Introduction to Asymptotic Theory for Representations and Characters of Symmetric Groups

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Lecture 1 : Overview of the Course

INTRODUCTION

$\mathfrak{S}_n$: the symmetric group of degree n

$\mathbb{Y}_n$: the set of Young diagrams of size n

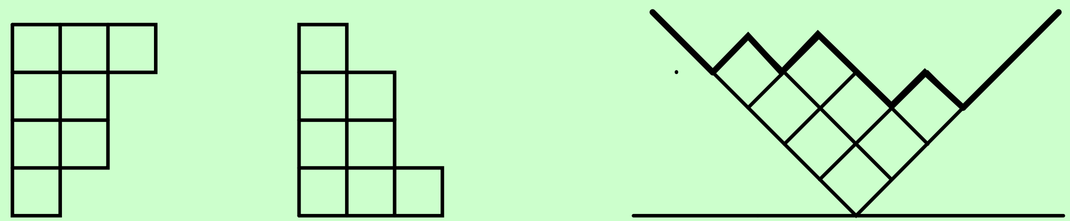

Fig. 1 Young diagram $(3 \geq 2 \geq 2 \geq 1) = (1^1 2^2 3^1)$ and its profile

λ' : transposed Young diagram

Growing Young diagram ($n \to \infty$) vs

behaviour of corresponding representation and character

Examples of the scales

- Vershik–Kerov condition

$$\lambda \in \mathbb{Y}_n; \quad \lambda_i,\ \lambda'_i \ \sim\ \text{(constant)}\ \times n$$

- balanced Young diagram

$$\lambda \in \mathbb{Y}_n; \quad \lambda_1,\ \lambda'_1 \ \leq\ \text{(constant)}\ \times \sqrt{n}$$

Today's overview:

I. Continuous Diagrams

Analytic description of Young diagrams

II. The Limit Shape and Fluctuations

$\longrightarrow$ balanced region

III. The Infinite Symmetric Group, Wreath Products, and Their Characters

$\longrightarrow$ VK region

Plan of the Course

§1. Young Diagrams and the Young Graph

[approx. $3 \sim 4$ lectures]

§2. The Limit Shape and Fluctuations

[approx. 5 lectures]

§3. The Infinite Symmetric Group, Wreath Products, and Their Characters

[approx. 5 lectures]

History, Names

Young, Frobenius, Schur, Weyl, $\cdots$

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Wigner : eigendistributions of random matrices

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Thoma : characters of $\mathfrak{S}_\infty$

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Voiculescu : characters of $U(\infty)$

Vershik–Kerov : limit shape of Young diagram

Logan–Shepp : limit shape of Young diagram

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Vershik–Kerov : characters of $\mathfrak{S}_\infty$, VK condition

Diaconis : card shuffling, cut-off phenomenon

Voiculescu : free probability, random matrices

Hirai : irreducible representations of $\mathfrak{S}_\infty$

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Obata : characters and irreducible characters of $\mathfrak{S}_\infty$

Vershik

Kerov

Olshanski

Tracy–Widom : fluctuation of eigenvalues in GUE

Speicher : combinatorics of free probability

Biane : permutation model for free probability

Okounkov

Baik–Deift–Johansson : fluctuation of rows in PE

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Johansson

Borodin

Ivanov

Fulman

Śniady

Collins

Matsumoto

I. CONTINUOUS DIAGRAMS

Examples of coordinates encoding a Young diagram

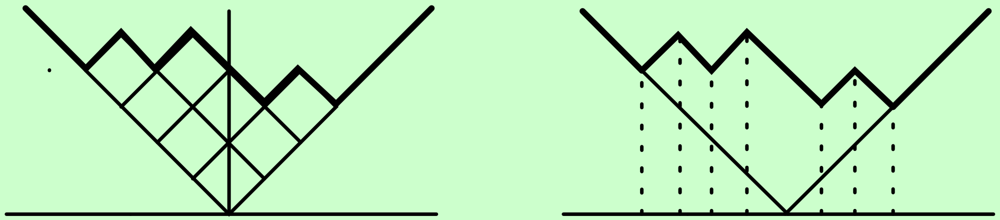

Fig. 2 coordinates of $\lambda = (3, 2, 2, 1)$, $(x_1 = -4, x_4 = 3)$

Frobenius : $a_i = \lambda_i - i + 1/2, \quad b_i = \lambda'_i - i + 1/2$

shifted row : $\lambda_i - i + 1/2, \quad i = 1, 2, \ldots$

min-max : $x_1 < y_1 < \cdots < y_{r-1} < x_r$

$\mathbb{Y} = \{\text{Young diagrams}\}$

$\mathbb{D}_0 = \{\text{rectangular diagrams}\}$

$$\mathbb{D} = \{\omega(x) : \text{ 1-Lipschitz continuous function on } \mathbb{R}$$
$$\text{such that } \omega(x) = |x| \text{ for sufficiently large } x\}$$
$$= \{\text{continuous diagrams}\} \quad \supset \mathbb{D}_0 \supset \mathbb{Y}$$

