Dynamical scaling limit of the restriction-induction chain on Young diagrams in terms of free probability

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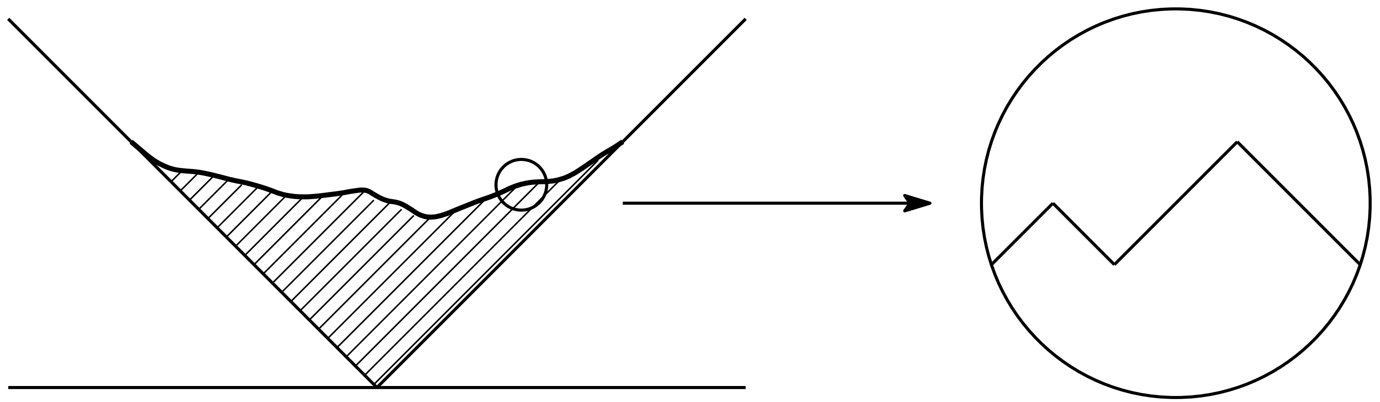


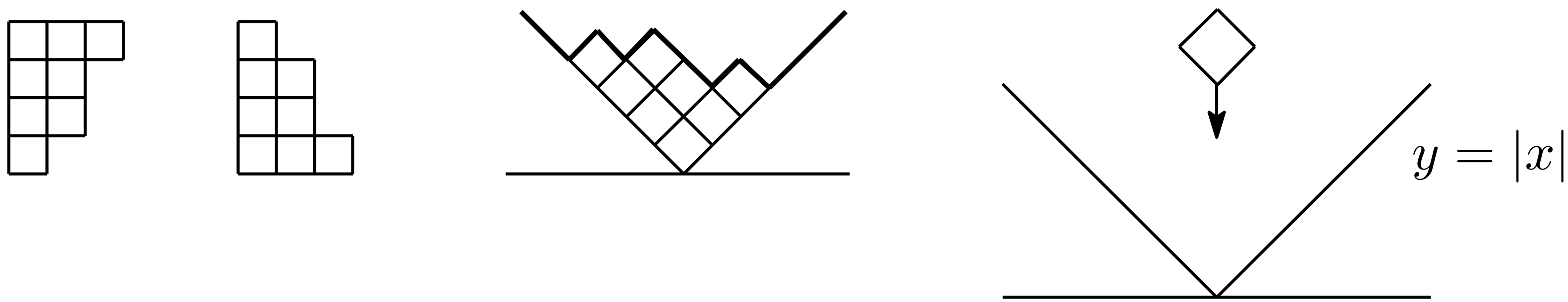
§1 Introduction

Representations of a group of matrices/permutations in a large scale have much to do with free probability theory.

