a uniformly distributed random permutaion of size n. We get a rough estimate

$$P(L_n = l) \leq \frac{1}{n!} \binom{n}{l}^2 (n-l)! \leq \frac{n^l}{(l!)^2} \tag{7.6}$$

[1] Communicated by P. Śniady in the lecture.

by first picking up members of a longest increasing subsequence and their positions. Combining (7.5) with (7.6), we have

$$\sum_{n=1}^{\infty} \mathfrak{P}_n(\{\lambda \in \mathbb{Y}_n \mid c(\lambda) > c\sqrt{n}\}) = \sum_{n=1}^{\infty} P(L_n > c\sqrt{n})$$
$$\leq \sum_{n=1}^{\infty} \sum_{l=c\sqrt{n}}^{n} \frac{n^l}{(l!)^2} = \sum_{l=c}^{\infty} \sum_{n=l}^{l^2/c^2} \frac{n^l}{(l!)^2} \leq \sum_{l=c}^{\infty} \frac{1}{(l!)^2} \Big(\frac{l^2}{c^2}\Big)^{l+1}.$$

The rightmost infinite series converges if $c > e$ since Stirling's formula ensures

$$\frac{1}{(l!)^2} \Big(\frac{l^2}{c^2}\Big)^{l+1} \sim \frac{l}{2\pi c^2} \Big(\frac{e}{c}\Big)^{2l}.$$

Estimate for $r(\lambda)$ is similar because of the symmetry. □

Remark 7.7. The Schensted correspondence describes a bijection between $\mathfrak{S}_n$ and the set of pairs of standard tableaux with the same shapes. For permutation $\pi \in \mathfrak{S}_n$, the first tableau $P(\pi)$ results from the bumping procedure while the second tableau $Q(\pi)$ records the history of the growth. For example, given permutation 325164 in $\mathfrak{S}_6$, we proceed as

$$3 \to \begin{matrix} 2 \\ 3 \end{matrix} \to \begin{matrix} 2 & 5 \\ 3 & \end{matrix} \to \begin{matrix} 1 & 5 \\ 2 & \\ 3 & \end{matrix} \to \begin{matrix} 1 & 5 & 6 \\ 2 & & \\ 3 & & \end{matrix} \to \begin{matrix} 1 & 4 & 6 \\ 2 & 5 & \\ 3 & & \end{matrix} = P, \qquad \begin{matrix} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & & \end{matrix} = Q.$$

Then the length of the longest increasing subsequence of π coincides with the length of the first row of $P(\pi)$ (though the first row itself need not be a longest increasing subsequence of π in general, just as seen above). This yields (7.5).

8. Limit shape of Young diagrams and fluctuations III (Lecture 8)

8.1. Kerov's CLT and fluctuations in the Plancherel ensemble.

Definition 8.1. For $\rho \in \mathbb{Y}$ set

$$\Sigma_\rho(\lambda) = \begin{cases} n(n-1)\cdots(n-j+1)\tilde{\chi}^\lambda_{(\rho,1^{n-j})}, & |\lambda| = n \geq j = |\rho|, \\ 0, & |\lambda| = n < j = |\rho|. \end{cases}$$

In particular, set $\Sigma_j = \Sigma_{(j)}$ for a one-row diagram.

Theorem 8.2 (Kerov's central limit theorem)**.** *Random variable Σ_j on $(\mathbb{Y}_n, \mathfrak{P}_n)$, having mean 0 and variance $jn(n-1)\cdots(n-j+1)$, satisfies*

$$\lim_{n\to\infty} \mathfrak{P}_n(\{\lambda \in \mathbb{Y}_n \,|\, n^{-j/2}\Sigma_j(\lambda) \leq x_j,\ j = 2, 3, \ldots, k\}) = \prod_{j=2}^{k} \int_{-\infty}^{x_j} \frac{1}{\sqrt{2\pi j}} e^{-x^2/2j} dx$$

for $k \geq 2$ and $x_2, \cdots, x_k \in \mathbb{R}$. In other words, $\{\Sigma_j\}_{j\geq 2}$ are asymptotically independent and Gaussian with respect to the Plancherel measure.

This subsection is devoted to explaining why Kerov's CLT is fundamental in describing fluctuation of random Young diagrams in the Plancherel ensemble. Proof of Theorem 8.2 will be included in a wider context of QCLT later.

For that purpose it is suitable to mention here polynomial functions on Young diagrams introduced by Kerov–Olshanski. Recall that $\{M_k(\tau_\lambda)\}$ and $\{M_k(\mathfrak{m}_\lambda)\}$ are mutually related in polynomial relations. $M_k(\tau_\lambda)$ is clearly a polynomial in x_i's and y_i's, the min-max coordinates of $\lambda \in \mathbb{Y}$.

Definition 8.3. The algebra generated by $\{M_k(\tau_\lambda)\}_{k\in\mathbb{N}}$ is called the algebra of polynomial functions on $\mathbb{Y}$ and denoted by $\mathbb{A}$. Polynomial function $M_k(\tau_\lambda)$ is by definition homogeneous with weight degree k. This weight degree gives a filtration in $\mathbb{A}$.

Classical, free and Boolean cumulants, denoted by κ_k, R_k and B_k respectively, of both of Rayleigh and transition measures are generators of $\mathbb{A}$ by virtue of the moment-cumulant formulas. A remarkable fact is that $\{\Sigma_k(\lambda)\}$ also forms a generator set of $\mathbb{A}$. We see more as the following.

Theorem 8.4. *We have*

$$\Sigma_k(\lambda) = R_{k+1}(\mathfrak{m}_\lambda) + P_k(R_2(\mathfrak{m}_\lambda), \cdots, R_{k-1}(\mathfrak{m}_\lambda)) = K_k(R_{k+1}(\mathfrak{m}_\lambda), R_{k-1}(\mathfrak{m}_\lambda), \cdots)$$

where P_k is a polynomial of $\mathbb{Z}$-coefficients in which each term has weight degree $\leq k-1$ and of the same parity with $k+1$, $R_j(\mathfrak{m}_\lambda)$ being regarded to have weight degree j.

Remark 8.5. K_k is called a Kerov's polynomial. Positivity of their coefficients is a famous open problem.

We know from Theorem 6.6 that scaled Young diagram $\lambda^{\sqrt{n}}$, $\lambda \in \mathbb{Y}_n$, converges to the limit shape:

$$\lambda^{\sqrt{n}}(x) - \Omega(x) \longrightarrow 0, \qquad n \to \infty$$

in the Plancherel ensemble. Note that, for $k \in \mathbb{N}$,

$$\int_{-\infty}^{\infty} x^k(\lambda^{\sqrt{n}}(x) - \Omega(x))dx = \frac{1}{(k+1)(k+2)}\{M_{k+2}(\tau_{\lambda^{\sqrt{n}}}) - M_{k+2}(\tau_\Omega)\}. \tag{8.1}$$

Incorporating (8.1) into Kerov's CLT, we deduce fluctuation of shapes of random Young diagrams picked up from the Plancherel ensemble as follows.[2]

Consider a family of (8.1) indexed by $k \in \mathbb{N}$. The kth moment of a $1/\sqrt{n}$-rescaled diagram gets $n^{-k/2}$-multiple. Theorem 8.4 (or a rather weaker version would be enough) yields that the right hand side of (8.1) is equivalently replaced by

$$n^{-(k+2)/2}P(\Sigma_{k+1}, \Sigma_k, \cdots) - \text{(constant)} \tag{8.2}$$

where a term in the polynomial $P(\Sigma_{k+1}, \Sigma_k, \cdots)$, say

$$\Sigma_{k_1}\Sigma_{k_2}\cdots\Sigma_{k_l}\Sigma_1^{\,r}, \qquad k_i \geq 2,\ r \geq 0 \tag{8.3}$$

has weight degree $(k_1+1)+\cdots+(k_l+1)+2r \leq k+2$. On the other hand, Kerov's CLT tells us that Σ_j's behave as independent $n^{j/2}$-multiple Gaussian random variables asymptotically as $n \to \infty$. Hence the contributing order of (8.3) in (8.2) is n to the power

$$-\frac{k+2}{2} + \frac{k_1}{2} + \cdots + \frac{k_l}{2} + r \leq -\frac{l}{2}.$$

Note that the constant term together with the $l = 0$ term in (8.2) vanishes as $n \to \infty$ since we already know that (8.2) tends to 0. The dominant terms are thus those of $l = 1$. Then, rescaled by $\sqrt{n}$-multiple, (8.2) and hence (8.1) converge to a sum of Gaussian random variables. This means that $\sqrt{n}(\lambda^{\sqrt{n}} - \Omega)$ viewed as a random variable in λ with respect to the Plancherel measure $\mathfrak{P}_n$ converges to a "Gaussian object" at least in some weak sense as $n \to \infty$.

Remark 8.6. The limiting Gaussian object above was captured as a generalized Gaussian process supported on $[-2, 2]$ by Kerov and Ivanov–Olshanski by considering Chebyshev polynomials as test functions in (8.1).

8.2. **Quantum decomposition of adjacency operators.** Since irreducible character values $\tilde{\chi}^\lambda_\rho$ appear as eigenvalues of adjacency operators A_ρ (e.g. Proposition 5.4), we can reformulate Kerov's CLT in terms of adjacency operators. This enables us to express the asymptotic Gaussian fluctuation in question by using creation and annihilation operators on the Boson Fock space. In this procedure, the notion of quantum decomposition appears to be of use.

Using spectral decomposition of A_ρ as Proposition 5.4, we have

$$\Big\langle \delta_e, \frac{A_{\rho_1}}{|C_{\rho_1}|}\cdots\frac{A_{\rho_r}}{|C_{\rho_r}|}\delta_e\Big\rangle = \sum_{\lambda\in\mathbb{Y}_n}\frac{(\dim\lambda)^2}{n!}\tilde{\chi}^\lambda_{\rho_1}\cdots\tilde{\chi}^\lambda_{\rho_r}. \tag{8.4}$$

Lemma 8.7. *Kerov's CLT is equivalent to the following convergence of mixed moments of adjacency operators with respect to the vacuum state δ_e:*

$$\lim_{n\to\infty}\Big\langle \delta_e, \Big(\frac{A_{(2,1^{n-2})}}{\sqrt{|C_{(2,1^{n-2})}|}}\Big)^{p_2}\cdots\Big(\frac{A_{(k,1^{n-k})}}{\sqrt{|C_{(k,1^{n-k})}|}}\Big)^{p_k}\delta_e\Big\rangle = \prod_{j=2}^{k}\int_{-\infty}^{\infty}x^{p_j}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx$$

for $k \geq 2$ and $p_2, \cdots, p_k \in \mathbb{N}$.

Proof. Straightforward from (8.4). □

[2]The argument in p.300 of our monograph [17], which concerns this deduction, should be replaced by the following, or more precisely, according to statements in the "corrections page" of that monograph which will appear in web pages of the authors.

We define quantum decomposition of an adjacency operator:

$$A_{(j,1^{n-j})} = A^+_{(j,1^{n-j})} + A^-_{(j,1^{n-j})} + A^o_{(j,1^{n-j})} \tag{8.5}$$

according to the length function l (up, down and in the same stratum). Then, quantum plus and minus components converge to creation and annihilation operators on the Boson Fock space. The limit picture of spectral structure is quite transparent and immediately yields Gaussian fluctuation. We do not have to be troubled over complications at finite levels.

Definition 8.8 (quantum decomposition)**.** (1) For $g \in \mathfrak{S}_n$ we define linear operators g^+, g^-, g^o on $\ell^2(\mathfrak{S}_n)$ by

$$g^\epsilon \delta_x = \begin{cases} \delta_{gx}, & l(\text{type}(gx)) \sharp_\epsilon \, l(\text{type}(x)), \\ 0, & \text{otherwise}, \end{cases} \qquad \sharp_\epsilon = \begin{cases} >, & \epsilon = + \\ <, & \epsilon = - \\ =, & \epsilon = o. \end{cases}$$

Immediately $(g^+ + g^- + g^o)\delta_x = \delta_{gx}$, $(g^+)^* = g^-$, $(g^o)^* = g^o$.
(2) For $\rho \in \mathbb{Y}_n$ we set

$$A^\epsilon_\rho = \sum_{g \in C_\rho} g^\epsilon, \qquad \epsilon \in \{+, -, o\}.$$

We call $A_\rho = A^+_\rho + A^-_\rho + A^o_\rho$ quantum decomposition of A_ρ. Clearly A^+_ρ and A^-_ρ are mutually adjoint while A^o_ρ is self-adjoint.

Recall ξ_σ, $\Phi(\sigma)$ and $\Gamma(\mathfrak{S}_n)$ in Section 5.1. However, we often express an element of $\mathbb{Y}_n$ as $(\sigma, 1^{n-|\sigma|})$ where $\sigma \in \overline{\mathbb{Y}}$, $|\sigma| \le n$ by indicating the one-box rows explicitly. This is useful when type σ is fixed and n tends to ∞. Recall the intersection numbers determined in (7.4). To avoid too heavy notations, we set

$$p^\tau_{\rho\sigma}(n) = p^{(\tau,1^{n-|\tau|})}_{(\rho,1^{n-|\rho|})\,(\sigma,1^{n-|\sigma|})}, \qquad \rho, \sigma, \tau \in \overline{\mathbb{Y}}, \quad |\rho|, |\sigma|, |\tau| \le n.$$

Choosing $x \in C_{(\tau,1^{n-|\tau|})}$ arbitrarily, we have

$$\begin{aligned} p^\tau_{\rho\sigma}(n) &= \left|\{(z,z') \,|\, z \in C_{(\rho,1^{n-|\rho|})}, z' \in C_{(\sigma,1^{n-|\sigma|})},\ zz' = x\}\right| \\ &= \left|\{z \in C_{(\rho,1^{n-|\rho|})} \,|\, z^{-1}x \in C_{(\sigma,1^{n-|\sigma|})}\}\right|. \end{aligned} \tag{8.6}$$

Lemma 8.9. *For $j \in \{2, \dots, n\}$ and $\sigma \in \overline{\mathbb{Y}}$ such that $|\sigma| \le n$, we have*

$$\begin{aligned} A^\pm_{(j,1^{n-j})}\xi_{(\sigma,1^{n-|\sigma|})} &= \sum_{i=1}^{j-1} \sum_{|\tau| \le n: l(\tau) = l(\sigma) \pm i} p^\tau_{(j)\sigma}(n)\xi_{(\tau,1^{n-|\tau|})}, \\ A^o_{(j,1^{n-j})}\xi_{(\sigma,1^{n-|\sigma|})} &= \sum_{|\tau| \le n: l(\tau) = l(\sigma)} p^\tau_{(j)\sigma}(n)\xi_{(\tau,1^{n-|\tau|})}. \end{aligned}$$

In particular, $\Gamma(\mathfrak{S}_n)$ is invariant under $A^\epsilon_{(j,1^{n-j})}$.