Embedding into a space of measures on $\mathbb{R}$

- atomic measure with unit mass at each point
- Rayleigh measure τ_λ
- Kerov's transition measure $\mathfrak{m}_\lambda$

Definition of transition measure $\mathfrak{m}_\lambda$ of $\lambda \in \mathbb{D}_0$

$$G_\lambda(z) = \frac{(z-y_1)\cdots(z-y_{r-1})}{(z-x_1)\cdots(z-x_r)} = \int_{-\infty}^{\infty} \frac{1}{z-x} \mathfrak{m}_\lambda(dx)$$

$$= \frac{1}{z}\sum_{n=0}^{\infty} \frac{M_n(\mathfrak{m}_\lambda)}{z^n} = \frac{1}{z}\exp\left\{\sum_{n=1}^{\infty} \frac{M_n(\tau_\lambda)}{n}\frac{1}{z^n}\right\}$$

where M_n : nth moment,

$$\tau_\lambda = \sum_{i=1}^{r} \delta_{x_i} - \sum_{i=1}^{r-1} \delta_{y_i} \quad \text{(Rayleigh measure of } \lambda\text{)}$$

By an approximation, transition measure $\mathfrak{m}_\omega$ for $\omega \in \mathbb{D}$

Example of a continuous diagram — limit shape

$$\Omega(x) = \begin{cases} \frac{2}{\pi}\left(x \arcsin\frac{x}{2} + \sqrt{4-x^2}\right), & |x| \le 2 \\ |x|, & |x| > 2 \end{cases}$$

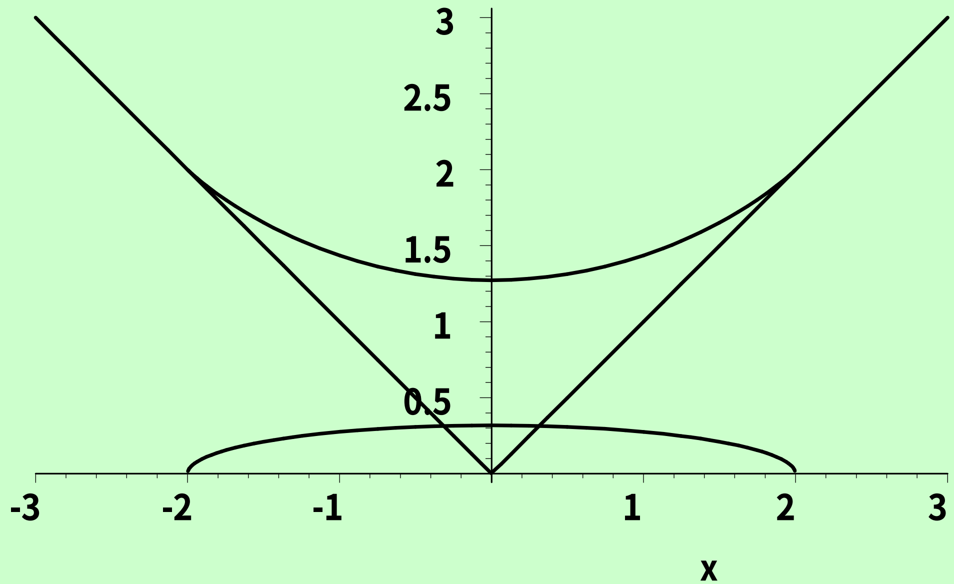

Fig. 3 limit shape and semi-circle distribution

Semicircle distribution as the transition measure of the limit shape

$$\frac{d\mathfrak{m}_\Omega}{dx} = \frac{1}{2\pi}\sqrt{4 - x^2}\,, \qquad |x| \le 2$$