From microscopic randomness of branching rule of irreducible representations

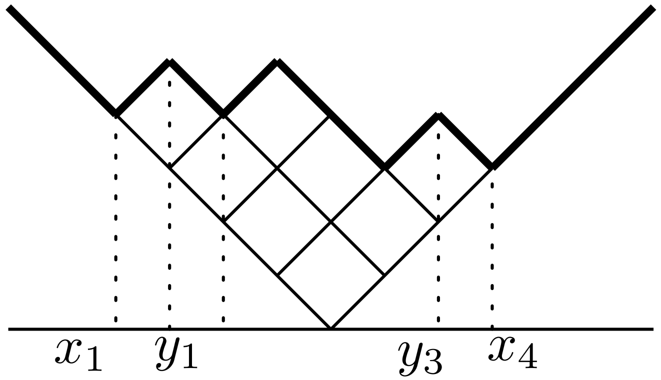
To macroscopic (deterministic) behavior with free-probabilistic structure

- scaling limit of continuous time Markov chain on Young diagram ensemble
- evolution of interfaces (formed by profile of Young diagram)



Young diagram λ is characterized by

its profile $y = \lambda(x)$ or transition measure $\mathfrak{m}_\lambda = \sum_{i=1}^{r} \mu_i \delta_{x_i} \in \mathcal{P}(\mathbb{R})$



$$\frac{(z-y_1)\cdots(z-y_{r-1})}{(z-x_1)\cdots(z-x_r)} = \frac{\mu_1}{z-x_1} + \cdots + \frac{\mu_r}{z-x_r}$$

$\Longrightarrow$ extended to continuous diagram: $y = \omega(x) \longleftrightarrow \mathfrak{m}_\omega$

(Markov transform)

★ Plancherel growth process ($\Longleftrightarrow$ induction chain) on Young diagrams

$\downarrow$ diffusive scaling limit, as effect of LLN

time evolution of macroscopic profile $\omega_0 \longrightarrow \Omega_t \longrightarrow \cdots$

$$\mathfrak{m}_{\Omega_t} = \mathfrak{m}_{\omega_0} \boxplus \mathfrak{m}_{\Omega_t^0}$$

$\mathfrak{m}_{\Omega_t^0}$: semi-circle distribution with mean 0 and variance t

$$\mathfrak{m}_{\Omega_t^0}(dx) = \frac{1}{2\pi t}\sqrt{4t-x^2}\,1_{[-2\sqrt{t},2\sqrt{t}]}(x)dx$$

$$\Omega_t^0(x) = \begin{cases} \frac{2}{\pi}\Big(x \arcsin\frac{x}{2\sqrt{t}} + \sqrt{4t-x^2}\Big), & |x| \leqq 2\sqrt{t} \\ |x|, & |x| > 2\sqrt{t} \end{cases}$$

(Ω_1^0 is limit shape due to Vershik–Kerov and Logan–Shepp)

$$\frac{\partial g}{\partial t} = -g\,\frac{\partial g}{\partial z} \qquad \text{for} \quad g(t,z) = \int_{\mathbb{R}} \frac{1}{z-x}\mathfrak{m}_{\Omega_t}(dx)$$

Aim of this talk

★ Restriction-induction chain on Young diagrams (canonical setting)

$\downarrow$ diffusive scaling limit, as effect of LLN

time evolution of macroscopic profile $\omega_0 \longrightarrow \omega_t \longrightarrow \Omega = \Omega_1^0$

$$\mathfrak{m}_{\omega_t} = (\mathfrak{m}_{\omega_0})_{e^{-t}} \boxplus (\mathfrak{m}_{\Omega})_{1-e^{-t}}$$

where initial profile ω_0 taken so that $\mathfrak{m}_{\omega_0}$ has mean 0 and variance 1