Proof. Straightforward from the definitions:

$$
\begin{aligned}
&A^+_{(j,1^{n-j})}\xi_{(\sigma,1^{n-|\sigma|})}\\
&= \sum_{x\in C_{(\sigma,1^{n-|\sigma|})}} \sum_{g\in C_{(j,1^{n-j})}} g^+\delta_x = \sum_{x\in C_{(\sigma,1^{n-|\sigma|})}} \sum_{g\in C_{(j,1^{n-j})}: l(\mathrm{type}(gx))>l(\sigma)} g^+\delta_x\\
&= \sum_{y\in\mathfrak{S}_n} \big|\{(g,x)\,|\, x\in C_{(\sigma,1^{n-|\sigma|})}, g\in C_{(j,1^{n-j})}, l(\mathrm{type}(y))>l(\sigma), gx=y\}\big|\delta_y\\
&= \sum_{\tau\in\overline{\mathbb{Y}}:|\tau|\le n, l(\tau)>l(\sigma)} \sum_{y\in C_{(\tau,1^{n-|\tau|})}} p^{\tau}_{(j)\sigma}(n)\delta_y = \sum_{i=1}^{j-1} \sum_{\tau\in\overline{\mathbb{Y}}:|\tau|\le n, l(\tau)=l(\sigma)+i} p^{\tau}_{(j)\sigma}(n)\xi_{(\tau,1^{n-|\tau|})}
\end{aligned}
$$

and similarly for $-$ and o. □

8.3. **The Fock space on $\overline{\mathbb{Y}}$.** Let $\{\Psi(\sigma)\,|\,\sigma\in\overline{\mathbb{Y}}\}$ denote the canonical orthonormal basis in $\ell^2(\overline{\mathbb{Y}})$. $\Psi(\varnothing)$ is referred to as the vacuum vector.

Definition 8.10. Set for $j\ge 2$

$$
\begin{aligned}
B_j^+\Psi(\sigma) &= \sqrt{m_j(\sigma)+1}\;\Psi(\sigma\cup(j))\\
B_j^-\Psi(\sigma) &= \sqrt{m_j(\sigma)}\;\Psi(\sigma\setminus(j))\quad (\,=0\ \text{ if } m_j(\sigma)=0).
\end{aligned}
$$

We call $(\ell^2(\overline{\mathbb{Y}}), \{B_j^+\}, \{B_j^-\})$ the Fock space on $\overline{\mathbb{Y}}$. As usual B_j^+ and B_j^- are called a creation operator and an annihilation operator respectively.

The commutation relation is immediately seen as

$$
[B_i^-, B_j^-] = [B_i^+, B_j^+] = [B_i^-, B_j^+] = 0,\ i\ne j;\quad [B_j^-, B_j^+] = 1
$$

on $\Gamma(\overline{\mathbb{Y}}) = \mathrm{span}\{\Psi(\sigma)\,|\,\sigma\in\overline{\mathbb{Y}}\}$.

Proposition 8.11. *The Fock space on $\overline{\mathbb{Y}}$ is isomorphic to the Boson Fock space over an infinite-dimensional separable Hilbert space. To be more precise, let $\{v_2, v_3, \cdots\}$ be an ONB in Hilbert space $\mathcal{H}$. The unitary operator from $\ell^2(\overline{\mathbb{Y}})$ to $\Gamma(\mathcal{H})$ determined by*

$$
\Psi(\sigma)\ \longmapsto\ \sqrt{\frac{|\sigma|!}{\prod_{j\ge 2} m_j(\sigma)!}}\ \underbrace{v_2\hat\otimes\cdots\hat\otimes v_2}_{m_2(\sigma)}\hat\otimes\cdots\hat\otimes\underbrace{v_j\hat\otimes\cdots\hat\otimes v_j}_{m_j(\sigma)}\hat\otimes\cdots
$$

intertwines the actions of $B_j^\pm$ and $A^\pm(v_j)$ (see below) respectively for $j = 2, 3, \ldots$. In particular, the one-particle space is $\mathrm{span}\{\Psi\big((j)\big)\,|\,j\ge 2\}$. *The number of particles in state ρ is indicated by $r(\rho)$.*

We recall the Boson Fock space over $\mathcal{H}$. Let $\Gamma(\mathcal{H}) = \bigoplus_{n=0}^\infty \mathcal{H}^{\hat\otimes n}$ where $\mathcal{H}^{\hat\otimes n}$ is spanned by such symmetric tensors as $u_1\hat\otimes\cdots\hat\otimes u_n = n!^{-1}\sum_{g\in\mathfrak{S}_n} u_{g(1)}\otimes\cdots\otimes u_{g(n)}$, $u_1,\cdots,u_n\in\mathcal{H}$, and equipped with the scalar product inherited from $\mathcal{H}^{\otimes n}$. ($\mathcal{H}^{\hat\otimes 0} = \mathbb{C}$.) The scalar product on $\Gamma(\mathcal{H})$ is just sum of those on components. Creation and annihilation operators are defined by

$$
A^+(u)\underbrace{v\hat\otimes\cdots\hat\otimes v}_{n} = \sqrt{n+1}\,u\hat\otimes\underbrace{v\hat\otimes\cdots\hat\otimes v}_{n},\quad A^-(u)\underbrace{v\hat\otimes\cdots\hat\otimes v}_{n} = \sqrt{n}\,\langle u,v\rangle\underbrace{v\hat\otimes\cdots\hat\otimes v}_{n-1}.
$$

$\Gamma_{\mathrm{boson}}(\mathcal{H}) = (\Gamma(\mathcal{H}), A^+, A^-)$ is called the Boson Fock space over $\mathcal{H}$.

9. Limit shape of Young diagrams and fluctuations IV (Lecture 9)

9.1. **Quantum central limit theorem for adjacency operators.** Consider the adjacency operator associated with j-cycles and its quantum decomposition

$$A_{(j,1^{n-j})} = A^+_{(j,1^{n-j})} + A^-_{(j,1^{n-j})} + A^o_{(j,1^{n-j})}, \qquad j \geq 2,$$

each quantum component acting on $\Gamma(\mathfrak{S}_n) = \mathrm{span}\{\xi_{(\sigma,1^{n-|\sigma|})} \,|\, \sigma \in \overline{\mathbb{Y}}\}$. Since its vacuum variance is given by

$$\langle \delta_e, A^2_{(j,1^{n-j})}\delta_e\rangle = |C_{(j,1^{n-j})}| = n(n-1)\cdots(n-j+1)/j,$$

we take normalization by the square root of this factor for CLT. Lemma 8.9 yields

(9.1)
$$\frac{A^{\pm}_{(j,1^{n-j})}}{\sqrt{|C_{(j,1^{n-j})}|}}\Phi_{(\sigma,1^{n-|\sigma|})} = \sum_{i=1}^{j-1}\sum_{|\tau|\leq n: l(\tau)=l(\sigma)\pm i} p^{\tau}_{(j)\sigma}(n)\sqrt{\frac{|C_{(\tau,1^{n-|\tau|})}|}{|C_{(j,1^{n-j})}|\,|C_{(\sigma,1^{n-|\sigma|})}|}}\Phi_{(\tau,1^{n-|\tau|})},$$

(9.2)
$$\frac{A^{o}_{(j,1^{n-j})}}{\sqrt{|C_{(j,1^{n-j})}|}}\Phi_{(\sigma,1^{n-|\sigma|})} = \sum_{|\tau|\leq n: l(\tau)=l(\sigma)} p^{\tau}_{(j)\sigma}(n)\sqrt{\frac{|C_{(\tau,1^{n-|\tau|})}|}{|C_{(j,1^{n-j})}|\,|C_{(\sigma,1^{n-|\sigma|})}|}}\Phi_{(\tau,1^{n-|\tau|})}.$$

Lemma 9.1. *Let $j \geq 2$ and $\sigma, \tau \in \overline{\mathbb{Y}}$ satisfy neither $\tau = (\sigma, j)$ nor $\sigma = (\tau, j)$. Then*

$$p^{\tau}_{(j)\sigma}(n)\sqrt{\frac{|C_{(\tau,1^{n-|\tau|})}|}{|C_{(j,1^{n-j})}|\,|C_{(\sigma,1^{n-|\sigma|})}|}} = O(1/\sqrt{n}), \qquad n \to \infty.$$

Proof. Here is just a hint. Recall

$$p^{\tau}_{(j)\sigma}(n) = \big|\{z \in C_{(j,1^{n-j})} \,|\, z^{-1}x \in C_{(\sigma,1^{n-|\sigma|})}\}\big|$$

where x is arbitrarily picked up from $C_{(\tau,1^{n-|\tau|})}$. For j-cycle $z = (a_1\ a_2\ \cdots\ a_j)$, we consider

$$z^{-1}x = (a_j\ a_{j-1})\cdots(a_3\ a_2)(a_2\ a_1)x \tag{9.3}$$

and the associated $(j-1)$-walk on $\overline{\mathbb{Y}}$ starting at τ and ending at σ. As done in analysis of moments of the Jucys-Murphy element, we perform order counting along a walk. Situations are a bit more complicated than before because we deal with a chain of transpositions like (9.3), not only of $(a\ *)$'s. Note that the square root part in question is of order $n^{(|\tau|-|\sigma|-j)/2}$. At every up/down step along the $(j-1)$-walk, we take into account gain/loss of the size also. For a step $\tau' \to \tau''$, it holds clearly that $|\tau''| = |\tau'| \pm i$, $i = 0, 1, 2$. Figure 9.1 illustrates the steps of left multiplication of transposition $(\circ\ \times)$ to x (denoted by a tableau). The left [resp. right] three ones are up [resp. down] steps. Gain or loss of size is $+2$, $+1$, ± 0, ± 0, -1 and -2 respectively from left to right. The actions of ± 2-size rarely happen and give a clue for classification of the walks. Look at the whole $(j-1)$-walk from τ to σ, count the order of growth at each step, and recognize the exceptional two cases to obtain the desired estimate. $\square$

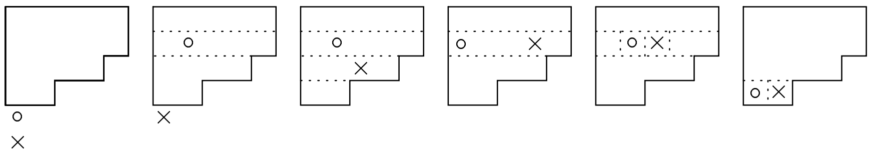

FIG. 9.1. Action of transposition $(\circ\ \times)$ with up/down of length and gain/loss of size

Theorem 9.2 (QCLT). *For $k \in \mathbb{N}$, $\epsilon_1, \cdots, \epsilon_k \in \{+,-,o\}$, $j_1, \cdots, j_k \in \{2,3,\dots,\}$ and $\tau, \sigma \in \overline{\mathbb{Y}}$, we have*

$$\lim_{n\to\infty}\Big\langle \Phi(\tau, 1^{n-|\tau|}), \frac{A^{\epsilon_1}_{(j_1,1^{n-j_1})}}{\sqrt{|C_{(j_1,1^{n-j_1})}|}}\cdots\frac{A^{\epsilon_k}_{(j_k,1^{n-j_k})}}{\sqrt{|C_{(j_k,1^{n-j_k})}|}}\Phi(\sigma,1^{n-|\sigma|})\Big\rangle_{\Gamma(\mathfrak{S}_n)} = \langle \Psi(\tau), B^{\epsilon_1}_{j_1}\cdots B^{\epsilon_k}_{j_k}\Psi(\sigma)\rangle_{\ell^2(\overline{\mathbb{Y}})}. \tag{9.4}$$

(Recall that $B_j \equiv 0$.)

Proof. We combine (9.1) and (9.2) with Lemma 9.1. A simple argument yields

$$p^{\tau}_{(j)\sigma}(n) = \begin{cases} m_j(\sigma)+1, & \tau = (\sigma, j), \\ (n-|\sigma|+j)\cdots(n-|\sigma|+1)/j, & \tau = \sigma\setminus(j). \end{cases}$$

Together with Lemma 9.1, we then have

$$\lim_{n\to\infty} p^{\tau}_{(j)\sigma}(n)\sqrt{\frac{|C_{(\tau,1^{n-|\tau|})}|}{|C_{(j,1^{n-j})}|\,|C_{(\sigma,1^{n-|\sigma|})}|}} = \begin{cases} \sqrt{m_j(\sigma)+1}, & \tau = (\sigma,j), \\ \sqrt{m_j(\sigma)}, & \tau = \sigma\setminus(j), \\ 0, & \text{otherwise}. \end{cases} \tag{9.5}$$

Equation (9.4) is now verified by induction on k. We just mention the inductive step for $\epsilon = +$ since the other arguments are more or less similar. Assume that (9.4) holds for k. Using (9.1) and (9.5), we have

$$\begin{aligned}
&\Big\langle \Phi(\tau, 1^{n-|\tau|}), \frac{A^{\epsilon_1}_{(j_1,1^{n-j_1})}}{\sqrt{|C_{(j_1,1^{n-j_1})}|}}\cdots\frac{A^{\epsilon_k}_{(j_k,1^{n-j_k})}}{\sqrt{|C_{(j_k,1^{n-j_k})}|}}\frac{A^{+}_{(j_{k+1},1^{n-j_{k+1}})}}{\sqrt{|C_{(j_{k+1},1^{n-j_{k+1}})}|}}\Phi(\sigma,1^{n-|\sigma|})\Big\rangle_{\Gamma(\mathfrak{S}_n)} \\
&= \sum_{i=1}^{j_{k+1}-1}\sum_{|\rho|\le n: l(\rho)=l(\rho)+i} p^{\rho}_{(j_{k+1})\sigma}(n)\sqrt{\frac{|C_{(\rho,1^{n-|\rho|})}|}{|C_{(j_{k+1},1^{n-j_{k+1}})}|\,|C_{(\sigma,1^{n-|\sigma|})}|}} \\
&\qquad \times \Big\langle \Phi(\tau, 1^{n-|\tau|}), \frac{A^{\epsilon_1}_{(j_1,1^{n-j_1})}}{\sqrt{|C_{(j_1,1^{n-j_1})}|}}\cdots\frac{A^{\epsilon_k}_{(j_k,1^{n-j_k})}}{\sqrt{|C_{(j_k,1^{n-j_k})}|}}\Phi(\rho,1^{n-|\rho|})\Big\rangle_{\Gamma(\mathfrak{S}_n)} \\
&\longrightarrow \sqrt{m_{j_{k+1}}(\sigma)+1}\,\langle \Psi(\tau), B^{\epsilon_1}_{j_1}\cdots B^{\epsilon_k}_{j_k}\Psi(\sigma,j)\rangle_{\ell^2(\overline{\mathbb{Y}})} \\
&= \langle \Psi(\tau), B^{\epsilon_1}_{j_1}\cdots B^{\epsilon_k}_{j_k}B^{+}_{j_{k+1}}\Psi(\sigma)\rangle_{\ell^2(\overline{\mathbb{Y}})}.
\end{aligned}$$

This shows (9.4) for $k+1$. ∎

Remark 9.3. Theorem 9.2 implies the convergence of a system of noncommutative random variables

$$\lim_{n\to\infty}\Big(\frac{A^{+}_{(j,1^{n-j})}}{\sqrt{|C_{(j,1^{n-j})}|}}, \frac{A^{-}_{(j,1^{n-j})}}{\sqrt{|C_{(j,1^{n-j})}|}}, \frac{A^{o}_{(j,1^{n-j})}}{\sqrt{|C_{(j,1^{n-j})}|}} \mid j=2,3,\dots\Big) = (B^{+}_j, B^{-}_j, B^{o}_j \mid j=2,3,\dots)$$

in the sense of matrix elements of mixed products. This is a common notion for convergence in QCLT.