Jucys–Murphy element

$$J_n = (1\ n+1) + (2\ n+1) + \cdots (n\ n+1) \ \in\ \mathbb{C}[\mathfrak{S}_{n+1}]$$

$$\widetilde{\chi}^\lambda(\mathbb{E}_n J_n^{\,k}) = M_k(\mathfrak{m}_\lambda) = \int_{\mathbb{R}} x^k \mathfrak{m}_\lambda(dx), \qquad k \in \mathbb{N}$$

$\widetilde{\chi}^\lambda$: normalized irreducible character for $\lambda \in \mathbb{Y}_n$

$\mathbb{E}_n : \mathbb{C}[\mathfrak{S}_{n+1}] \longrightarrow \mathbb{C}[\mathfrak{S}_n]$ conditional expectation

Plancherel ensemble of random Young diagrams (PE)

Plancherel measure $\mathfrak{P}_n$ on $\mathbb{Y}_n$

$$\mathfrak{P}_n(\lambda) = \frac{(\dim \lambda)^2}{n!}, \qquad \lambda \in \mathbb{Y}_n$$

Plancherel measure $\mathfrak{P}$ on the path space $\mathfrak{T}$

$$\mathfrak{P}\big(\{t \in \mathfrak{T} \mid t(1) = \lambda^{(1)}, \cdots, t(n) = \lambda^{(n)}\}\big) = \frac{\dim \lambda^{(n)}}{n!}$$

where $\lambda^{(j)} \in \mathbb{Y}_j$,

path $t = (\varnothing = t(0) \nearrow t(1) \nearrow \cdots \nearrow t(n) \nearrow \cdots) \in \mathfrak{T}$

Plancherel growth process on $\mathbb{Y}$

$$\mathfrak{P}\Big(t(n+1)=\Lambda \,\Big|\, t(1)=\lambda^{(1)},\dots,t(n-1)=\lambda^{(n-1)},t(n)=\lambda\Big)$$
$$=\begin{cases}\dfrac{\dim\Lambda}{(n+1)\dim\lambda}, & \text{if } \lambda\nearrow\Lambda,\\ 0, & \text{otherwise}\end{cases}$$

“transition measure”

$\lambda=(x_1<y_1<\dots<y_{r-1}<x_r)\in\mathbb{Y}_n,$

$\lambda\nearrow\Lambda^{(j)}$ (one box at the jth valley)

$$\frac{\dim\Lambda^{(j)}}{(n+1)\dim\lambda}=\mathfrak{m}_\lambda(\{x_j\}),\quad j=1,\dots,r$$

II. THE LIMIT SHAPE AND FLUCTUATIONS

$$\lambda \in \mathbb{Y}_n \longrightarrow \text{ profile } \lambda(x)$$

$$\longrightarrow \text{ scaled } \lambda^{\sqrt{n}}(x) = \frac{1}{\sqrt{n}}\lambda(\sqrt{n}x)$$

Theorem II.1 (Vershik–Kerov, Logan–Shepp 1977)

Along a.s. path $t \in \mathfrak{T}$ w.r.t. Plancherel measure $\mathfrak{P}$,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left|t(n)^{\sqrt{n}}(x) - \Omega(x)\right| = 0$$

Typically, $t(n) \sim \sqrt{n}$-extension of Ω

Concentration at Ω-component in irreducible decomposition of the regular representation (character)

$$\delta_e = \sum_{\lambda \in \mathbb{Y}_n} \mathfrak{P}_n(\lambda) \widetilde{\chi}^{\lambda}$$

Derivation of the limit shape

(i) hook formula $\quad \dim \lambda = n!/\prod_{b\in\lambda} h_\lambda(b), \quad \lambda \in \mathbb{Y}_n$

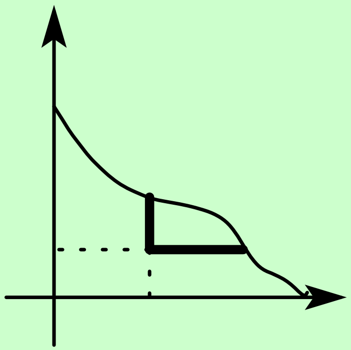

Fig. 4 continuous version of hook

Variational problem : Minimize

$$\iint_{D_f} \log(f(x)-y+f^{-1}(y)-x)dxdy \text{ under } \iint_{D_f} dxdy = 1$$

(ii) moment sequence of the transition measure

(iii) 1-point function of the point process

$$u \in \mathbb{R}, \;\; x_n \in \mathbb{Z}, \;\; \lim_{n\to\infty} \frac{x_n}{\sqrt{n}} = u,$$

$$\lim_{n\to\infty} \mathfrak{P}_n(\{\lambda \in \mathbb{Y}_n | x_n \in \{\lambda_i - i\}\}) = \rho_\infty(u)$$