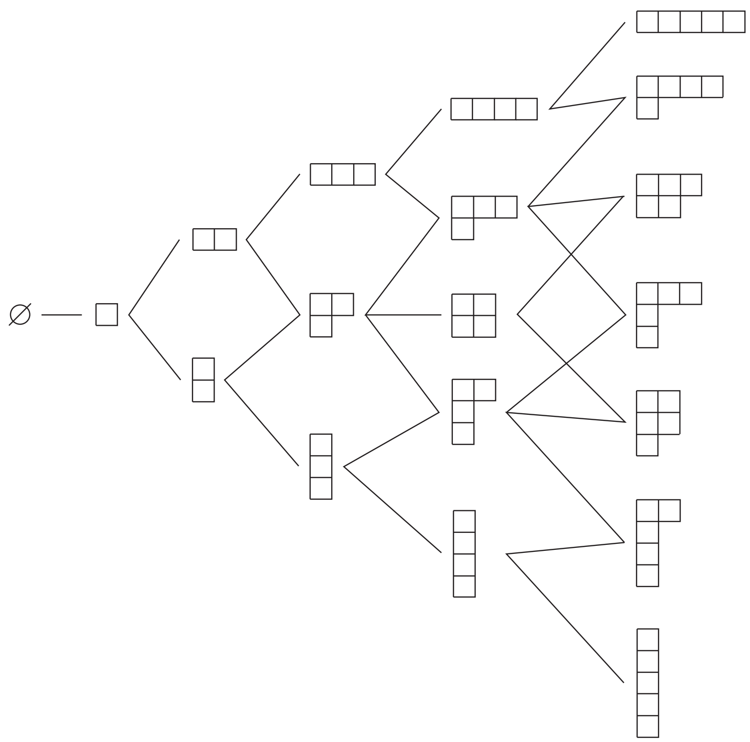
$$\frac{\partial G}{\partial t} = -G\,\frac{\partial G}{\partial z} + \frac{1}{G}\,\frac{\partial G}{\partial z} + G \qquad \text{for} \quad G(t,z) = \int_{\mathbb{R}} \frac{1}{z-x}\mathfrak{m}_{\omega_t}(dx)$$

- two remarks : grand canonical setting, evolution of global fluctuation
- two problems I cannot solve yet :
  - behavior of logarithmic energy for ω_t
  - derivation of PDE for $\omega(t,x) = \omega_t(x)$

§2 Plancherel measure and Plancherel growth process

(review) : Kerov, Vershik–Kerov, ...

Young graph vertices: $\mathbb{Y} = \bigsqcup_{n=0}^{\infty} \mathbb{Y}_n, \quad \mathbb{Y}_0 = \{\varnothing\}$



Plancherel growth process is Markov chain (Z_n) on Young diagrams with transition matrix $P^{\uparrow}$ and initial distribution $\delta_{\varnothing}$ s.t.

$$P^{\uparrow}_{\lambda,\mu} = p^{\uparrow}(\lambda,\mu) : \text{ proportional to } \dim\mu$$
$$= \frac{\dim\mu}{(|\lambda|+1)\dim\lambda}, \qquad \lambda,\mu\in\mathbb{Y}, \quad \lambda\nearrow\mu$$

Irreducible decomposition of induction of irreducible representation

$$\mathrm{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}\pi^{\lambda} \cong \bigoplus_{\mu\in\mathbb{Y}_n:\lambda\nearrow\mu}\pi^{\mu}$$

Then, the distribution after n step is

$$p_n(\varnothing,\lambda) = \mathbb{P}(Z_n=\lambda) = \frac{(\dim\lambda)^2}{n!} = \mathbb{M}_{\mathrm{Pl}}^{(n)}(\lambda)$$

called Plancherel measure on $\mathbb{Y}_n$

macroscopic profile : $1/\sqrt{n}$ both horizontally and vertically

$$\lambda \in \mathbb{Y}_n \quad \longrightarrow \quad [\lambda]^{\sqrt{n}}(x) = \frac{1}{\sqrt{n}}\lambda(\sqrt{n}x), \quad [\lambda]^{\sqrt{n}} \in \mathbb{D}_0 \subset \mathbb{D}$$

• rectangular diagram

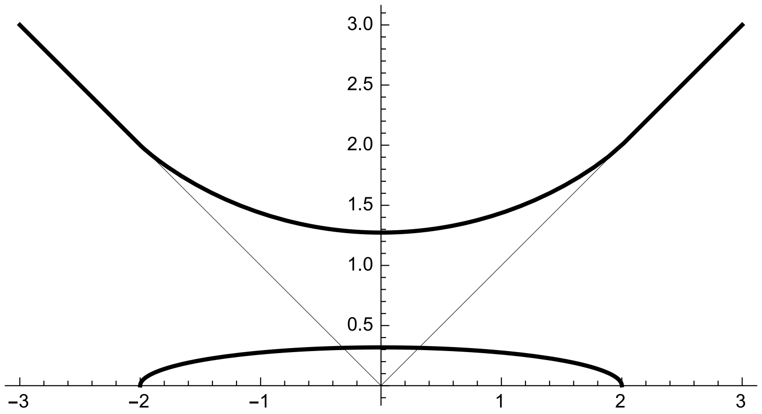
$$\mathbb{D}_0 = \Big\{\lambda : \mathbb{R} \longrightarrow \mathbb{R} \,\Big|\, \text{continuous, piecewise linear,} \\ \lambda'(x) = \pm 1,\ \lambda(x) = |x| \ (|x| \text{ large enough})\Big\}$$