Kerov's CLT, or equivalently Lemma 8.7, follows from Theorem 9.2 through a so-called classical reduction. In fact, since $\{B^{+}_j + B^{-}_j \mid j=2,3,\dots\}$ is a system of independent Gaussian random variables with respect to the vacuum state, we have

$$\begin{aligned}
&\lim_{n\to\infty}\Big\langle \delta_e, \Big(\frac{A_{(2,1^{n-2})}}{\sqrt{|C_{(2,1^{n-2})}|}}\Big)^{p_2}\cdots\Big(\frac{A_{(k,1^{n-k})}}{\sqrt{|C_{(k,1^{n-k})}|}}\Big)^{p_k}\delta_e\Big\rangle\\
&= \langle \Psi(\varnothing), (B^{+}_2+B^{-}_2)^{p_2}\cdots(B^{+}_k+B^{-}_k)^{p_k}\Psi(\varnothing)\rangle\\
&= \langle \Psi(\varnothing), (B^{+}_2+B^{-}_2)^{p_2}\Psi(\varnothing)\rangle\cdots\langle\Psi(\varnothing),(B^{+}_k+B^{-}_k)^{p_k}\Psi(\varnothing)\rangle\\
&= \prod_{j=2}^{k}\int_{-\infty}^{\infty} x^{p_j}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx.
\end{aligned}$$

For adjacency operators associated with general conjugacy classes, we have the following asymptotic (Gaussian) behaviour.

Theorem 9.4. *For $k\in\mathbb{N}$, $\tau,\sigma\in\overline{\mathbb{Y}}$, $r_1,\cdots,r_k\in\mathbb{N}$ and $\rho^{(1)},\cdots,\rho^{(k)}\in\overline{\mathbb{Y}}$, we have*

$$\begin{aligned}
&\lim_{n\to\infty}\Big\langle\Phi(\tau,1^{n-|\tau|}), \Big(\frac{A_{(\rho^{(1)},1^{n-|\rho^{(1)}|})}}{\sqrt{|C_{(\rho^{(1)},1^{n-|\rho^{(1)}|})}|}}\Big)^{r_1}\cdots\Big(\frac{A_{(\rho^{(k)},1^{n-|\rho^{(k)}|})}}{\sqrt{|C_{(\rho^{(k)},1^{n-|\rho^{(k)}|})}|}}\Big)^{r_k}\Phi(\sigma,1^{n-|\sigma|})\Big\rangle_{\Gamma(\mathfrak{S}_n)}\\
&= \Big\langle\Psi(\tau), \prod_{j\ge 2}\Big(\frac{H_{m_j(\rho^{(1)})}(B^{+}_j+B^{-}_j)}{\sqrt{m_j(\rho^{(1)})!}}\Big)^{r_1}\cdots\Big(\frac{H_{m_j(\rho^{(k)})}(B^{+}_j+B^{-}_j)}{\sqrt{m_j(\rho^{(k)})!}}\Big)^{r_k}\Psi(\sigma)\Big\rangle_{\ell^2(\overline{\mathbb{Y}})}
\end{aligned}$$

where $H_r(x)$ is a monic Hermite polynomial determined by

$$H_0(x)=1,\quad H_1(x)=x,\quad xH_r(x)=H_{r+1}(x)+rH_{r-1}(x)\quad(r\ge 1).$$

Remark 9.5. It may be useful to note in the above expression that

$$H_r(B^{+}_j+B^{-}_j)=\sum_{i=0}^{r}\binom{r}{i}(B^{+}_j)^i(B^{-}_j)^{r-i}.$$

To see the vacuum expectation explicitly, set $\tau = \sigma = \varnothing$ in Theorem 9.4. Then the right hand side is

$$\Big\langle \Psi(\varnothing), \prod_{j\geq 2} \Big(\frac{H_{m_j(\rho^{(1)})}(B_j^+ + B_j^-)}{\sqrt{m_j(\rho^{(1)})!}}\Big)^{r_1} \cdots \Big(\frac{H_{m_j(\rho^{(k)})}(B_j^+ + B_j^-)}{\sqrt{m_j(\rho^{(k)})!}}\Big)^{r_k} \Psi(\varnothing)\Big\rangle_{\ell^2(\mathbb{Y})}$$
$$= \prod_{j\geq 2} \Big\langle \Psi(\varnothing), \Big(\frac{H_{m_j(\rho^{(1)})}(B_j^+ + B_j^-)}{\sqrt{m_j(\rho^{(1)})!}}\Big)^{r_1} \cdots \Big(\frac{H_{m_j(\rho^{(k)})}(B_j^+ + B_j^-)}{\sqrt{m_j(\rho^{(k)})!}}\Big)^{r_k} \Psi(\varnothing)\Big\rangle_{\ell^2(\mathbb{Y})}$$
$$= \prod_{j\geq 2} \int_{-\infty}^{\infty} \Big(\frac{H_{m_j(\rho^{(1)})}(x)}{\sqrt{m_j(\rho^{(1)})!}}\Big)^{r_1} \cdots \Big(\frac{H_{m_j(\rho^{(k)})}(x)}{\sqrt{m_j(\rho^{(k)})!}}\Big)^{r_k} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx.$$

9.2. **LLN and CLT in several ensembles.** We collect some remarks on LLN and CLT in other statistical ensembles of Young diagrams.

9.2.1. *Berry-Esseen type CLT in the Plancherel ensemble.*

Theorem 9.6 (Fulman)**.** *There exists constant $C > 0$ such that*

$$\Big|\mathfrak{P}_n\big(\{\lambda \in \mathbb{Y}_n \,|\, \frac{1}{\sqrt{2n}}\Sigma_2(\lambda) \leq x\}\big) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2} dy\Big| \leq Cn^{-1/4}.$$

9.2.2. *Fluctuation of the longest row and column in the Plancherel ensemble.* Recall $\lambda_1, \lambda_1' \sim 2\sqrt{n}$ for $\lambda \in (\mathbb{Y}_n, \mathfrak{P}_n)$.

Theorem 9.7 (Baik–Deift–Johansson)**.**

$$\lim_{n\to\infty} \mathfrak{P}_n\Big(\Big\{\lambda \in \mathbb{Y}_n \,|\, n^{1/3}\Big(\frac{\lambda_1}{\sqrt{n}} - 2\Big) \leq x\Big\}\Big) = F(x).$$

The right hand side is the Tracy–Widom distribution function expressed by using the solution of Painlevé II equation with Airy asymptotics, which appeared in describing the fluctuation of the largest eigenvalue of a random matrix in GUE.

Furthermore, similar asymptotics for an arbitrary number of rows and colums were conjectured by Baik–Deift–Johansson. Several proofs of this result are due to Okounkov, Johansson, and Borodin–Okounkov–Olshanski.

9.2.3. *Representations with approximate character factorization property.* Given a representation ρ of $\mathfrak{S}_n$, we have probability P_n on $\mathbb{Y}_n$ by assigning to λ the total dimension of isotypic components of irreducible representation U_λ normalized by $\dim \rho$. Biane introduced the notion of approximate factorization property for a family of representations $\{\rho_n\}$ and their characters $\{\chi_n\}$, which tells that

$$\big|\widetilde{\chi_n}(g_1 g_2 \cdots g_k) - \widetilde{\chi_n}(g_1)\widetilde{\chi_n}(g_2)\cdots\widetilde{\chi_n}(g_k)\big|$$

is of an appropriate small order as $n \to \infty$ for $g_1, \cdots, g_k \in \mathfrak{S}_\infty$ with disjoint supports. Those systems of ensembles which have such asymptotically vanishing correlations are rich sources of models admitting LLN and CLT. They include as examples

- The ensembles determined by the Littlewood–Richardson coefficients
- The Thoma characters restricted onto $\mathfrak{S}_n$ where both parameters $\alpha_i^{(n)}$ and $\beta_i^{(n)}$ tend to 0 as $n \to \infty$ (appropriately fast).

Actually Biane showed LLN for these models. Extending approximate character factorization property to certain decay conditions of classical cumulants $\kappa(g_1, \cdots, g_k)$, Śniady showed CLT and obtained the universal feature of Gaussian fluctuation in a wide variety of ensembles.

9.2.4. *Kerov–Olshanski–Vershik's representation of* $\mathfrak{S}_\infty$. Kerov–Olshanski–Vershik introduced an interesting generalization of the regular representation of $\mathfrak{S}_\infty$. The generalized regular representation $L^{(z)}$ depneds on complex parameter z, which reduces to the regular representation L as $z \to \infty$. Contrasted with factoriality of L, factorial decomposition of $L^{(z)}$ gives rise to an interesting probability (z-measure) on the Thoma parameter space. Taking irreducible decomposition of $L_n^{(z)} = L^{(z)}\big|_{\mathfrak{S}_n}$, we have probability $P_n^{(z)}$ on $\mathbb{Y}_n$. This ensemble would miss the character factorization property above. Nevertheless it seems to be nice to seek for limiting objects under a balanced rescale as $n, z \to \infty$.

9.2.5. *Jack ensemble.* The Jack measure on $\mathfrak{T}$ (the path space on the Young graph) is given by replacing the role of Schur functions s_λ by Jack symmetric functions $P_\lambda^{(\alpha)}$. Namely, the coefficient $\kappa^{(\alpha)}(\lambda, \mu)$ appearing in Pieri's formula for Jack symmetric functions assigns edge multiplicity to $\lambda \nearrow \mu$. This causes deformation of the notions of harmonic functions on $\mathbb{Y}$ and central probabilities on $\mathfrak{T}$. A Jack version of Theorem 9.6 is due to Fulman, which involves α-deformed character value at a 2-cycle. However, a desctiption of the whole fluctuation of the Jack ensemble which is as satisfactory as Kerov's CLT for the Plancherel one seems to be still open.

10. Characters of infinite wreath product groups I (Lecture 10)

10.1. **Factor representations and characters.** We mention some generalities mainly for terminologies.

Let G be a Hausdorff topological group. Similarly in Sect. 4, set

- $\mathcal{K}(G) = \{\text{continuous positive-definite central functions on } G\}$,
- $\mathcal{K}(G) = \{f \in \mathcal{K}(G) \,|\, f(e) = 1\}$,
- $E(G) = \{f \in \mathcal{K}_1(G) \,|\, f \text{ is extremal}\}$.

Let π be a continuous unitary representation (UR) of G and $\mathcal{A} = \pi(G)''$ the von Neumann algebra generated by $\pi(G)$. If $\mathcal{A}$ is a factor, π is called a factorial UR. Moreover if $\mathcal{A}$ is of finite type (I_n with $n < \infty$ or II_1), there exists a unique finite faithful normal normalized trace on the cone $\mathcal{A}^+$ of the nonnegative elements, linear extension of which to $\mathcal{A}$ is denoted by ϕ. Then,

$$f_\pi(g) = \phi(\pi(g)), \qquad g \in G, \tag{10.1}$$

is called the character of π.

Theorem 10.1. *There exists a bijective correspondence between $E(G)$ and the set of quasi-equivalence classes of finite factorial URs of G. $[\pi] \mapsto f$ is given by* (10.1) *while $f \mapsto [\pi]$ by the Gelfand–Raikov representation.*

Definition 10.2. Taking Theorem 10.1 into account, we refer to an element of $E(G)$ as a character of G.

10.2. **Finite and infinite wreath products of a compact group.**

Definition 10.3 (wreath product)**.** Let T be a compact group with identity element e_T. For $n \in \mathbb{N} \cup \{\infty\}$, set $D_n = D_n(T) = \{d = (t_i)_{i=1}^n \,|\, t_i \in T,\ t_i = e_T \text{ with finite exceptions}\}$. $\mathfrak{S}_n$ acts onto $D_n(T)$ from the left as usual:

$$d = (t_i)_{i=1}^n \mapsto \sigma(d) = (t_{\sigma^{-1}(i)})_{i=1}^n.$$

Under this action the semidirect product $\mathfrak{S}_n(T) = D_n(T) \rtimes \mathfrak{S}_n$ is defined. $D_n(T)$ and $\mathfrak{S}_n$ are often regarded as subgroups (the former is normal) of $\mathfrak{S}_n(T)$. Under this identification, we have $\sigma(d) = \sigma d \sigma^{-1}$. An element of $\mathfrak{S}_n(T)$ is expressed as $g = (d, \sigma) = d\sigma$. The identity element of $\mathfrak{S}_n$ is denoted by $\mathbf{1}$.

Remark 10.4. If T is an infinite set and $n = \infty$, $\mathfrak{S}_\infty(T)$ equipped with the natural inductive limit topology is a topological group which is not locally compact. Although theory of finite factorial representations and characters works well in a case of general compact group T, we mainly keep the simplest case of $T = \mathbb{Z}_2$ in mind in what follows.

We begin with an extension of cycle decomposition in $\mathfrak{S}_n$.

Definition 10.5. Let $n \in \mathbb{N} \cup \{\infty\}$. For cycle $\sigma \in \mathfrak{S}_n$, $c(\sigma)$ denotes the cardinality of its support.

(1) $g = (d, \sigma) \in \mathfrak{S}_n(T)$ is called a basic element if

(i) σ is a cycle with $c(\sigma) \geq 2$ and it holds $\mathrm{supp} d \subset \mathrm{supp} \sigma$, or

(ii) $\sigma = \mathbf{1}$ and $d = (t_i)$ with $t_i = e_T$ except one index $i = q$.

For a basic element of (i), set $c(g) = c(\sigma)$. For a basic element of (ii), we use the notation of $\xi_q = (t_q, (q))$ and call $\{q\}$ the support of ξ_q. Set $c(\xi_q) = 1$.

(2) $g \in \mathfrak{S}_n(T)$ is uniquely (up to the order) decomposed into product of basic elements with disjoint supports, which we call the standard decomposition of g:

$$g = \xi_{q_1} \cdots \xi_{q_r} g_1 \cdots g_m, \qquad \xi_{q_i} = (t_{q_i}, (q_i)),\ g_j = (d_j, \sigma_j). \tag{10.2}$$

Note that $\sigma = \sigma_1 \cdots \sigma_m$ gives cycle decomposition.

We describe the conjugacy classes of $\mathfrak{S}_n(T)$, $n \in \mathbb{N} \cup \{\infty\}$, through standard decompositions (10.2).

Lemma 10.6. (1) *Conjugation of a basic element of* $\mathfrak{S}_n(T)$ *produces again a basic element.*
(2) *Basic element* (d, σ) *can be tranferred to* (d_0, σ) *after conjugation by* d' *with* $\mathrm{supp} d' \subset \mathrm{supp}\sigma$, *where* d_0 *has a unique nontrivial entry* t_{q_0}. *The position* q_0 *can be chosen arbitrarily in* $\mathrm{supp} d$.