More generally, the s-point function in a determinantal form

(Borodin–Okounkov–Olshanski 2000)

CASE (ii)

random variable $M_k(\mathfrak{m}_{\lambda^{\sqrt{n}}})$ on $(\mathbb{Y}_n, \mathfrak{P}_n)$

constant $M_k(\mathfrak{m}_\Omega)$ (= Catalan number for even k)

$$\sum_{\lambda \in \mathbb{Y}_n} (M_k(\mathfrak{m}_{\lambda^{\sqrt{n}}}) - M_k(\mathfrak{m}_\Omega))^p \mathfrak{P}_n(\lambda)$$

$$= \delta_e((\mathbb{E}_n(\frac{J_n}{\sqrt{n}})^k - M_k(\mathfrak{m}_\Omega))^p)$$

since $\widetilde{\chi}^\lambda(\mathbb{E}_n J_n^k) = M_k(\mathfrak{m}_\lambda)$

Polynomial functions on $\mathbb{Y}$ (Kerov–Olshanski 1994, Ivanov–Olshanski 2002) generated by the moments $M_k(\mathfrak{m}_\lambda)$ or the cumulants $\kappa_k(\mathfrak{m}_\lambda)$, $R_k(\mathfrak{m}_\lambda)$ or the irreducible characters $\Sigma_k(\lambda)$ ($\longrightarrow$ Kerov polynomial)

Fluctuation in the Plancherel ensemble (1)

$\lambda_1 \sim 2\sqrt{n}$ for $\lambda \in \mathbb{Y}_n$ w.r.t. $\mathfrak{P}_n$

Theorem II.2 (CLT for the longest row in PE, Baik–Deift–Johansson 1999)

$$\lim_{n\to\infty} \mathfrak{P}_n\Big(\Big\{\lambda \in \mathbb{Y}_n \,\Big|\, \widetilde{\lambda}_1 = n^{1/3}\Big(\frac{\lambda_1}{\sqrt{n}} - 2\Big) \leq x\Big\}\Big) = F(x)$$

$F(x)$: Tracy–Widom distribution function

PE : $\lambda_1 = 2\sqrt{n} + n^{1/6}X$

GUE : $E_1 = \sqrt{2N} + \frac{1}{\sqrt{2}}N^{-1/6}X$

where X obeys the Tracy–Widom distribution

$$\widetilde{\lambda}_i = n^{1/3}\Big(\frac{\lambda_i}{\sqrt{n}} - 2\Big), \quad i = 1, 2, \dots$$

Theorem II.3 (conjectured by Baik–Deift–Johansson; Okounkov 2000, Borodin–Okounkov–Olshanski 2000, Johansson 2001)

$\widetilde{\lambda} = (\widetilde{\lambda}_1 \geq \widetilde{\lambda}_2 \geq \cdots)$ converges in joint distribution to the Airy ensemble as $n \to \infty$, i.e. the correlation functions are in a determinantal form with the Airy kernel

Fluctuation in the Plancherel ensemble (2)

Kerov's CLT : irreducible characters at different cycles are asymptotically independent and Gaussian w.r.t. the Plancherel measure

Theorem II.4 (Kerov 1993)

$$\lim_{n\to\infty} \mathfrak{P}_n\Big(\Big\{\lambda \in \mathbb{Y}_n \Big| n^{k/2}\widetilde{\chi}^{\lambda}_{(k,1^{n-k})} \leq x_k \ (2 \leq k \leq m)\Big\}\Big) = \prod_{k=2}^{m} \frac{1}{\sqrt{2\pi k}} \int_{-\infty}^{x_k} e^{-y^2/2k}\, dy$$

Change of generators in polynomial functions on $\mathbb{Y}$

$\Longrightarrow$ CLT for moments and cumulants (classical, free, $\cdots$)

Theorem II.5 (Kerov 1993, Ivanov–Olshanski 2002)

$$\lambda^{\sqrt{n}}(x) = \Omega(x) + \frac{2}{\sqrt{n}} \Phi_n(x)$$

$\Phi_n(x)$ on $(\mathbb{Y}_n, \mathfrak{P}_n)$ converges to a generalized Gaussian process supported by $[-2, 2]$ as $n \to \infty$