• continuous diagram

$$\mathbb{D} = \Big\{\omega : \mathbb{R} \longrightarrow \mathbb{R} \,\Big|\, |\omega(x) - \omega(y)| \leqq |x-y|,\ \omega(x) = |x| \ (|x| \text{ large enough})\Big\}$$

$$\Omega(x) = \begin{cases} \frac{2}{\pi}\big(x \arcsin\frac{x}{2} + \sqrt{4-x^2}\big), & |x| \leqq 2 \\ |x|, & |x| > 2 \end{cases} \qquad \text{limit shape}$$

The following LLN holds (static scaling limit for the Plancherel measure)



Vershik – Kerov 1977, Logan – Shepp 1977

$$\mathbb{M}_{\mathrm{Pl}}^{(n)}\Big(\Big\{\lambda \in \mathbb{Y}_n \;\Big|\; \sup_{x\in\mathbb{R}} |[\lambda]^{\sqrt{n}}(x) - \Omega(x)| \geqq \epsilon\Big\}\Big) = \mathbb{P}\big(\|[Z_n]^{\sqrt{n}} - \Omega\|_{\sup} \geqq \epsilon\big)$$

$$\xrightarrow[n\to\infty]{} 0 \qquad\qquad (\forall \epsilon > 0)$$

Namely, $[Z_n]^{\sqrt{n}}$ converges to Ω in probability as $n \to \infty$.

Continuous time Plancherel growth process $\tilde{Z}_s = Z_{N_s}$

– $(N_s)_{s \geqq 0}$: Poisson process on $\{0, 1, \cdots\}$, $N_0 = 0$ a.s., independent of (Z_n)

– initial distribution $\delta_{\varnothing}$

– transition matrix $e^{s(P^{\uparrow} - I)}$

$$\tilde{\mathbb{P}}(\tilde{Z}_s = \lambda) = \sum_{n=0}^{\infty} \frac{e^{-s} s^n}{n!} \mathbb{M}_{\mathrm{Pl}}^{(n)}(\lambda), \qquad \lambda \in \mathbb{Y}$$

(Poissonization of the Plancherel measures)

Dynamical scaling limit

s: microscopic time, t: macroscopic time $\quad s = tn$

Then $\quad [\tilde{Z}_{tn}]^{\sqrt{n}} \xrightarrow[n \to \infty]{} ?$

$$\tilde{\mathbb{P}}\big(\|[\tilde{Z}_{tn}]^{\sqrt{n}} - \Omega_t^0\|_{\sup} \geqq \epsilon\big) = \tilde{\mathbb{P}}^{\tilde{Z}_{tn}}\big(\|[\lambda]^{\sqrt{n}} - \Omega_t^0\|_{\sup} \geqq \epsilon\big)$$
$$= \sum_{k=0}^{\infty} \frac{e^{-tn}(tn)^k}{k!} \mathbb{M}_{\mathrm{Pl}}^{(k)}\big(\|[\lambda]^{\sqrt{n}} - \Omega_t^0\|_{\sup} \geqq \epsilon\big)$$

The above Poisson distribution has mean tn and standard deviation $\sqrt{tn}$

Under $\mathbb{M}_{\mathrm{Pl}}^{(\lfloor tn \rfloor)}$, $[\lambda]^{\sqrt{tn}} \to \Omega_1^0 \iff [\lambda]^{\sqrt{n}} \to \Omega_t^0$ where

$$\Omega_t^0(x) = \begin{cases} \frac{2}{\pi}\big(x \arcsin \frac{x}{2\sqrt{t}} + \sqrt{4t - x^2}\big), & |x| \leqq 2\sqrt{t} \\ |x|, & |x| > 2\sqrt{t} \end{cases}$$