Proof. (1) First $\sigma\xi_q\sigma^{-1} = \sigma(t_q, (q))\sigma^{-1} = (t_q, (\sigma(q))$. For basic element (d, σ), we see that $\tau(d, \sigma)\tau^{-1} = (\tau d\tau^{-1}, \tau\sigma\tau^{-1})$ is basic. Indeed, $\tau(i_1\ \cdots\ i_p)\tau^{-1} = (\tau(i_1)\ \cdots\ \tau(i_p))$. Moreover, $d'(d, \sigma)d'^{-1} = (d'd\sigma d'^{-1}\sigma^{-1}, \sigma)$ satisfies that, if $q \notin \mathrm{supp}\sigma$, the q-entry of the D_n-part is

$$(d'd\sigma d'^{-1}\sigma^{-1})_q = (d')_q(d)_q(\sigma d'^{-1}\sigma^{-1})_q = (d')_q e_T (d'^{-1})_q = e_T.$$

(2) Given basic element (d, σ), take any $q \in \mathrm{supp}\sigma$ and set $d' = ((d)_q, (\sigma^{-1}(q)))$. For any entry index q', $(d'd\sigma d'^{-1}\sigma^{-1})_{q'} = (d')_{q'}(d)_{q'}(\sigma d'^{-1}\sigma^{-1})_{q'}$ satisfies

$$(d')_{q'} = \begin{cases} (d)_q, & q' = \sigma^{-1}(q), \\ e_T, & \text{otherwise}, \end{cases} \qquad (\sigma d'^{-1}\sigma^{-1})_{q'} = \begin{cases} (d^{-1})_q, & q' = q, \\ e_T, & \text{otherwise}. \end{cases}$$

Conjugation by this d' thus acts as

$$d = (\cdots, (d)_{\sigma^{-1}(q)}, \cdots, (d)_q, \cdots) \longmapsto (\cdots, (d)_q(d)_{\sigma^{-1}(q)}, \cdots, e_T, \cdots),$$

changing only the q- and $\sigma^{-1}(q)$-entries. Repeating such conjugations, we get the desired form in which $(d)_q(d)_{\sigma^{-1}(q)}(d)_{\sigma^{-2}(q)} \cdots$ lies at the end position of cycle $(q\ \sigma^{-1}(q)\ \sigma^{-2}(q)\ \cdots)$. This position is arbitrary according to the choice of the initial q. □

Example 10.7 ($T = \mathbb{Z}_2 = \{1, -1\}$)**.** A basic element $(d, \sigma) \in \mathfrak{S}_n(\mathbb{Z}_2)$ is called a positive [resp. negative] cycle, or p-cycle [resp. n-cycle] for short, if the number of (-1)'s in $d \in D_\infty$ is even [resp. odd]. A basic element $\xi_q = (t_q, (q))$, $t_q = -1$, is regarded as a negative (or n-) cycle of length 1. Then, Lemma 10.6 combined with standard decomposition tells us that a conjugacy class of $\mathfrak{S}_n(\mathbb{Z}_2)$ is characterized by the type of p/n-cycle decomposition. Putting together positive and negative cycles separately, we have a pair of Young diagrams, one consisting of p-cycle rows and the other consisting of n-cycle rows. If $n < \infty$, the sum of two sizes is n. On the other hand, the conjugacy classes of $\mathfrak{S}_\infty(\mathbb{Z}_2)$ are parametrized by

$$\{(\rho_+, \rho_-) \mid \rho_+ \in \overline{\mathbb{Y}}, \rho_- \in \mathbb{Y}\},$$

infinitely many 1-p-cycles being removed.

Lemma 10.6 enables us to get parametrization of the conjugacy classes of general $\mathfrak{S}_n(T)$ also (for $n \in \mathbb{N} \cup \{\infty\}$). In fact, we have only to extend p and n for $\mathfrak{S}_n(\mathbb{Z}_2)$ to $|T/\sim|$ colours, where $T/\sim$ denotes the conjugacy classes of T. The colour of basic element (d, σ), where $d = (t_i)$ and $\sigma = (i_1\ \cdots\ i_k)$, is given by the conjugacy class of T which contains $t_{i_k} \cdots t_{i_1}$. See the conjugation in (2) of the proof of Lemma 10.6.

The colour of basic element ξ_q is obviously given by the conjugacy class containing the unique nontrivial element t_q.

10.3. **IURs of $\mathfrak{S}_n(T)$ ($n < \infty$).** Recall the definition of an induced representation. Let H be a subgroup of G and π a UR of H on $V(\pi)$. Assume for simplicity $[G : H] < \infty$ and $\dim V(\pi) < \infty$. The induced representation $\Pi = \mathrm{Ind}_H^G \pi$ on $V(\Pi)$ is a UR defined as follows. Set

$$V(\Pi) = \big\{\varphi : \text{ continuous } V(\pi)\text{-valued function on } G \,\big|\, \varphi(hx) = \pi(x)(\varphi(x)),\ h \in H,\ x \in G\big\},$$
$$\|\varphi\|^2_{V(\Pi)} = \frac{1}{[G : H]} \sum_{\bar{x} \in H\backslash G} \|\varphi(x)\|^2_{V(\pi)},$$

and $(\Pi(g)\varphi)(x) = \varphi(xg)$ for $g, x \in G$. Note that $\dim V(\Pi) = [G : H] \dim V(\pi)$.

All IURs of finite wreath product $\mathfrak{S}_n(T)$ are constructed by using induced representations. Here we mention the procedure only for the simplest case of $T = \mathbb{Z}_2$. Let $G_n = \mathfrak{S}_n(\mathbb{Z}_2) = D_n \rtimes \mathfrak{S}_n$ where $D_n = (\mathbb{Z}_2)^n$. We express as $\widehat{\mathbb{Z}_2} = \{\zeta_1, \zeta_{-1}\}$ by setting $\zeta_1(\pm 1) = 1$ and $\zeta_{-1}(\pm 1) = \pm 1$. $\mathfrak{S}_n$ acts on $\widehat{D_n}$ through the action on D_n, i.e.

$$\eta \in \widehat{D_n} \mapsto {}^\sigma\eta \in \widehat{D_n}, \quad {}^\sigma\eta(d) = \eta(\sigma^{-1}(d)) = \eta(\sigma^{-1} d \sigma).$$

Set $S_\eta = \{\sigma \in \mathfrak{S}_n \,|\, {}^\sigma\eta = \eta\}$. Each $\eta \in \widehat{D_n}$ determines decomposition $\{1, 2, \ldots, n\} = I_{n,1} \sqcup I_{n,-1}$ according as the kth entry of η is $\zeta_{\pm 1}$ if $k \in I_{n,\pm 1}$. For example, $I_{n,-1} = \varnothing$ for the trivial character η. We may write as $\eta = \zeta_1^{I_{n,1}} \zeta_{-1}^{I_{n,-1}}$. Then, $S_\eta = \mathfrak{S}_{I_{n,1}} \times \mathfrak{S}_{I_{n,-1}}$ where $\mathfrak{S}_{I_{n,\pm 1}}$ is canonically embedded in $\mathfrak{S}_n$. Consider the subgroup of G_n defined by $H_n = D_n \rtimes S_\eta = \mathfrak{S}_{I_{n,1}}(\mathbb{Z}_2) \times \mathfrak{S}_{I_{n,-1}}(\mathbb{Z}_2)$. η determines a character (one-dimensional UR) of H_n by

$$\eta(d, \sigma) = \eta(d), \qquad d \in D_n,\ \sigma \in S_\eta.$$

Indeed, the definition of S_η yields $\eta(d\sigma d'\sigma^{-1}) = \eta(d)({}^{\sigma^{-1}}\eta)(d') = \eta(d)\eta(d')$. IUR ξ of S_η on $V(\xi)$ also determines an IUR of H_n by

$$\xi(d, \sigma) = \xi(\sigma), \qquad d \in D_n,\ \sigma \in S_\eta.$$

Then, $\pi = \eta \otimes \xi$ is an IUR of H_n on $\mathbb{C} \otimes V(\xi) = V(\xi)$. We set

$$\Pi = \Pi^{\eta,\xi} = \mathrm{Ind}_{H_n}^{G_n} \pi, \qquad \eta \in \widehat{D_n},\ [\xi] \in \widehat{S_\eta}.$$

$\mathfrak{S}_n$ acting on $\widehat{D_n}$ as before, we have $S_{{}^\sigma\eta} = \sigma S_\eta \sigma^{-1}$ for $\eta \in \widehat{D_n}$ and $\sigma \in \mathfrak{S}_n$. Then, the equation

$$({}^\sigma\xi)(\tau) = \xi(\sigma^{-1}\tau\sigma), \qquad \sigma \in \mathfrak{S}_n,\ \tau \in S_{{}^\sigma\eta},$$

defines an IUR ${}^\sigma\xi$ of $S_{{}^\sigma\eta}$. Thus $\mathfrak{S}_n$ acts on the bundle $\{(\eta, [\xi]) \,|\, \eta \in \widehat{D_n},\ [\xi] \in \widehat{S_\eta}\} \simeq \bigsqcup_{\eta \in \widehat{D_n}} \widehat{S_\eta}$.

Theorem 10.8. (1) *$\Pi^{\eta,\xi}$ is an IUR of G_n.*
(2) *$\Pi^{\eta,\xi} \cong \Pi^{\eta',\xi'}$ if and only if $(\eta, [\xi])$ and $(\eta', [\xi'])$ belong to the same $\mathfrak{S}_n$-orbit i.e. there exists $\sigma \in \mathfrak{S}_n$ such that $\eta' = {}^\sigma\eta$, $\xi' \cong {}^\sigma\xi$.*
(3) *For any IUR Π of G_n, there exist $\eta \in \widehat{D_n}$ and $[\xi] \in \widehat{S_\eta}$ such that $\Pi \cong \Pi^{\eta,\xi}$. Hence $\widehat{G_n} \simeq \mathfrak{S}_n \backslash \{(\eta, [\xi]) \,|\, \eta \in \widehat{D_n},\ [\xi] \in \widehat{S_\eta}\} \simeq \{(\lambda, \mu) \,|\, \lambda, \mu \in \mathbb{Y},\ |\lambda| + |\mu| = n\}$.*

Proof. After proving (1) – (3), the last part immediately follows from that a complete system of representatives of $\mathfrak{S}_n\backslash\widehat{D_n}$ is given by $\{\zeta_1^{\{1,\dots,m\}}\zeta_{-1}^{\{m+1,\dots,n\}} \,|\, m \in \{1,\dots,n\}\}$ and that the IURs of $\mathfrak{S}_{\{1,\dots,m\}} \times \mathfrak{S}_{\{m+1,\dots,n\}}$ are covered by $\{\lambda \boxtimes \mu \,|\, \lambda \in \mathbb{Y}_m,\ \mu \in \mathbb{Y}_{n-m}\}$.

(1) We show that the dimension of the intertwiners for Π is equal to 1. $L \in \mathcal{L}(V(\Pi))$, a linear operator on $V(\Pi)$, is expressed as an integral operator

$$(L\varphi)(g) = \int_{G_n} K(g,g')\mu_{G_n}(dg'), \qquad \varphi \in V(\Pi),$$

with kernel $K : G_n \times G_n \longrightarrow \mathcal{L}(V(\pi))$. (In our case of $T = \mathbb{Z}_2$, a finite group, this is just a matrix representation.) K satisfies the following properties: first

$$K(hg, h'g') = \pi(h)K(g,g')\pi(h')^{-1}, \qquad g, g' \in G_n,\ h, h' \in H_n, \tag{10.3}$$

which comes from the invariance property of elements of $V(\Pi)$, and secondly

$$K(gg'', g'g'') = K(g,g'), \qquad g, g', g'' \in G_n, \tag{10.4}$$

which results from the intertwining property of $\Pi(g'')L = L\Pi(g'')$. Set

$$k(g) = K(g,e), \qquad g \in G_n \quad (e : \text{identity of } G_n).$$

Then, K is recovered by $K(g,g') = k(gg'^{-1})$ and it holds that

$$k(hgh') = \pi(h)k(g)\pi(h'), \qquad g \in G_n,\ h, h' \in H_n. \tag{10.5}$$

We thus look at k on the double cosets $H_n\backslash G_n/H_n \simeq S_\eta\backslash\mathfrak{S}_n/S_\eta$. For a representative $\tau \in \mathfrak{S}_n$ of a double coset, we have

$$\pi(h)k(\tau) = k(\tau)\pi(\tau^{-1}h\tau), \qquad h \in H_n \cap \tau H_n \tau^{-1}. \tag{10.6}$$

• If $[\tau] \neq H_n$, ${}^\tau\eta \neq \eta$ on D_n. However, $D_n \subset H_n \cap \tau H_n \tau^{-1}$ and (10.6) imply $k(\tau) = 0$.
• If $[\tau] = H_n$ i.e. $\tau = \mathbf{1}$, (10.6) with irreducibility of π implies that $k(\mathbf{1})$ is scalar. The choice of k is thus one-dimensional. Hence so are K and L.

(2) If $\eta' = {}^\sigma\eta$ and $\xi' \cong {}^\sigma\xi$ for $\sigma \in \mathfrak{S}_n$, then $\Pi^{\eta,\xi} \cong \Pi^{{}^\sigma\eta,{}^\sigma\xi} \cong \Pi^{\eta',\xi'}$. To show the converse, let $\{x_0, \cdots, x_{m-1}\}$ be a complete system of representatives of $H_n\backslash G_n \simeq S_\eta\backslash\mathfrak{S}_n$ where $x_0 = \mathbf{1}, x_1, \cdots, x_{m-1} \in \mathfrak{S}_n$, $m = [\mathfrak{S}_n : S_\eta]$. We have

$$\Pi^{\eta,\xi}\big|_{D_n} \cong \bigoplus_{j=0}^{m-1} [\dim V(\xi)]({}^{x_j^{-1}}\eta). \tag{10.7}$$

In fact, taking an ONB $\{v_1, \cdots, v_l\}$ of $V(\xi) = V(\pi)$, set

$$\varphi_{ij}(x) = \begin{cases} \pi(h)v_i, & x \in Hx_j,\ x = hx_j, \\ 0, & x \notin Hx_j, \end{cases} \qquad i = 1,\dots,l,\ j = 0,\dots,m-1.$$

Then we have $\Pi(d)\varphi_{ij} = ({}^{x_j^{-1}}\eta)(d)\varphi_{ij}$, which implies (10.7). Also

$$\Pi^{\eta',\xi'}\big|_{D_n} \cong \bigoplus_{j=0}^{m'-1} [\dim V(\xi')]({}^{x_j'^{-1}}\eta'). \tag{10.8}$$

Assume $\Pi^{\eta,\xi} \cong \Pi^{\eta',\xi'}$. Restricting them onto D_n and comparing (10.7) and (10.8), we see $\eta' = {}^{x_0^{-1}}\eta' = {}^{x_j^{-1}}\eta$ for some j, i.e. there exists $\sigma \in \mathfrak{S}_n$ such that $\eta' = {}^\sigma\eta$. Then, $\Pi^{\eta,\xi} \cong \Pi^{{}^\sigma\eta,\xi'} \cong \Pi^{\eta,{}^{\sigma^{-1}}\xi'}$. We will see $\xi \cong {}^{\sigma^{-1}}\xi'$. In fact, (10.7) yields that, in irreducible decomposition $\Pi^{\eta,\xi}\big|_{H_n} \cong \eta \otimes \xi \oplus (*)$, $(*)\big|_{D_n}$ does not contain η any more. Hence $(*)$ does not contain $\eta\otimes\xi$. Similarly for $\Pi^{\eta,{}^{\sigma^{-1}}\xi'}\big|_{H_n} \cong \eta\otimes{}^{\sigma^{-1}}\xi'\oplus(**)$. Since $\Pi^{\eta,{}^{\sigma^{-1}}\xi'}\big|_{H_n} \cong \Pi^{\eta,\xi}\big|_{H_n}$, it necessarily holds that $\eta\otimes\xi \cong_{H_n} \eta\otimes{}^{\sigma^{-1}}\xi'$. Both sides being restricted onto S_η, we have $\xi \cong_{S_\eta} {}^{\sigma^{-1}}\xi'$.