Gaussian fluctuation of the scaled transition measures around the semicircle distribution

Adjacency operator for conjugacy class C :

$$(A_C f)(x) = \sum_{g \in C} f(g^{-1}x)$$

Kerov's CLT again :

$$\lim_{n\to\infty} \left\langle \delta_e, \left(\frac{A_{(2,1^{n-2})}}{\sqrt{|C_{(2,1^{n-2})}|}} \right)^{p_2} \cdots \left(\frac{A_{(m,1^{n-m})}}{\sqrt{|C_{(m,1^{n-m})}|}} \right)^{p_m} \delta_e \right\rangle$$

$$= \prod_{k=2}^{m} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{p_k} e^{-x^2/2} dx$$

Quantum decomposition

$$A_{(k,1^{n-k})} = A^{+}_{(k,1^{n-k})} + A^{-}_{(k,1^{n-k})} + A^{\circ}_{(k,1^{n-k})}$$

$$\lim_{n\to\infty}(A^{+}, A^{-}, A^{\circ}) = (B^{+}, B^{-}, B^{\circ})$$

in the sense of convergence of matrix elements of any mixed product

See

Hora–Obata : Quantum Probability and Spectral Analysis of Graphs, TMP Series, Springer, 2007 (May?)

Other ensembles

- Littlewood–Richardson : Biane 1998, Śniady 2006
- Schur : Okounkov 2001, Matsumoto 2005
- discrete orthogonal polynomials : Johansson 2001
- Kerov–Olshanski–Vershik representation of $\mathfrak{S}_\infty$ ($\to z$-measure) : Borodin–Olshanski 1998–
- Jack : Fulman 2004

1-parameter deformation of the coefficients in quantum decomposition (Hora–Obata monograph)

III. THE INFINITE SYMMETRIC GROUP, WREATH PRODUCTS AND CHARACTERS

Character of a finite factor representation

G : topological group

π : continuous unitary representation of G

$\mathfrak{A} = \pi(G)''$: factor, finite type

ϕ : linear extension of the faithful normal normalized trace on $\mathfrak{A}$

$\Longrightarrow f(g) = \phi(\pi(g))$: character of π

$K_1(G) = \{$ continuous, positive definite, normalized class function on $G\}$

$E(G)$: the extremal points of $K_1(G)$

Theorem III.1 (Hirai–Hirai 2005)

Bijective correspondence between

- $E(G)$
- the quasi-equivalence classes of finite factor representations of G

Hence $E(G) = \{$ characters of $G\}$

$\mathfrak{S}_\infty = \bigcup_{n=1}^\infty \mathfrak{S}_n$: the infinite symmetric group

Theorem III.2 (Thoma 1964) $E(\mathfrak{S}_\infty)$ is parametrized by $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ and $\beta = (\beta_i)_{i\in\mathbb{N}}$ such that

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \ \beta_1 \geq \beta_2 \geq \cdots \geq 0, \ \sum_{i=1}^\infty (\alpha_i + \beta_i) \leq 1$$

$\alpha, \beta \mapsto f_{\alpha,\beta} \in E(\mathfrak{S}_\infty)$: factorizable

$$f_{\alpha,\beta}(c_k) = \sum_{i=1}^\infty (\alpha_i^k + (-1)^{k-1}\beta_i^k), \quad c_k : k\text{-cycle}, k = 2, 3, \ldots$$

Theorem III.3 (Vershik–Kerov 1981)

Given $f \in E(\mathfrak{S}_\infty)$, there exists path $t \in \mathfrak{T}$ such that

$$\lim_{n\to\infty} \frac{t(n)_i}{n} = \alpha_i, \ \lim_{n\to\infty} \frac{t(n)'_i}{n} = \beta_i, \ f = f_{\alpha,\beta} = \lim_{n\to\infty} \widetilde{\chi}^{t(n)}$$

Character $f \in E(\mathfrak{S}_\infty)$ can be approximated also by reducible characters of $\mathfrak{S}_n$ (Hirai 2004)

Dominant irreducible components

Wreath product of compact group T with $\mathfrak{S}_\infty$

$D_\infty(T)$: restricted direct product of T

$\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty \ni g = (d, \sigma) = d\sigma$