Proposition $[\tilde{Z}_{tn}]^{\sqrt{n}} \xrightarrow[n\to\infty]{} \Omega_t^0$ in probability ■

§3 Restriction-induction chain

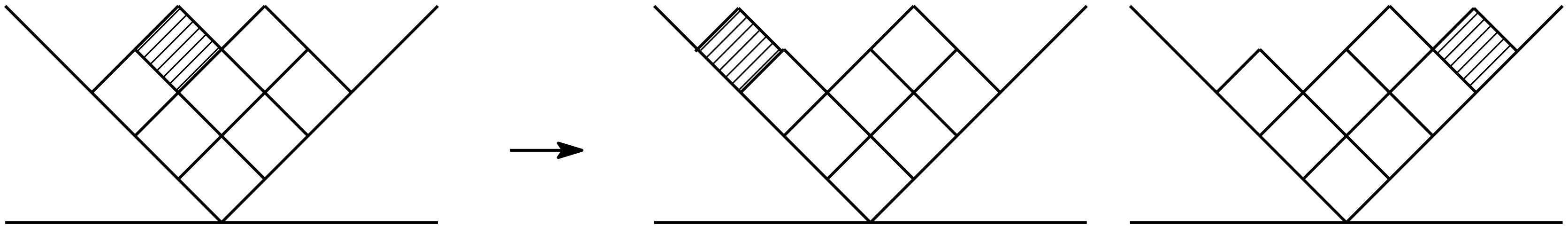
$$p^{\downarrow}(\lambda,\mu) \quad \text{(proportional to } \dim\mu) \; = \begin{cases} \frac{\dim\mu}{\dim\lambda}, & \mu\nearrow\lambda \\ 0, & \text{otherwise} \end{cases}$$

$$p^{\uparrow}(\lambda,\mu) \quad \text{as before (proportional to } \dim\mu)$$

Irreducible decomposition of restriction and induction of irreducible representation

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}\pi^{\lambda} \cong \bigoplus_{\nu\in\mathbb{Y}_{n-1}:\nu\nearrow\lambda}\pi^{\nu}, \qquad \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}\pi^{\nu} \cong \bigoplus_{\mu\in\mathbb{Y}_{n}:\nu\nearrow\mu}\pi^{\mu}$$

restriction $\leftrightarrow$ removing 1 box, induction $\leftrightarrow$ adding 1 box



Res-Ind chain $(X_m^{(n)})_{m=0,1,2,\dots}$ on $\mathbb{Y}_n$ has transition matrix

$P^{(n)} = P^{\downarrow} P^{\uparrow} = \big(p^{(n)}(\lambda,\mu)\big)_{\lambda,\mu\in\mathbb{Y}_n}$

$$p^{(n)}(\lambda,\mu) = \sum_{\nu\in\mathbb{Y}_{n-1}:\nu\nearrow\lambda,\nu\nearrow\mu} p^{\downarrow}(\lambda,\nu)p^{\uparrow}(\nu,\mu), \qquad \lambda,\mu\in\mathbb{Y}_n$$

Lemma Res-Ind chain is symmetric w.r.t. the Plancherel measure:

$$\mathbb{M}_{\mathrm{Pl}}^{(n)}(\lambda)p^{(n)}(\lambda,\mu) = \mathbb{M}_{\mathrm{Pl}}^{(n)}(\mu)p^{(n)}(\mu,\lambda), \quad \lambda,\mu\in\mathbb{Y}_n,$$

hence the Plancherel measure is invariant distribution for Res-Ind chain ∎

Restriction-induction chain is formerly dealt with e.g. in

- Fulman 2004, 2005 :

  to construct exchangeable r.v.s distributed in Plancherel measure to apply Stein method

- Borodin – Olshanski 2009 :

  to construct diffusion process on Thoma simplex under rescale of time $t = s/n^2$, space $1/n$ capturing factorial representations of $\mathfrak{S}_\infty$ (instead of limit shape)