(3) In the case of finite T, especially for $T = \mathbb{Z}_2$, we know the number of the conjugacy classes of $\mathfrak{S}_n(\mathbb{Z}_2)$ in Sect. 10.2 (Example 10.7). We get the same number of non-equivalent IURs of $\mathfrak{S}_n(\mathbb{Z}_2)$ in (2). This completes the proof. □

11. Characters of infinite wreath product groups II (Lecture 11)

11.1. **Character formula for $\mathfrak{S}_n(\mathbb{Z}_2)$.** We establish a formula for irreducible characters of $G_n = \mathfrak{S}_n(\mathbb{Z}_2) = D_n \rtimes \mathfrak{S}_n$ where $D_n = D_n(\mathbb{Z}_2) = (\mathbb{Z}_2)^n$.

The conjugacy classes of $\mathfrak{S}_n(\mathbb{Z}_2)$ are described by p/n (coloured)-cycle decomposition. They are parametrized by

$$\{((\rho, 1^{n-|\rho|-|\sigma|}), \sigma) \mid \rho \in \overline{\mathbb{Y}}, \sigma \in \mathbb{Y}, |\rho| + |\sigma| \leq n\},$$

where ρ and $(1^{n-|\rho|-|\sigma|})$ indicate a nontrivial and trivial p-cycle respectively while σ indicates an n-cycle including 1-n-cycle like $(-1, (q))$.

The equivalence classes of IURs of $\mathfrak{S}_n(\mathbb{Z}_2)$ are parametrized by

$$\mathbb{Y}_n(\mathbb{Z}_2) = \{(\lambda, \mu) \mid \lambda, \mu \in \mathbb{Y}, |\lambda| + |\mu| = n\}.$$

A corresponding IUR to $(\lambda, \mu) \in \mathbb{Y}_n(\mathbb{Z}_2)$ is constructed as follows. Set $|\lambda| = m$, $|\mu| = n - m$, and let $\widehat{\mathbb{Z}_2} = \{\zeta_1, \zeta_{-1}\}$. Take $\eta = \zeta_1^{\{1,\dots,m\}} \zeta_{-1}^{\{m+1,\dots,n\}} \in \widehat{D_n}$ and IUR $\xi = \lambda \boxtimes \mu$ of $S_\eta = \{\sigma \in \mathfrak{S}_n \mid {}^\sigma\eta = \eta\} = \mathfrak{S}_{\{1,\dots,m\}} \times \mathfrak{S}_{\{m+n,\dots,m\}}$. Extending η and ξ trivially to $H_n = D_n \rtimes S_\eta$, set $\pi = \eta \otimes \xi$, which is an IUR of H_n. The desired IUR of $G_n = \mathfrak{S}_n(\mathbb{Z}_2)$ is given by $\Pi^{(\lambda,\mu)} = \mathrm{Ind}_{H_n}^{G_n} \pi$.

Recall the induced character formula. It ensures that normalized character $\tilde{\chi}_\Pi$ is a centralization of the trivial extension of normalized character $\tilde{\chi}_\pi$:

$$\tilde{\chi}_\Pi(g) = \int_{G_n} \tilde{\chi}_\pi(g' g g'^{-1}) \mu_{G_n}(dg')$$

where the integrand is by definition 0 outside of H_n. In particular, $\tilde{\chi}_\pi(g) = 0$ if g is not conjugate to an element of H_n. In our case of $\Pi = \Pi^{(\lambda,\mu)}$, $(\lambda, \mu) \in \mathbb{Y}_n(\mathbb{Z}_2)$, we have

$$\tilde{\chi}^{(\lambda,\mu)}_{((\rho,1^{n-|\rho|-|\sigma|}),\sigma)} = \sum_{g' \in \mathfrak{S}_n(\mathbb{Z}_2)} \frac{1}{2^n n!} \eta(g' g g'^{-1})\, \tilde{\chi}^{\lambda\boxtimes\mu}(g' g g'^{-1}), \tag{11.1}$$

where η and $\lambda \boxtimes \mu$ are regarded as representations of H_n.

Let us consider a standard (or p/n-cycle) decomposition of g as

$$g = \xi_{q_1} \cdots \xi_{q_r} (d_1 s_1) \cdots (d_p s_p) = \xi_{q_1} \cdots \xi_{q_r} d_1 \cdots d_p s_1 \cdots s_p$$

where s_j is a nontrivial cycle. Conjugation of g by $g' = (d', s')$ gives

$$\begin{aligned} g' g g'^{-1} = {} & d' s' \xi_{q_1} \cdots \xi_{q_r} s'^{-1} s' d_1 \cdots d_p s'^{-1} \\ & \times\ s' s_1 \cdots s_p s'^{-1} d'^{-1} (s' s_1 \cdots s_p s'^{-1})^{-1} s' s_1 \cdots s_p s'^{-1}. \end{aligned} \tag{11.2}$$

Hence $g' g g'^{-1} \in H_n$ if and only if $s' s_1 \cdots s_p s'^{-1} \in \mathfrak{S}_{\{1,\dots,m\}} \times \mathfrak{S}_{\{m+1,\cdots,n\}}$. Conjugation by g' thus induces re-enumeration of the letters in $s_1, \cdots, s_p$ satisfying this condition. Putting $+$ and $-$ to the cycles filled with $\{1, \dots, m\}$ and $\{m+1, \dots, n\}$ respectively, we consider $\rho = \rho^+ \sqcup \rho^-$ and $\overline{\sigma} = \overline{\sigma}^+ \sqcup \overline{\sigma}^-$ where $\overline{\sigma} = \sigma \setminus (1^{m_1(\sigma)})$. Applying η to the D_n-component of (11.2), we consider re-enumeration of $q_1, \cdots, q_r$ also, either from $\{1, \dots, m\}$ or from $\{m+1, \dots, n\}$. This gives rise to the decomposition $(1^{m_1(\sigma)}) = (1^{m_1^+}) \sqcup (1^{m_1^-})$. Altogether we have decomposition $\rho = \rho^+ \sqcup \rho^-$ and $\sigma = \sigma^+ \sqcup \sigma^-$ independently. The D_n-component of (11.2) is mapped by η to

$$\eta(d')(-1)^{m_1^-}(-1)^{r(\overline{\sigma}^-)} \eta(d'^{-1}) = (-1)^{r(\sigma^-)}.$$

The remaining part is to count all possibilities of the re-enumeration of the letters. We then get the following.

Theorem 11.1. *For* $(\lambda,\mu)\in\mathbb{Y}_n(\mathbb{Z}_2)$ *and* $\rho\in\overline{\mathbb{Y}}$, $\sigma\in\mathbb{Y}$ *such that* $|\rho|+|\sigma|\le n$, *it holds*

$$\tilde{\chi}^{(\lambda,\mu)}_{((\rho,1^{n-|\rho|-|\sigma|}),\sigma)} = \sum_{\rho^+,\rho^-,\sigma^+,\sigma^-:\ \rho=\rho^+\sqcup\rho^-,\sigma=\sigma^+\sqcup\sigma^-} \frac{|\lambda|(|\lambda|-1)\cdots(|\lambda|-(|\rho^+|+|\sigma^+|)+1)\cdot|\mu|(|\mu|-1)\cdots(|\mu|-(|\rho^-|+|\sigma^-|)+1)}{n(n-1)\cdots(n-(|\rho|+|\sigma|)+1)} \times(-1)^{r(\sigma^-)}\tilde{\chi}^{\lambda}_{(\rho^+,\sigma^+,1^{|\lambda|-(|\rho^+|+|\sigma^+|)})}\tilde{\chi}^{\mu}_{(\rho^-,\sigma^-,1^{|\mu|-(|\rho^-|+|\sigma^-|)})}.$$

11.2. **Branching graph for $\mathfrak{S}_n(\mathbb{Z}_2)$.** For $(\lambda,\mu)\in\mathbb{Y}_n(\mathbb{Z}_2)$ and $(\nu,\theta)\in\mathbb{Y}_{n-1}(\mathbb{Z}_2)$, we set $(\nu,\theta)\nearrow(\lambda,\mu)$ if either $\nu\nearrow\lambda$, $\theta=\mu$ or $\nu=\lambda$, $\theta\nearrow\mu$ holds.

Theorem 11.2.

$$\Pi^{(\lambda,\mu)}\big|_{\mathfrak{S}_{n-1}(\mathbb{Z}_2)} \cong \bigoplus_{(\nu,\theta)\in\mathbb{Y}_{n-1}(\mathbb{Z}_2):\ (\nu,\theta)\nearrow(\lambda,\mu)} \Pi^{(\nu,\theta)}.$$

Proof. It might be desirable to give a proof by way of structure of representations (bases of representation spaces). Here we just verify

$$\chi^{(\lambda,\mu)}\big|_{\mathfrak{S}_{n-1}(\mathbb{Z}_2)} = \sum_{(\nu,\theta)\nearrow(\lambda,\mu)} \chi^{(\nu,\theta)}. \tag{11.3}$$

Theorem 11.1 (in non-normalized forms) gives

$$\chi^{(\lambda,\mu)}_{((\rho,1^{n-|\rho|-|\sigma|}),\sigma)} = \sum_{\rho=\rho^+\sqcup\rho^-,\sigma=\sigma^+\sqcup\sigma^-}\frac{(n-(|\rho|+|\sigma|))!}{(|\lambda|-(|\rho^+|+|\sigma^+|))!\cdot(|\mu|-(|\rho^-|+|\sigma^-|))!} \times(-1)^{r(\sigma^-)}\chi^{\lambda}_{(\rho^+,\sigma^+,1^{|\lambda|-(|\rho^+|+|\sigma^+|)})}\chi^{\mu}_{(\rho^-,\sigma^-,1^{|\mu|-(|\rho^-|+|\sigma^-|)})}.$$

for $|\rho|+|\sigma|\le n-1$. Dividing the binomial term in the second line, we have on one hand

$$\sum_{\rho=\rho^+\sqcup\rho^-,\sigma=\sigma^+\sqcup\sigma^-}\frac{(n-1-(|\rho|+|\sigma|))!}{(|\lambda|-1-(|\rho^+|+|\sigma^+|))!\cdot(|\mu|-(|\rho^-|+|\sigma^-|))!} \times(-1)^{r(\sigma^-)}\sum_{\nu\nearrow\lambda}\chi^{\nu}_{(\rho^+,\sigma^+,1^{|\nu|-(|\rho^+|+|\sigma^+|)})}\chi^{\mu}_{(\rho^-,\sigma^-,1^{|\mu|-(|\rho^-|+|\sigma^-|)})} = \sum_{\nu\nearrow\lambda}\chi^{(\nu,\mu)}_{((\rho,1^{n-1-(|\rho|+|\sigma|)}),\sigma)}$$

and a similar equation on the other hand. Hence

$$\chi^{(\lambda,\mu)}_{((\rho,1^{n-(|\rho|+|\sigma|)}),\sigma)} = \sum_{\nu\nearrow\lambda}\chi^{(\nu,\mu)}_{((\rho,1^{n-1-(|\rho|+|\sigma|)}),\sigma)} + \sum_{\theta\nearrow\mu}\chi^{(\lambda,\theta)}_{((\rho,1^{n-1-(|\rho|+|\sigma|)}),\sigma)},$$

i.e. (11.3). ∎

11.3. **Characters of $\mathfrak{S}_\infty(\mathbb{Z}_2)$ I.** We write down the irreducible character values at basic elements for $\mathfrak{S}_n(\mathbb{Z}_2)$.

▶ For 1-n-cycle: $\rho = \varnothing$, $\sigma = (1)$,

$$\tilde{\chi}^{(\lambda,\mu)}_{((1^{n-1}),(1))} = \frac{|\lambda| - |\mu|}{n}. \tag{11.4}$$

▶ For k-p-cycle: $\rho = (k)$, $\sigma = \varnothing$, $k \geq 2$,

$$\tilde{\chi}^{(\lambda,\mu)}_{((k,1^{n-k}),\varnothing)} = \frac{|\lambda|(|\lambda|-1)\cdots(|\lambda|-k+1)}{n(n-1)\cdots(n-k+1)}\tilde{\chi}^{\lambda}_{(k,1^{|\lambda|-k})} + \frac{|\mu|(|\mu|-1)\cdots(|\mu|-k+1)}{n(n-1)\cdots(n-k+1)}\tilde{\chi}^{\mu}_{(k,1^{|\mu|-k})}. \tag{11.5}$$

▶ For k-n-cycle: $\rho = \varnothing$, $\sigma = (k)$, $k \geq 2$,

$$\tilde{\chi}^{(\lambda,\mu)}_{((1^{n-k}),(k))} = \frac{|\lambda|(|\lambda|-1)\cdots(|\lambda|-k+1)}{n(n-1)\cdots(n-k+1)}\tilde{\chi}^{\lambda}_{(k,1^{|\lambda|-k})} - \frac{|\mu|(|\mu|-1)\cdots(|\mu|-k+1)}{n(n-1)\cdots(n-k+1)}\tilde{\chi}^{\mu}_{(k,1^{|\mu|-k})}. \tag{11.6}$$

We introduce the Vershik–Kerov condition for growing Young diagrams $(\lambda,\mu) \in \mathbb{Y}_n(\mathbb{Z}_2)$:

$$\lim_{n\to\infty}\frac{|\lambda|}{n} = c_+, \qquad \lim_{n\to\infty}\frac{|\mu|}{n} = c_-, \tag{11.7}$$

$$\lim_{n\to\infty}\frac{\lambda_i}{n} = \alpha_i,\ \lim_{n\to\infty}\frac{\lambda'_i}{n} = \beta_i,\ \lim_{n\to\infty}\frac{\mu_i}{n} = \alpha_{-i},\ \lim_{n\to\infty}\frac{\mu'_i}{n} = \beta_{-i}, \quad i \in \mathbb{N}. \tag{11.8}$$

Note that $c_+ + c_- = 1$ and $\sum_{i=1}^\infty(\alpha_{\pm i} + \beta_{\pm i}) \leq c_\pm$. Set

$$\gamma_\pm = c_\pm - \sum_{i=1}^\infty(\alpha_{\pm i} + \beta_{\pm i}). \tag{11.9}$$

We then have $\sum_{i=1}^\infty(\alpha_i + \beta_i + \alpha_{-i} + \beta_{-i}) + \gamma_+ + \gamma_- = 1$. Since irreducible character values at cycles for $\mathfrak{S}_n$ enjoy an asymptotic formula

$$\tilde{\chi}^{\lambda}_{(k,1^{n-k})} = \sum_{i=1}^\infty\Big\{\Big(\frac{a_i(\lambda)}{n}\Big)^k + (-1)^{k-1}\Big(\frac{b_i(\lambda)}{n}\Big)^k\Big\} + O\big(1/n\big), \qquad n = |\lambda| \to \infty, \tag{11.10}$$

the limits of (11.4) – (11.6) under the Vershik–Kerov condition yield

$$\lim_{n\to\infty}\tilde{\chi}^{(\lambda,\mu)}_{((1^{n-1}),(1))} = \sum_{i=1}^\infty(\alpha_i + \beta_i) + \gamma_+ - \sum_{i=1}^\infty(\alpha_{-i} + \beta_{-i}) - \gamma_-, \tag{11.11}$$

$$\lim_{n\to\infty}\tilde{\chi}^{(\lambda,\mu)}_{((k,1^{n-k}),\varnothing)} = \sum_{i=1}^\infty(\alpha_i^k + (-1)^{k-1}\beta_i^k) + \sum_{i=1}^\infty(\alpha_{-i}^k + (-1)^{k-1}\beta_{-i}^k), \tag{11.12}$$

$$\lim_{n\to\infty}\tilde{\chi}^{(\lambda,\mu)}_{((1^{n-k}),(k))} = \sum_{i=1}^\infty(\alpha_i^k + (-1)^{k-1}\beta_i^k) - \sum_{i=1}^\infty(\alpha_{-i}^k + (-1)^{k-1}\beta_{-i}^k), \tag{11.13}$$

for $k \geq 2$.