$e \neq g \in \mathfrak{S}_\infty(T)$, standard decomposition of g:

$$g = \xi_{q_1} \cdots \xi_{q_r} g_1 \cdots g_m,$$

$\xi_{q_i} = (t_i, (q_i))$; $g_j = (d_j, \sigma_j)$, σ_j: cycle, supp$d_j \subset$ suppσ_j

Conjugacy classes of $\mathfrak{S}_\infty(T)$ are described by

$$\{[t_{q_i}]\}_{i=1,\dots,r} \text{ and } \{(P_{\sigma_j}(d_j), l(\sigma_j))\}_{j=1,\dots,m},$$

cycle $\sigma = (i_1 \ \cdots \ i_l)$, $d = (t_{i_1}, \cdots, t_{i_l})$, $P_\sigma(d) = [t_{i_l} \cdots t_{i_1}]$

Theorem III.4 (Hirai–Hirai 2002–)

$E(\mathfrak{S}_\infty(T))$ is parametrized by

$$A = \left((\alpha_{\zeta,\epsilon,i})_{\zeta\in\widehat{T},\epsilon\in\{0,1\},i\in\mathbb{N}}, (\mu_\zeta)_{\zeta\in\widehat{T}}\right)$$

such that

$$\alpha_{\zeta,\epsilon,1} \geq \alpha_{\zeta,\epsilon,2} \geq \cdots \geq 0, \quad \mu_\zeta \geq 0,$$

$$\sum_{\zeta\in\widehat{T}}\Big(\sum_{\epsilon\in\{0,1\}}\sum_{i=1}^{\infty}\alpha_{\zeta,\epsilon,i} + \mu_\zeta\Big) = 1$$

$A \mapsto f_A \in E(\mathfrak{S}_\infty(T))$: factorizable

$$f_A(s,(q)) = \sum_{\zeta \in \widehat{T}} \Big(\sum_{\epsilon \in \{0,1\}} \sum_{i=1}^{\infty} \frac{\alpha_{\zeta,\epsilon,i}}{\dim \zeta} + \frac{\mu_\zeta}{\dim \zeta} \Big) \chi_\zeta(s), \quad s \in T,$$

$$f_A(d,\sigma) = \sum_{\zeta \in \widehat{T}} \Big\{ \sum_{\epsilon \in \{0,1\}} \sum_{i=1}^{\infty} (-1)^{\epsilon(k-1)} \Big(\frac{\alpha_{\zeta,\epsilon,i}}{\dim \zeta} \Big)^k \Big\} \chi_\zeta(P_\sigma(d))$$

σ: k-cycle, $k \geq 2$, supp$d \subset$ suppσ

Given $f = f_A \in E(\mathfrak{S}_\infty(T))$, construction of a finite factor representation of $\mathfrak{S}_\infty(T)$ such that

- A is visible
- $f = f_A$ is a diagonal matrix element

(Hirai–Hirai–Hora 2006)

Branching graph for $\mathfrak{S}_n(T)$'s

Equivalence classes of irreducible representations of $\mathfrak{S}_n(T)$ are parametrized by

$$\mathbb{Y}_n(T) = \Big\{ \Lambda = (\lambda^\zeta)_{\zeta \in \widehat{T}} \,\Big|\, \lambda^\zeta \in \mathbb{Y},\ \sum_{\zeta \in \widehat{T}} |\lambda^\zeta| = n \Big\}$$

$\Lambda = (\lambda^\zeta) \in \mathbb{Y}_n(T)$, $\mathrm{M} = (\mu^\zeta) \in \mathbb{Y}_{n+1}(T)$

$\Lambda \nearrow \mathrm{M} \iff \exists \zeta_0 \in \widehat{T};\ \lambda^{\zeta_0} \nearrow \mu^{\zeta_0}$

($\zeta_0 = \zeta_{\Lambda,\mathrm{M}}$: uniquely determined)

$\mathbb{Y}(T) = \bigcup_{n=0}^{\infty} \mathbb{Y}_n(T)$: vertex set

edge $\Lambda \nearrow \mathrm{M}$ with multiplicities $\dim \zeta_{\Lambda,\mathrm{M}}$