Remove or add one box, treating each corner equally

Funaki – Sasada 2010 :

hydrodynamic limit for an evolutional model

Recall scheme of the problem

For continuous time Markov chain $(Y_s^{(n)})_{s \geqq 0}$ on $\mathbb{Y}_n$,

limiting behavior as $n \to \infty$ and $s \to \infty$ under scaling in space vs time

– macroscopic profile : $1/\sqrt{n}$ both horizontally and vertically

$$\lambda \in \mathbb{Y}_n \quad \longrightarrow \quad [\lambda]^{\sqrt{n}}(x) = \frac{1}{\sqrt{n}}\lambda(\sqrt{n}x), \qquad [\lambda]^{\sqrt{n}} \in \mathbb{D}$$

– macroscopic time : $t = s/n$ (diffusive scale)

Letting $n \to \infty$, as an effect of LLN, the distribution of $[Y_{tn}^{(n)}]^{\sqrt{n}}$ concentrates at a point ω_t, depending on t.

ω_t : macroscopic profile at macroscopic time t

Describe evotuion of ω_t along t !

Continuous time Res-Ind chain $\tilde{X}^{(n)}_s = X^{(n)}_{N_s}$ on $\mathbb{Y}_n$ with

- $(N_s)_{s \geqq 0}$: Poisson process independent of $(X^{(n)}_m)$
- transition matrix $e^{s(P^{(n)}-I)}$,
- initial distribution $\delta_{\lambda^{(n)}}$,
- invariant distribution $\mathbb{M}^{(n)}_{\mathrm{Pl}}$

Dynamic scaling limit

s: microscopic time, t: macroscopic time $\quad s = tn$

Then $[\tilde{X}_{tn}{}^{(n)}]^{\sqrt{n}} \xrightarrow[n\to\infty]{} ?$ (macroscopic profile depending on t)

Let $\mathbb{M}^{(n)}_t = \tilde{\mathbb{P}}^{\tilde{X}^{(n)}_{tn}}$: distribution of $\tilde{X}^{(n)}_{tn}$ on $\mathbb{Y}_n$

Theorem (PRIMS 2015, SBMP 2016)

If initial condition satisfies $[\lambda^{(n)}]^{\sqrt{n}} \longrightarrow \omega_0 \in \mathbb{D}$ as $n \to \infty$, then, for $\forall t > 0$, there exists $\omega_t \in \mathbb{D}$ s.t. LLN

$$\mathbb{M}_t^{(n)}\Big(\Big\{\lambda \in \mathbb{Y}_n \;\Big|\; \|[\lambda]^{\sqrt{n}} - \omega_t\|_{\sup} \geqq \epsilon\Big\}\Big) \xrightarrow[n\to\infty]{} 0 \quad (\forall \epsilon > 0)$$

holds. ■

– ω_0 can be taken arbitrarily in $\mathbb{D}$ s.t. $\displaystyle\int_{\mathbb{R}} \big(\omega_0(x) - |x|\big)dx = 2$

– ω_t converges to Ω (limit shape) in $\mathbb{D}$ as $t \to \infty$

– The area is kept invariant: $\displaystyle\int_{\mathbb{R}} (\omega_t(x) - |x|)dx = 2 \quad$ for $\forall t$

– ω_t is described precisely by using free probability (as seen later)

Remark For a sequence of probability spaces $(\mathbb{Y}_n, \mathbb{M}^{(n)})$, we know some sufficient condition for LLN

$$\mathbb{M}^{(n)}\Big(\Big\{\lambda \in \mathbb{Y}_n \;\Big|\; \|[\lambda]^{\sqrt{n}} - \psi\|_{\sup} \geqq \epsilon\Big\}\Big) \xrightarrow[n\to\infty]{} 0 \quad (\forall \epsilon > 0)$$

to hold with some continuous diagram $\psi \in \mathbb{D}$, which we call a concentration property at ψ (approximate factorization property of Biane 2001).