These give us characters of $\mathfrak{S}_\infty(\mathbb{Z}_2)$ depending on the parameters $(\alpha_{\pm i})_{i\in\mathbb{N}}$, $(\beta_{\pm i})_{i\in\mathbb{N}}$ and $\gamma_\pm$ if we notice the factorizability of a character and structure of the conjugacy classes. Actually, these cover the whole set of characters of $\mathfrak{S}_\infty(\mathbb{Z}_2)$.

Together with a closer look at the convergence in the Vershik–Kerov regions, we show this fact in the next section.

12. Characters of infinite wreath product groups III (Lecture 12)

12.1. Harmonic functions and central probabilities on a branching graph.

We recall and extend some notions in Sect. 4.

Definition 12.1. A branching graph has by definition a stratified vertex set $\mathbb{G} = \bigsqcup_{n=0}^{\infty} \mathbb{G}_n$, $\mathbb{G}_0 = \{\varnothing\}$. An edge is denoted by $\alpha \nearrow \beta$ where $\alpha \in \mathbb{G}_n$ and $\beta \in \mathbb{G}_{n+1}$ lie in consecutive strata. The following conditions are postulated.

(1) For any $\beta \in \mathbb{G} \setminus \mathbb{G}_0$, it holds $0 < \big|\{\alpha \in \mathbb{G} \,|\, \alpha \nearrow \beta\}\big| < \infty$.
(2) For any $\alpha \in \mathbb{G}$, it holds $0 < \big|\{\beta \in \mathbb{G} \,|\, \alpha \nearrow \beta\}\big| \le \infty$.
(3) Edge multiplicities are given by $\kappa(\alpha, \beta) > 0$ for $\alpha \nearrow \beta$.

Namely, every vertex (except the root $\varnothing$) has finitely many ingoing edges and possibly infinitely many outgoing edges. For a notational convenience, we set $\kappa(\alpha, \beta) = 0$ if α and β are in consecutive strata but not adjacent.

Example 12.2. Let $\mathbb{G}_n = \widehat{\mathfrak{S}_n(T)} = \mathbb{Y}_n(T)$ and $\kappa(\Lambda, \mathrm{M}) = \dim \zeta_{\Lambda,\mathrm{M}}$. Here $\zeta_{\Lambda,\mathrm{M}}$ denotes the unique element in $\widehat{T}$ determined by the adjacency $\Lambda \nearrow \mathrm{M}$. If T is a continuous compact group, outgoing edges of a vertex are infinitely many (and can be of an arbitrary cardinality).

In what follows, we assume that $\mathbb{G}$ is countable for simplicity though this restriction can be removed by appropriate modifications ([14]).

Let $\mathfrak{T} = \mathfrak{T}(\mathbb{G})$ denote the set of paths $t = (t(0) \nearrow t(1) \nearrow \cdots \nearrow t(n) \nearrow \cdots)$, where $t(n) \in \mathbb{G}_n$, on branching graph $\mathbb{G}$. For finite path $u = (\alpha \nearrow \cdots \nearrow \beta)$, $\alpha \in \mathbb{G}_m$, $\beta \in \mathbb{G}_n$, set $w_u = \prod_{i=m}^{n-1} \kappa(u(i), u(i+1))$ and call it the weight of u. Their sum

$$d(\alpha, \beta) = \sum_{u=(\alpha \nearrow \cdots \nearrow \beta)} w_u$$

is called a combinatorial dimension function of $\mathbb{G}$. Furthermore, let $\mathfrak{T}(\alpha)$ be the set of paths from $\varnothing$ to α for any $\alpha \in \mathbb{G}$ and $\mathfrak{S}_{\mathfrak{T}(\alpha)}$ the set of permutations on $\mathfrak{T}(\alpha)$. Note that $\mathfrak{T}(\alpha)$ is a finite set by Definition 12.1. Cylindrical subset C_u of $\mathfrak{T}$ is assigned to $u \in \mathfrak{T}(\alpha)$. Each element $g \in \mathfrak{S}_{\mathfrak{T}(\alpha)}$ is considered to act on $\mathfrak{T}$ through

$$g(t) = \begin{cases} (g(t_n) \nearrow t(n+1) \nearrow \cdots), & \text{if } t(n) = \alpha \text{ for some } n, \\ t, & \text{otherwise.} \end{cases}$$

Here $t_n \in \mathfrak{T}(\alpha)$ denotes the truncated finite path $(t(0) \nearrow t(1) \nearrow \cdots \nearrow t(n) = \alpha)$.

Definition 12.3. (1) $\mathbb{C}$-valued function φ on $\mathbb{G}$ is said to be harmonic if it satisfies

$$\varphi(\alpha) = \sum_{\beta:\, \alpha \nearrow \beta} \kappa(\alpha, \beta) \varphi(\beta), \qquad \alpha \in \mathbb{G}.$$

The set of nonnegative normalized ($\varphi(\varnothing) = 1$) harmonic functions on $\mathbb{G}$ is denoted by $\mathcal{H}(\mathbb{G})$.

(2) Probability M on $\mathfrak{T}$ is said to be central if it satisfies such a quasi-invariance with respect to $\bigcup_{\alpha \in \mathbb{G}} \mathfrak{S}_{\mathfrak{T}(\alpha)}$ that

$$M(g^{-1}B) = \int_B \frac{w_{g^{-1}(t_n)}}{w_{t_n}} M(dt), \qquad B \in \mathfrak{B}(\mathfrak{T}),$$

for any $g \in \mathfrak{S}_{\mathfrak{T}(\alpha)}$, $\alpha \in \mathbb{G}_n$, $n \in \mathbb{N}$. The set of central probabilities on $\mathfrak{T}$ is denoted by $\mathcal{M}_1(\mathfrak{T})$.

Proposition 12.4. *There exists a bijection between* $\mathcal{H}_1(\mathbb{G})$ *and* $\mathcal{M}_1(\mathfrak{T})$ *through*

$$\varphi(\alpha) = \frac{M(C_u)}{w_u}, \qquad u \in \mathfrak{T}(\alpha),\ \alpha \in \mathbb{G}.$$

Central probability M is often said to be ergodic if it is extremal in $\mathcal{M}_1(\mathfrak{T})$.

12.2. **Convergence of the Martin kernels.** We call

$$K(\alpha,\beta) = \frac{d(\alpha,\beta)}{d(\varnothing,\beta)}, \qquad \alpha,\beta \in \mathbb{G},$$

a Martin kernel on $\mathbb{G}$.

Theorem 12.5. *Let* $M \in \mathcal{M}_1(\mathfrak{T})$ *be ergodic and* φ *the corresponding extremal element of* $\mathcal{H}_1(\mathbb{G})$*. Then, for* M*-a.s.* $t \in \mathfrak{T}$*, we have*

$$\varphi(\alpha) = \lim_{n\to\infty} K(\alpha, t(n)), \qquad \alpha \in \mathbb{G}.$$

Proof. Let $X_n : \mathfrak{T} \longrightarrow \mathbb{G}_n$ be the projection such that $X_n(t) = t(n)$ and $\mathfrak{B}_n$ the σ-subfield of $\mathfrak{B}$ generated by $X_n, X_{n+1}, \cdots$. Take an arbitrary $\alpha \in \mathbb{G}$. We see that $(K(\alpha, X_n))_n$ is a backwards $(\mathfrak{B}_n)$-martingale satisfying

$$E^M[K(\alpha,X_n) \,|\, \mathfrak{B}_{n+1}] = K(\alpha, X_{n+1}), \qquad M\text{-a.s.},$$
$$E^M[K(\alpha,X_n)] = \varphi(\alpha).$$

The backwards martingale convergence theorem tells that there exists

$$\lim_{n\to\infty} K(\alpha, X_n) = Z_\infty^\alpha, \qquad M\text{-a.s. and in } L^1(M).$$

Z_∞^α is measurable with respect to $\mathfrak{B}_\infty = \bigcap_{n=1}^\infty \mathfrak{B}_n$. Since M is ergodic and hence tail-trivial, Z_∞^α is constant M-a.s. where the constant is equal to

$$E^M[Z_\infty^\alpha] = \lim_{n\to\infty} E^M[K(\alpha,X_n)] = \varphi(\alpha).$$

We take an exceptional set independently of countable α's. □

12.3. **Characters of $\mathfrak{S}_\infty(\mathbb{Z}_2)$ II.** We specialize the arguments in the previous two subsections to the case of a countable branching graph $\mathbb{G} = \mathbb{Y}(\mathbb{Z}_2)$. Set $\mathfrak{T} = \mathfrak{T}(\mathbb{Z}_2) = \mathfrak{T}(\mathbb{Y}(\mathbb{Z}_2))$. Each vertex $\alpha \in \mathbb{Y}(\mathbb{Z}_2)$ is a pair of Young diagrams. We express it as $\alpha = (\alpha^{\zeta_1}, \alpha^{\zeta_{-1}})$ or (α^+, α^-) for short. The root vertex is $\varnothing = (\varnothing^+, \varnothing^-)$.

Proposition 12.6. *There exists a bijection between* $\mathcal{K}_1(\mathfrak{S}_\infty(\mathbb{Z}_2))$ *and* $\mathcal{H}_1(\mathbb{Y}(\mathbb{Z}_2))$ *through*

$$f\big|_{\mathfrak{S}_n(\mathbb{Z}_2)} = \sum_{\alpha \in \mathbb{Y}_n(\mathbb{Z}_2)} \varphi(\alpha)\chi^\alpha, \qquad n \in \mathbb{N},$$

where χ^α *is a nonnormalized irreducible character of* $\mathfrak{S}_n(\mathbb{Z}_2)$*.*

Theorem 12.7. *Given* $f \in E(\mathfrak{S}_\infty(\mathbb{Z}_2))$*, let* M *be the corresponding ergodic element of* $\mathcal{M}_1(\mathfrak{T})$ *in Proposition 12.6 and Proposition 12.4. Then, for* M*-a.s.* $t \in \mathfrak{T}$*, we have*

$$\lim_{n\to\infty} \tilde{\chi}^{t(n)} = f.$$

Proof. Iterating restrictions onto lower levels one by one, we have

$$\tilde{\chi}^{t(n)}\big|_{\mathfrak{S}_k(\mathbb{Z}_2)} \sum_{\alpha \in \mathbb{Y}_k(\mathbb{Z}_2)} \frac{d(\alpha, t(n))}{d(\varnothing, t(n))}\chi^\alpha.$$

Incorporate this with Theorem 12.5 and Proposition 12.6. □

Combining Theorem 12.7 with computations done in Sect. 11.3, we deduce the Vershik–Kerov condition.

Theorem 12.8. *Along M-a.s. path $t \in \mathfrak{T}$ in Theorem 12.7, there exist:*

$$c_\pm = \lim_{n\to\infty} \frac{|t(n)^\pm|}{n} \qquad (c_+ + c_- = 1), \tag{12.1}$$

$$\alpha_{\pm i} = \lim_{n\to\infty} \frac{a_i(t(n)^\pm)}{n}, \quad \beta_{\pm i} = \lim_{n\to\infty} \frac{b_i(t(n)^\pm)}{n}, \qquad i \in \mathbb{N}. \tag{12.2}$$

Here $a_i(\lambda) = \lambda_i - i$ and $b_i(\lambda) = \lambda'_i - i$ denote the Frobenius coordinates of $\lambda \in \mathbb{Y}$.

Proof. Recall the computation of irreducible character values at basic elements. Equation (11.4) for a 1-n-cycle gives

$$\tilde{\chi}^{(t(n)^+, t(n)^-)}_{((1^{n-1}),(1))} = \frac{|t(n)^+| - |t(n)^-|}{n}. \tag{12.3}$$

Since $|t(n)^+| + |t(n)^-| = 1$ always holds, convergence of (12.3) yields (12.1).

Equations (11.5) for k-p-cycle and (11.6) for k-n-cycle give

$$\begin{aligned}\tilde{\chi}^{(t(n)^+, t(n)^-)}_{k\text{-p/n-cycle}} &= \frac{|t(n)^+|(|t(n)^+|-1)\cdots(|t(n)^+|-k+1)}{n(n-1)\cdots(n-k+1)} \tilde{\chi}^{t(n)^+}_{(k,1^{|t(n)^+|-k})} \\ &\pm \frac{|t(n)^-|(|t(n)^-|-1)\cdots(|t(n)^-|-k+1)}{n(n-1)\cdots(n-k+1)} \tilde{\chi}^{t(n)^-}_{(k,1^{|t(n)^-|-k})},\end{aligned} \tag{12.4}$$

where p and n on the left hand side correspond to + and − in the second line respectively. Convergence of (12.4) yields that there exist

$$\lim_{n\to\infty} \frac{|t(n)^\pm|(|t(n)^\pm|-1)\cdots(|t(n)^\pm|-k+1)}{n(n-1)\cdots(n-k+1)} \tilde{\chi}^{t(n)^\pm}_{(k,1^{|t(n)^\pm|-k})}.$$

Combining this with (11.10), we have existence of

$$\lim_{n\to\infty} \sum_{i=1}^\infty \Big\{ \Big(\frac{a_i(t(n)^\pm)}{n}\Big)^k + (-1)^{k-1} \Big(\frac{b_i(t(n)^\pm)}{n}\Big)^k \Big\}. \tag{12.5}$$

We check that the four sequences $\{a_i(t(n)^\pm)/n\}_n$, $\{b_i(t(n)^\pm)/n\}_n$ have unique limit points respectively for any $i \in \mathbb{N}$. Take their limit points $\alpha_{\pm i}$, $\beta_{\pm i}$ respectively. Then, if $k \geq 2$,

$$\sum_{i=1}^\infty \big\{ (\alpha_{\pm i})^k + (-1)^{k-1} (\beta_{\pm i})^k \big\} \tag{12.6}$$

agree with (12.5). Hence (12.6) does not depend on the choice of limit points. However, (12.6) determines $\alpha_{\pm i}$ and $\beta_{\pm i}$ uniquely since it holds that

$$\exp\Big\{ \sum_{k=2}^\infty \Big(\sum_{i=1}^\infty (\alpha_i^k + (-1)^{k-1}\beta_i^k) \Big) \frac{z^k}{k} \Big\} = \exp\Big\{ -z \sum_{i=1}^\infty (\alpha_i + \beta_i) \Big\} \prod_{i=1}^\infty \frac{1+\beta_i z}{1-\alpha_i z}, \quad z \in \mathbb{C}$$

and similarly for α_{-i}, β_{-i}. □

Recall that the conjugacy classes of $\mathfrak{S}_\infty(\mathbb{Z}_2)$ are parametrized by

$$\{(\rho, \sigma) \mid \rho \in \overline{\mathbb{Y}}, \sigma \in \mathbb{Y}\} \tag{12.7}$$

by way of positive/negative cycle decomposition. As a slight extension of Theorem 4.9, we have the following.