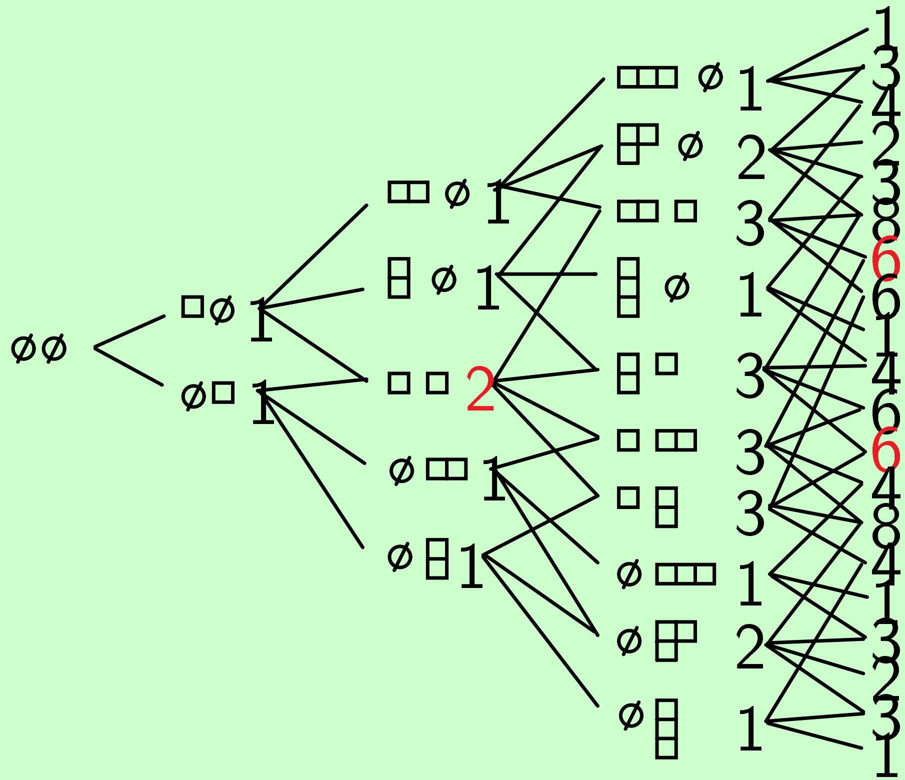

Fig. 5 Branching graph for $\mathfrak{S}_n(\mathbb{Z}_2)$ (with dimensions)

Bijective correspondence between:

(a) continuous, positive definite, normalized class functions on $\mathfrak{S}_\infty(T)$

(b) positive, normalized harmonic functions on $\mathbb{Y}(T)$

(c) certain quasi-invariant probabilities on the path space $\mathfrak{T}(T)$

Theorem III.5 (Boyer 2005 (finite T), Hirai–Hirai–Hora)

Given $f \in E(\mathfrak{S}_\infty(T))$, a.s. path $t = (t(0) \nearrow t(1) \nearrow \cdots)$, $t(n) = (t(n)^\zeta)_{\zeta \in \widehat{T}} \in \mathbb{Y}_n(T)$, w.r.t the corresponding probability admits

$$f = \lim_{n\to\infty} \widetilde{\chi}^{t(n)} \quad \text{(uniformly on each } \mathfrak{S}_k(T))$$

$$B_\zeta = \lim_{n\to\infty} \frac{|t(n)^\zeta|}{n}, \quad \zeta \in \widehat{T}, \qquad \sum_{\zeta\in\widehat{T}} B_\zeta = 1,$$

$$\alpha_{\zeta,0,i} = \lim_{n\to\infty} \frac{(t(n)^\zeta)_i}{n}, \ \alpha_{\zeta,1,i} = \lim_{n\to\infty} \frac{(t(n)^\zeta)'_i}{n}, \quad \zeta \in \widehat{T}, \ i \in \mathbb{N}$$

Compared with Theorem III.4 (character formula of Hirai),

$$\mu_\zeta = B_\zeta - \sum_{\epsilon \in \{0,1\}} \sum_{i=1}^{\infty} \alpha_{\zeta,\epsilon,i}$$

Analysis of convergence process of reducible characters of $\mathfrak{S}_n(T)$ to character of $\mathfrak{S}_\infty(T)$

Specifying dominant irreducible components

SUMMARY

- A history of asymptotic representation theory concerning symmetric groups
- Analytic description of growing Young diagrams and their limiting objects
- Limit shape of random Young diagram in the Plancherel ensemble and some fluctuations
- Character of $\mathfrak{S}_\infty(T)$ and ergodic approach
- Understanding symmetry in a huge structure through limit theorems in probability theory

($\longrightarrow$ infinite-dimensional harmonic analysis)

SEE YOU NEXT TIME