Examples

- $\mathbb{M}^{(n)} = \delta_{\lambda^{(n)}}$ for $[\lambda^{(n)}]^{\sqrt{n}} \to \omega_0 \in \mathbb{D}$ as $n \to \infty$

  (ω_0 then satisfies $\displaystyle\int_{\mathbb{R}} (\omega_0(x) - |x|)dx = 2$)

- $\mathbb{M}^{(n)} = \mathbb{M}^{(n)}_{\mathrm{Pl}}$ (Plancherel measure)

Initial distribution can be generalized to one satisfying this concentration property

Theorem$^\sharp$ (PRIMS 2015, SBMP 2016)

The concentration property is propagated as time goes by; i.e. if initial distributions $\mathbb{M}_0^{(n)}$ satisfy the concentration property at $\omega_0 \in \mathbb{D}$, then $\mathbb{M}_t^{(n)}$ also satisfy the concentration property for $\forall t > 0$, hence there exists $\omega_t \in \mathbb{D}$ s.t. LLN

$$\mathbb{M}_t^{(n)}\left(\left\{\lambda \in \mathbb{Y}_n \;\middle|\; \|[\lambda]^{\sqrt{n}} - \omega_t\|_{\sup} \geqq \epsilon\right\}\right) \xrightarrow[n\to\infty]{} 0 \quad (\forall \epsilon > 0)$$

holds.

Here ω_t is determined by

$$\mathfrak{m}_{\omega_t} = (\mathfrak{m}_{\omega_0})_{e^{-t}} \boxplus (\mathfrak{m}_{\Omega})_{1-e^{-t}}$$

(free convolution of free compressions of transition measures).

Furthermore time evolution of the distribution is described through its Stieltjes transform

$$G(t,z) = \int_{\mathbb{R}} \frac{1}{z-x} \mathfrak{m}_{\omega_t}(dx).$$

PDE describing time evolution of transition measure $\mathfrak{m}_{\omega_t}$

$$\frac{\partial G}{\partial t} = -G\,\frac{\partial G}{\partial z} + \frac{1}{G}\,\frac{\partial G}{\partial z} + G, \qquad t > 0,\ z \in \mathbb{C}^+$$

■

Remark For Plancherel growth process also, consider initial distribution $\mathbb{M}_0^{(n)}$ on $\mathbb{Y}_{\lfloor an \rfloor} \subset \mathbb{Y}$ satisfying the concentration property at $\omega_0 \in \mathbb{D}$ where $\frac{1}{2}\int_{\mathbb{R}}(\omega_0(x) - |x|)dx = a$. Then, $[Z_{N_{tn}}]^{\sqrt{n}} \longrightarrow \Omega_t$ in probability as $n \to \infty$.

Transition measure of Ω_t^0 (limit shape of Plancherel growth process at time t) is semicircle distribution of mean 0 and variance t,

$$\int_{\mathbb{R}} \frac{1}{z-x}\mathfrak{m}_{\Omega_t^0}(dx) = \frac{z-\sqrt{z^2-4t}}{2t}$$

$$\mathfrak{m}_{\Omega_t} = \mathfrak{m}_{\omega_0} \boxplus \mathfrak{m}_{\Omega_t^0}$$

Stieltjes transform $g(t,z) = \int_{\mathbb{R}} \frac{1}{z-x}\mathfrak{m}_{\Omega_t}(dx)$ also satisfies PDE :

$$\frac{\partial g}{\partial t} = -g\,\frac{\partial g}{\partial z}$$

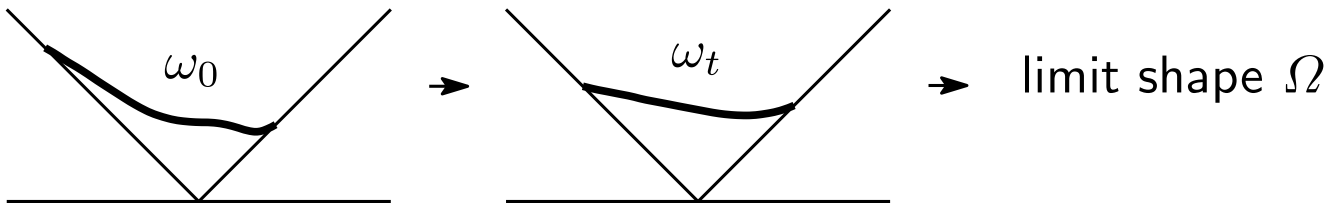
initial distribution $\mathbb{M}_0^{(n)} \quad \longrightarrow \quad \omega_0$