Theorem 12.9. *It holds that*

$$E(\mathfrak{S}_\infty(\mathbb{Z}_2)) = F(\mathfrak{S}_\infty(\mathbb{Z}_2)).$$

Namely, extremality and factorizability are equivalent in $\mathcal{K}_1(\mathfrak{S}_\infty(\mathbb{Z}_2))$.

Summing up, we have now all characters of $\mathfrak{S}_\infty(\mathbb{Z}_2)$. They are parametrized by

$$\begin{aligned}(12.8)\quad \big\{(\alpha,\beta,\gamma)\,\big|\,& \alpha=(\alpha_i)_{i=\pm1,\pm2,\dots},\ \beta=(\beta_i)_{i=\pm1,\pm2,\dots},\ \gamma=(\gamma_\pm),\\ &\alpha_{\pm1}\geq\alpha_{\pm2}\geq\cdots\geq 0,\ \beta_{\pm1}\geq\beta_{\pm2}\geq\cdots\geq 0,\ \gamma_\pm\geq 0,\\ &\sum_{i=1}^{\infty}(\alpha_i+\beta_i+\alpha_{-i}+\beta_{-i})+\gamma_++\gamma_-=1\big\}.\end{aligned}$$

Here we set $\gamma_\pm$ as (11.9). As modifications of power sums, we introduce

$$(12.9)\quad p_k^{\pm}(\alpha,\beta,\gamma)=\sum_{i=1}^{\infty}(\alpha_i^k+(-1)^{k-1}\beta_i^k)\pm\sum_{i=1}^{\infty}(\alpha_{-i}^k+(-1)^{k-1}\beta_{-i}^k),\qquad k\geq 2,$$

$$(12.10)\quad p_1^{\pm}(\alpha,\beta,\gamma)=\sum_{i=1}^{\infty}(\alpha_i+\beta_i)+\gamma_+\pm\sum_{i=1}^{\infty}(\alpha_{-i}+\beta_{-i})\pm\gamma_-.$$

Note that $p_1^+(\alpha,\beta,\gamma)\equiv 1$. The values of character $f_{\alpha,\beta,\gamma}$ of $\mathfrak{S}_\infty(\mathbb{Z}_2)$ corresponding to the parameter (α,β,γ) in (12.8) at basic elements are given by

$$f_{\alpha,\beta,\gamma}(\text{1-n-cycle})=p_1^-(\alpha,\beta,\gamma),$$

$$f_{\alpha,\beta,\gamma}(k\text{-p-cycle})=p_k^+(\alpha,\beta,\gamma),\qquad f_{\alpha,\beta,\gamma}(k\text{-n-cycle})=p_k^-(\alpha,\beta,\gamma),\qquad k\geq 2.$$

At a general conjugacy class indicated by (ρ,σ), where $\rho\in\overline{\mathbb{Y}}$, $\sigma\in\mathbb{Y}$, we have

$$(12.11)\qquad f_{\alpha,\beta,\gamma}((\rho,\sigma))=p_{(\rho,\sigma)}(\alpha,\beta,\gamma)=p_{\rho_1}^+p_{\rho_2}^+\cdots p_{\sigma_1}^-p_{\sigma_2}^-\cdots.$$

13. Realization of finite factorial representations of infinite wreath product groups (Lecture A)

This section is the content of the seminar talk, which is based on the joint paper with T.Hirai and E.Hirai [12]. We concentrate on the simplest case among those where compact group T is nontrivial, namely we treat the group

$$G = \mathfrak{S}_\infty(\mathbb{Z}_2) = D_\infty \rtimes \mathfrak{S}_\infty, \quad D_\infty = D_\infty(\mathbb{Z}_2) = \text{restricted direct product of } \mathbb{Z}_2.$$

The conjugacy classes of $\mathfrak{S}_\infty(\mathbb{Z}_2)$ are parametrized by

$$\{(\rho, \sigma) \,|\, \rho \in \overline{\mathbb{Y}},\ \sigma \in \mathbb{Y}\}$$

where ρ is the collection of types of nontrivial positive cycles while σ is that of negative cycles. The characters (= extremal normalized positive-definite central functions) of $\mathfrak{S}_\infty(\mathbb{Z}_2)$ are parametrized by the simplex

$$\begin{aligned}\bigl\{(\alpha,\beta,\gamma)\,\bigm|\,&\alpha = (\alpha_i)_{i=\pm1,\pm2,\dots},\ \beta = (\beta_i)_{i=\pm1,\pm2,\dots},\ \gamma = (\gamma_\pm),\\ &\alpha_{\pm1} \geq \alpha_{\pm2} \geq \cdots \geq 0,\ \beta_{\pm1} \geq \beta_{\pm2} \geq \cdots \geq 0,\ \gamma_\pm \geq 0,\\ &\sum_{i=1}^\infty (\alpha_i + \beta_i + \alpha_{-i} + \beta_{-i}) + \gamma_+ + \gamma_- = 1\bigr\}.\end{aligned}$$

Let $f_{\alpha,\beta,\gamma}$ denote the character of $\mathfrak{S}_\infty(\mathbb{Z}_2)$ corresponding to the classifying parameter (α, β, γ). Setting

$$p_k^\pm(\alpha,\beta,\gamma) = \sum_{i=1}^\infty (\alpha_i^k + (-1)^{k-1}\beta_i^k) \pm \sum_{i=1}^\infty (\alpha_{-i}^k + (-1)^{k-1}\beta_{-i}^k), \qquad k \geq 2,$$

$$p_1^\pm(\alpha,\beta,\gamma) = \sum_{i=1}^\infty (\alpha_i + \beta_i) + \gamma_+ \pm \sum_{i=1}^\infty (\alpha_{-i} + \beta_{-i}) \pm \gamma_-,$$

we know

$$f_{\alpha,\beta,\gamma}((\rho,\sigma)) = p_{(\rho,\sigma)}(\alpha,\beta,\gamma) = p^+_{\rho_1} p^+_{\rho_2} \cdots p^-_{\sigma_1} p^-_{\sigma_2} \cdots .$$

We construct the following representation Π in which the role of parameter (α,β,γ) is quite explicit.

Theorem 13.1. *Given character $f_{\alpha,\beta,\gamma}$ of $\mathfrak{S}_\infty(\mathbb{Z}_2)$, there exists a finite factorial UR $\Pi = \Pi^{\alpha,\beta,\gamma}$ of $\mathfrak{S}_\infty(\mathbb{Z}_2)$ on Hilbert space $\mathcal{H} = \mathcal{H}^{\alpha,\beta,\gamma}$ with cyclic vector $\Omega \in \mathcal{H}$ such that*

$$f_{\alpha,\beta,\gamma}(g) = \langle \Pi(g)\Omega, \Omega\rangle_{\mathcal{H}}, \qquad g \in \mathfrak{S}_\infty(\mathbb{Z}_2). \tag{13.1}$$

Main ingredients for the construction are

- twisting by a 1-cocycle for the β-parts
- inflation by a space of multiplicities.

We need, however, careful (and a bit cumbersome) construction of notations even in the simplest case of $T = \mathbb{Z}_2$. Once we establish (13.1), finiteness and factoriality of Π immediately follows from a general theory of group representations since we already know that $f_{\alpha,\beta,\gamma}$ is the character of a finite factorial representation of $\mathfrak{S}_\infty(\mathbb{Z}_2)$.

13.1. **Step1: Representation π.** Set (see Figure 13.1)

$$\begin{aligned} &\mathcal{X}_{\text{disc}} = \mathcal{N}_+^\alpha \sqcup \mathcal{N}_+^\beta \sqcup \mathcal{N}_-^\alpha \sqcup \mathcal{N}_-^\beta, && \mathcal{N}_{+,-}^{\alpha,\beta} \simeq \mathbb{N}, \\ &\mathcal{X}_{\text{cont}} = \Gamma_+ \sqcup \Gamma_-, && \Gamma_\pm = [0, \gamma_\pm], \\ &\mathcal{X}_\pm = \mathcal{N}_\pm^\alpha \sqcup \mathcal{N}_\pm^\beta \sqcup \Gamma_\pm. \end{aligned}$$

Recall that we set $\widehat{\mathbb{Z}_2} = \{\zeta_1, \zeta_{-1}\}$. The parts indexed by $\pm$ in Figure 13.1 correspond to $\zeta_{\pm 1}$ respectively. We set

$$\mathcal{X} = \mathcal{X}_{\text{disc}} \sqcup \mathcal{X}_{\text{cont}} = \mathcal{X}_+ \sqcup \mathcal{X}_-.$$

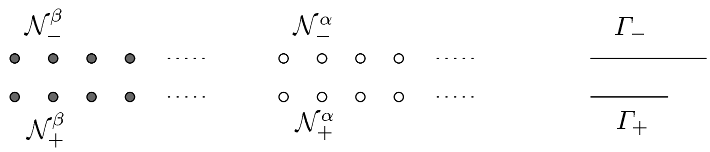

FIG. 13.1. Underlying space $\mathcal{X}$

Define probability $\nu = \nu_{\alpha,\beta,\gamma}$ on $\mathcal{X}$ by

$$\nu\big|_{\mathcal{N}_\pm^\alpha} = \sum_{i \in \mathbb{N}} \alpha_{\pm i} \delta_i, \qquad \nu\big|_{\mathcal{N}_\pm^\beta} = \sum_{i \in \mathbb{N}} \beta_{\pm i} \delta_i,$$
$$\nu\big|_{\Gamma_\pm} = \text{Lebesgue measure on } \Gamma_\pm$$

where all $\mathcal{N}_{+,-}^{\alpha,\beta}$ are identified with $\mathbb{N}$ and set ν^∞ to be its product measure on $\mathcal{X}^\infty$.

For measurable vector field $v = (v(x))_{x \in \mathcal{X}}$ where the fibre at x is $V(x) \simeq \mathbb{C}$, set

$$\|v\|_{V(\mathcal{X})}^2 = \int_{\mathcal{X}} \|v(x)\|_{V(x)}^2 \nu(dx)$$

and $V(\mathcal{X}) = \{v = (v(x))_{x \in \mathcal{X}} \mid \|v\|_{V(\mathcal{X})} < \infty\}$. The constant field $1_{\mathcal{X}} = (1_x)_{x \in \mathcal{X}}$, where $1_x \equiv 1$, has norm 1. Let $\mathbb{Z}_2$ act on $V(\mathcal{X})$ componentwise:

$$Z(t)v = (Z_x(t)v(x))_{x \in \mathcal{X}}, \quad t \in \mathbb{Z}_2, \qquad \text{where } Z_x(t) = \begin{cases} \zeta_1(t), & x \in \mathcal{X}_+, \\ \zeta_{-1}(t), & x \in \mathcal{X}_-. \end{cases}$$

Remark 13.2. Since $\mathbb{Z}_2$ is abelian, we have only to deal with one-dimensional fibre $V(x)$ at each $x \in \mathcal{X}$. This simplifies the construction of desired representations to some extent. In the case of general compact group T instead of $\mathbb{Z}_2$, we put a representation space of the IUR ζ of T at x, where ζ depends on base point x.

Let $\bigotimes_{i \in \mathbb{N}} V(\mathcal{X}_i)$, $\mathcal{X}_i \equiv \mathcal{X}$, be the tensor product with respect to the reference vector $(1_{\mathcal{X}_i})_{i \in \mathbb{N}}$. We set for decomposable element $\otimes_i v_i$, $v_i \in V(\mathcal{X}_i)$,

$$\begin{aligned} &\pi(d)(\otimes_i v_i) = \otimes_i Z(t_i) v_i, && d = (t_i) \in D_\infty, \\ &\pi(\sigma)(\otimes_i v_i) = \otimes_i v_{\sigma^{-1}(i)}, && \sigma \in \mathfrak{S}_\infty \end{aligned}$$

and then

$$\pi(g)w = \pi(d)\pi(\sigma)w, \qquad w \in \bigotimes_{i \in \mathbb{N}} V(\mathcal{X}_i), \quad g = (d, \sigma) \in \mathfrak{S}_\infty(\mathbb{Z}_2).$$

Lemma 13.3. *π is a UR of $\mathfrak{S}_\infty(\mathbb{Z}_2)$.*

Proof. Since π gives URs of D_∞ and $\mathfrak{S}_\infty$, it suffices to show $\pi(\sigma)\pi(d)\pi(\sigma)^{-1} = \pi(\sigma(d))$ to see compatibility with the semidirect product structure, which is now directly verified. □

13.2. **Step2: Representation $\tilde{\pi}$.** A decomposable element of $\bigotimes_{i\in\mathbb{N}} V(\mathcal{X}_i)$, $w = \otimes_i v_i$, is regarded as a measurable vector field on $\mathcal{X}^\infty$ through

$$\boldsymbol{x} = (x_i) \in \mathcal{X}^\infty \longmapsto w(\boldsymbol{x}) = \otimes_i v_i(x_i) \in \bigotimes_i V(x_i) = W(\boldsymbol{x}) \simeq \mathbb{C}$$

and satisfies

$$\|w\|^2 = \prod_i \|v_i\|^2 = \int_{\mathcal{X}^\infty} \prod_i \|v_i(x_i)\|^2 \nu^\infty(d\boldsymbol{x}) = \int_{\mathcal{X}^\infty} \|w(\boldsymbol{x})\|^2 \nu^\infty(d\boldsymbol{x}).$$

We rephrase π as a UR of $\mathfrak{S}_\infty(\mathbb{Z}_2)$ on

$$W(\mathcal{X}) = \Big\{ w = w(\boldsymbol{x}) \ \Big|\ \|w\|^2 = \int_{\mathcal{X}^\infty} \|w(\boldsymbol{x})\|^2_{W(\boldsymbol{x})} \nu^\infty(d\boldsymbol{x}) < \infty \Big\}$$

by

$$(\pi(d)w)(\boldsymbol{x}) = Z_{\boldsymbol{x}}(d)(w(\boldsymbol{x})), \qquad \boldsymbol{x} = (x_i), \quad d \in D_\infty, \tag{13.2}$$

$$(\pi(\sigma)w)(\boldsymbol{x}) = w(\sigma^{-1}\boldsymbol{x}), \qquad \boldsymbol{x} = (x_i), \quad \sigma \in \mathfrak{S}_\infty, \tag{13.3}$$

where we set $Z_{\boldsymbol{x}}(d) = \otimes_i Z_{x_i}(t_i)$ for $d = (t_i) \in D_\infty$.

Remark 13.4. If T is not abelian and $V(x_i)$ is multi-dimensional, the value spaces of both sides of (13.3) are not consistent. Then we should replace the right side by $\kappa(\sigma)w(\sigma^{-1}\boldsymbol{x})$, introducing

$$\kappa(\sigma) : \otimes_i v_i(x_i) \in \bigotimes_i V(x_i) \longmapsto \otimes_i v_{\sigma^{-1}(i)}(x_{\sigma^{-1}(i)}) \in \bigotimes_i V(x_{\sigma^{-1}(i)}).$$

In (13.2) also, we can simply use a product notation since $Z_{x_i}(t_i)$ is a scalar acting on one-dimensional space $V(x_i)$.

Definition 13.5. For $\boldsymbol{x} = (x_i) \in \mathcal{X}^\infty$ set $J(\boldsymbol{x}) = \{i \in \mathbb{N} \,|\, x_i \in \mathcal{N}^\beta = \mathcal{N}^\beta_+ \sqcup \mathcal{N}^\beta_-\}$ and

$$\mathrm{inv}(\sigma, \boldsymbol{x}) = \text{the number of inversions in } (\sigma(i))_{i\in J(\boldsymbol{x})}.$$

Lemma 13.6. $(-1)^{\mathrm{inv}(\sigma,\boldsymbol{x})}$ *is a multiplicative* 1*-cocycle, namely it holds that*

$$(-1)^{\mathrm{inv}(\sigma\tau,\boldsymbol{x})} = (-1)^{\mathrm{inv}(\sigma,\tau\boldsymbol{x})}(-1)^{\mathrm{inv}(\tau,\boldsymbol{x})}, \qquad \sigma,\ \tau \in \mathfrak{S}_\infty.$$

Combining π with this 1-cocycle, we set

$$(\tilde{\pi}(g)w)(\boldsymbol{x}) = (-1)^{\mathrm{inv}(\sigma^{-1},\boldsymbol{x})} Z_{\boldsymbol{x}}(d) w(\sigma^{-1}\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathcal{X}^\infty,\ g = (d,\sigma) \in \mathfrak{S}_\infty(\mathbb{Z}_2).$$

Lemma 13.7. $\tilde{\pi}$ *is a UR of* $\mathfrak{S}_\infty(\mathbb{Z}_2)$.

Proof. Lemma 13.6 yields that $\tilde{\pi}$ gives a UR of $\mathfrak{S}_\infty$. Then it suffices to verify $\tilde{\pi}(\sigma)\tilde{\pi}(d)\tilde{\pi}(\sigma)^{-1} = \tilde{\pi}(\sigma(d))$. □

For instance, take a constant field $\mathbf{1} = \otimes 1_{\mathcal{X}_i} \in W(\mathcal{X})$ and consider its matrix element:

$$\langle \tilde{\pi}(g)\mathbf{1}, \mathbf{1} \rangle = \int_{\mathcal{X}^\infty} (-1)^{\mathrm{inv}(\sigma^{-1},\boldsymbol{x})} \Big(\prod_i Z_{x_i}(t_i)\Big) \nu^\infty(d\boldsymbol{x}), \qquad g = (d,\sigma),\ d = (t_i).$$

We have no multiplicative structure on the 1-cocycle part with respect to cycle decomposition. This leads us to another trick in the next subsection.

13.3. **Step3: Representation $\tilde{\Pi}$.** We inflate the representation space of $\tilde{\pi}$ by putting a multiplicity parameter. Let $w = w(\boldsymbol{x},\boldsymbol{y})$ be a measurable vector field on $\mathcal{X}^\infty \times \mathcal{X}^\infty$ taking a value $w(\boldsymbol{x},\boldsymbol{y}) \in W(\boldsymbol{x}) = \bigotimes_i V(x_i) \simeq \mathbb{C}$ where the tensor product is taken with respect to reference vector $(1_{x_i})_{i\in\mathbb{N}}$, $1_{x_i} \equiv 1$ though it is trivial here for one-dimensional $V(x_i)$'s. For $\boldsymbol{x},\boldsymbol{y} \in \mathcal{X}^\infty$, we express $\boldsymbol{x} \sim \boldsymbol{y}$ if there exists $\tau \in \mathfrak{S}_\infty$ such that $\boldsymbol{x} = \tau(\boldsymbol{y})$. Setting

$$\|w\|_{\tilde{\mathcal{H}}}^2 = \int_{\mathcal{X}^\infty} \sum_{\boldsymbol{y}\,:\,\boldsymbol{y}\sim\boldsymbol{x}} \|w(\boldsymbol{x},\boldsymbol{y})\|_{W(\boldsymbol{x})}^2 \nu^\infty(d\boldsymbol{x}) \tag{13.4}$$

for vector field $w = w(\boldsymbol{x},\boldsymbol{y})$, consider

$$\tilde{\mathcal{H}} = \big\{\text{measurable vector field } w = w(\boldsymbol{x},\boldsymbol{y}) \text{ on } \mathcal{X}^\infty \times \mathcal{X}^\infty \;\big|\; \|w\|_{\tilde{\mathcal{H}}} < \infty\big\}.$$

Set for $g = (d,\sigma) \in \mathfrak{S}_\infty(\mathbb{Z}_2)$

$$(\tilde{\Pi}(g)w)(\boldsymbol{x},\boldsymbol{y}) = (-1)^{\mathrm{inv}(\sigma^{-1},\boldsymbol{x})} Z_{\boldsymbol{x}}(d) w(\sigma^{-1}\boldsymbol{x},\boldsymbol{y}), \qquad w \in \tilde{\mathcal{H}}. \tag{13.5}$$

Lemma 13.8. *$\tilde{\Pi}$ is a UR of $\mathfrak{S}_\infty(\mathbb{Z}_2)$.*

Proof. Since $\tilde{\Pi}$ gives URs of D_∞ and $\mathfrak{S}_\infty$, it suffices to show $\tilde{\Pi}(\sigma)\tilde{\Pi}(d)\tilde{\Pi}(\sigma)^{-1} = \tilde{\Pi}(\sigma(d))$ as before. □

Remark 13.9. If we consider a general case and put $\kappa(\sigma)$ into the right side of (13.5) as Remark 13.4, we use

$$\kappa(\sigma) Z_{\sigma^{-1}\boldsymbol{x}}(d)\kappa(\sigma)^{-1} = Z_{\boldsymbol{x}}(\sigma(d))$$

to show Lemma 13.8.

13.4. **Step4: Representation $\Pi = \Pi^{\alpha,\beta,\gamma}$.** Consider the diagonal $\Delta = \{(\boldsymbol{x},\boldsymbol{x}) \,|\, \boldsymbol{x} \in \mathcal{X}^\infty\} \subset \mathcal{X}^\infty \times \mathcal{X}^\infty$ and its indicator function 1_Δ, which takes value $1_\Delta(\boldsymbol{x},\boldsymbol{y}) \in W(\boldsymbol{x}) = \bigotimes_i V(x_i) \simeq \mathbb{C}$. We have

$$\|1_\Delta\|_{\tilde{\mathcal{H}}}^2 = \int_{\mathcal{X}^\infty} \|1_\Delta(\boldsymbol{x},\boldsymbol{x})\|_{W(\boldsymbol{x})}^2 \nu^\infty(d\boldsymbol{x}) = 1.$$

Let $\mathcal{H}$ denote the closed linear subspace of $\tilde{\mathcal{H}}$ spanned by $\tilde{\Pi}(\mathfrak{S}_\infty(\mathbb{Z}_2))1_\Delta$. We set

$$\Pi(g) = \Pi^{\alpha,\beta,\gamma}(g) = \tilde{\Pi}(g)\big|_{\mathcal{H}}, \qquad g \in \mathfrak{S}_\infty(\mathbb{Z}_2). \tag{13.6}$$

Then Π is a UR of $\mathfrak{S}_\infty(\mathbb{Z}_2)$ on $\mathcal{H}$ with cyclic unit vector 1_Δ.

13.5. **Step5: Matrix element and character.** Equation (13.6) yields that it holds for $g = (d,\sigma)$

$$\begin{aligned}\langle \Pi(g)1_\Delta,\, 1_\Delta\rangle_{\mathcal{H}} &= \int_{\mathcal{X}^\infty} \sum_{\boldsymbol{y}\,:\,\boldsymbol{y}\sim\boldsymbol{x}} \langle (\Pi(g)1_\Delta)(\boldsymbol{x},\boldsymbol{y}),\, 1_\Delta(\boldsymbol{x},\boldsymbol{y})\rangle_{W(\boldsymbol{x})} \nu^\infty(d\boldsymbol{x}) \\ &= \int_{\mathcal{X}^\infty} \sum_{\boldsymbol{y}\,:\,\boldsymbol{y}\sim\boldsymbol{x}} \langle (-1)^{\mathrm{inv}(\sigma^{-1},\boldsymbol{x})} Z_{\boldsymbol{x}}(d) 1_\Delta(\sigma^{-1}\boldsymbol{x},\boldsymbol{y}),\, 1_\Delta(\boldsymbol{x},\boldsymbol{y})\rangle_{W(\boldsymbol{x})} \nu^\infty(d\boldsymbol{x}) \\ &= \int_{\{\boldsymbol{x}\in\mathcal{X}^\infty \,|\, \sigma^{-1}\boldsymbol{x}=\boldsymbol{x}\}} (-1)^{\mathrm{inv}(\sigma^{-1},\boldsymbol{x})} Z_{\boldsymbol{x}}(d) \nu^\infty(d\boldsymbol{x}).\end{aligned} \tag{13.7}$$

Let $g = \xi_{q_1}\cdots\xi_{q_r} g_1\cdots g_m$ be a standard decomposition where $\xi_{q_i} = (-1,(q_i))$ and $g_j = (d_j,\sigma_j)$. For $\boldsymbol{x} \in \mathcal{X}^\infty$, $\sigma^{-1}\boldsymbol{x} = \boldsymbol{x}$ holds if and only if $\boldsymbol{x}$ is constant on any $\mathrm{supp}\sigma_j$. Then, it holds either $J(\boldsymbol{x}) \supset \mathrm{supp}\sigma_j$ or $J(\boldsymbol{x}) \cap \mathrm{supp}\sigma_j = \varnothing$ for any j. Hence $(-1)^{\mathrm{inv}(\sigma^{-1},\boldsymbol{x})}$ can be considered on $J(\boldsymbol{x})$ and then gets multiplicative since it is just the usual sign of a permutation. We thus find multiplicative structure in (13.7) and continue as

$$(13.8)\quad \int_{\mathcal{X}^{\mathbb{N}}\setminus\sqcup_{j=1}^{m}\mathrm{supp}\sigma_j} Z_{\boldsymbol{x}'}(d)\nu^{\infty}(d\boldsymbol{x}') \prod_{j=1}^{m}\int_{\mathcal{Y}_j}(-1)^{\mathrm{inv}(\sigma_j^{-1},\boldsymbol{x}^{(j)})}Z_{\boldsymbol{x}^{(j)}}(d)\nu^{(j)}(d\boldsymbol{x}^{(j)})$$

$$= (*) \times \prod_{j=1}^{m}(**),$$

setting $\mathcal{Y}_j = \{\boldsymbol{x}^{(j)} = (x_i)_{i\in\mathrm{supp}\sigma_j} \,|\, \boldsymbol{x}^{(j)}$ is constant on $\mathrm{supp}\sigma_j\}$.

Equation $(*)$ in (13.8) is equal to

$$\prod_{p=1}^{r}\int_{\mathcal{X}} Z_x(-1)\nu(dx) = \prod_{p=1}^{r}(\nu(\mathcal{X}_+) - \nu(\mathcal{X}_-)) = p_1^-(\alpha,\beta,\gamma)^r.$$

Expressing $\mathcal{Y}_j$ as a disjoint union of $\{\boldsymbol{x}^{(j)} = (x_i)_{i\in\mathrm{supp}\sigma_j} \,|\, x_i = c$, for any $i \in \mathrm{supp}\sigma_j\}$ and noting $c(\sigma_j) = |\mathrm{supp}\sigma_j| \geq 2$, we see that integration for $c \in \Gamma_+ \sqcup \Gamma_-$ in $(**)$ of (13.8) vanishes. Hence $(**)$ is equal to

$$\begin{aligned}
&\sum_{c\in\mathcal{N}_+^{\alpha}} \nu^{(j)}(\{\boldsymbol{x}^{(j)} \,|\, x_i = c \text{ for any } i \in \mathrm{supp}\sigma_j\}) \\
&\quad + \sum_{c\in\mathcal{N}_-^{\alpha}} \zeta_{-1}(d^{(j)})\, \nu^{(j)}(\{\boldsymbol{x}^{(j)} \,|\, x_i = c \text{ for any } i \in \mathrm{supp}\sigma_j\}) \\
&\quad + \sum_{c\in\mathcal{N}_+^{\beta}} (\mathrm{sgn}\sigma_j)\, \nu^{(j)}(\{\boldsymbol{x}^{(j)} \,|\, x_i = c \text{ for any } i \in \mathrm{supp}\sigma_j\}) \\
&\quad + \sum_{c\in\mathcal{N}_-^{\beta}} (\mathrm{sgn}\sigma_j)\, \zeta_{-1}(d^{(j)})\, \nu^{(j)}(\{\boldsymbol{x}^{(j)} \,|\, x_i = c \text{ for any } i \in \mathrm{supp}\sigma_j\}) \\
&= \sum_{i=1}^{\infty}\alpha_i^{c(\sigma_j)} + \sum_{i=1}^{\infty}\zeta_{-1}(d^{(j)})\,\alpha_{-i}^{c(\sigma_j)} \\
&\quad + \sum_{i=1}^{\infty}(\mathrm{sgn}\sigma_j)\,\beta_i^{c(\sigma_j)} + \sum_{i=1}^{\infty}(\mathrm{sgn}\sigma_j)\,\zeta_{-1}(d^{(j)})\,\beta_{-i}^{c(\sigma_j)} \\
&= p_k^{\pm}(\alpha,\beta,\gamma)
\end{aligned}$$

according as g_j is a k-p/n-cycle ($k \geq 2$). Hence

$$\begin{aligned}
(13.7) &= p_1^-(\alpha,\beta,\gamma)^{\sharp(\text{1-n-cycles})}\prod_{k\geq 2}\big(p_k^+(\alpha,\beta,\gamma)^{\sharp(k\text{-p-cycles})}p_k^-(\alpha,\beta,\gamma)^{\sharp(k\text{-n-cycles})}\big) \\
&= p_{(\rho,\sigma)}(\alpha,\beta,\gamma) \qquad \text{where} \quad g \in C_{(\rho,\sigma)} \subset \mathfrak{S}_{\infty}(\mathbb{Z}_2) \\
&= f_{\alpha,\beta,\gamma}(g).
\end{aligned}$$

We have thus obtained

$$(13.9)\qquad f_{\alpha,\beta,\gamma}(g) = \langle \Pi^{\alpha,\beta,\gamma}(g)1_{\Delta},\ 1_{\Delta}\rangle_{\mathcal{H}}, \qquad g \in \mathfrak{S}_{\infty}(\mathbb{Z}_2).$$

Combined with a general theory of the Gelfand–Raikov representation, (13.9) and extremality of $f_{\alpha,\beta,\gamma}$ yield that $\Pi^{\alpha,\beta,\gamma}$ is a finite factorial representation of $\mathfrak{S}_{\infty}(\mathbb{Z}_2)$.

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Notes

- The first version on 10 June 2007.
- Minor corrections in Theorem 8.4, Definition 8.10, and References on 27 February 2008. Still incomplete citations in the text.

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