$\downarrow \qquad 1/\sqrt{n}, \quad n \to \infty$

distribution at time tn $\mathbb{M}_t^{(n)} \quad \longrightarrow \quad \omega_t$

$\downarrow$

invariant distribution $\mathbb{M}_{\mathrm{Pl}}^{(n)} \quad \longrightarrow \quad \Omega$



§4 Remarks and Problems

♣ Evolution of global fluctuation (in progress)

fluctuation for other (non-Plancherel) ensembles

"character factorization property"   c.f. Śniady 2005

▶ In the Res-Ind model, character factorization property is propagated at any macroscopic time t.

Hence, if initial ensemble $(\mathbb{Y}_n, \mathbb{M}_0^{(n)})$ has character factorization property, $\sqrt{n}\big(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_{\omega_t}\big)$ on $(\mathbb{Y}_n, \mathbb{M}_t^{(n)})$ converges as $n \to \infty$ to the fluctuation at t, namely $\big\{\langle x^j, \sqrt{n}(\mathfrak{m}_{\lambda^{\sqrt{n}}} - \mathfrak{m}_{\omega_t})\rangle\big\}_j$ converges as $n \to \infty$ to Gaussian system with mean 0 and some covariance with complicated t-dependence.

♣ Res-Ind model in grand canonical setting (in progress)

Poissonization of the Plancherel measure

$$\mathbb{M}_{\mathrm{PP}}^{(\xi)} = \sum_{n=0}^{\infty} \frac{e^{-\xi}\xi^n}{n!} \mathbb{M}_{\mathrm{Pl}}^{(n)}, \qquad \xi > 0$$

is kept invariant under transition probability $P^{(\xi)}$ on $\mathbb{Y}$:

$$P^{(\xi)} = \alpha_\xi(n) P^{\uparrow(n)} + (1 - \alpha_\xi(n)) P^{\downarrow(n)},$$

$$\alpha_\xi(n) = \int_0^1 \xi e^{-\xi x}(1-x)^n dx$$

Continuous time Markov chain $(X_{N_s}^{(\xi)})_{s \geqq 0}$

Rescale for time $s = t\xi$, for space $[\lambda]^{\sqrt{\xi}} = \frac{1}{\sqrt{\xi}} \lambda(\sqrt{\xi} x)$ ($\lambda \in \mathbb{Y}$)

▶ Let initial ensemble $(\mathbb{Y}, \mathbb{M}_0^{(\xi)})$ satisfy the concentration property as $\xi \to \infty$, for example Poissonization of $(\mathbb{Y}_n, \mathbb{M}_0^{(n)})$, each satisfying concentration property as $n \to \infty$. Then $[X_{N_{t\xi}}^{(\xi)}]^{\sqrt{\xi}}$ converges to some ω_t in probability as $\xi \to \infty$.

Macroscopic profile ω_t is characterized by

$$\mathfrak{m}_{\omega_t} = (\mathfrak{m}_{\omega_0})_{e^{-t/2}} \boxplus (\mathfrak{m}_{\Omega})_{1-e^{-t/2}}, \qquad t > 0$$

♠ Logarithmic energy : taking

$$\mathbb{M}_{\mathrm{Pl}}^{(n)}(\lambda) = (1+o(1))\sqrt{2\pi n}\exp\Big\{-n\Big(1+\frac{1}{2}\iint_{\{s>t\}}\big(1-([\lambda]^{\sqrt{n}})'(s)\big) \\ \big(1+([\lambda]^{\sqrt{n}})'(t)\big)\log(s-t)dsdt + O\big(\frac{1}{\sqrt{n}}\big)\Big)\Big\}$$

into account, set for $\omega \in \mathbb{D}$

$$\theta(\omega) = 1+\frac{1}{2}\iint_{\{s>t\}}\big(1-\omega'(s)\big)\big(1+\omega'(t)\big)\log(s-t)dsdt$$

We know

- Limit shape Ω is unique minimizer for θ in $\mathbb{D}$ $(\theta(\Omega)=0)$.
- In Res-Ind model, ω_t converges to Ω as $t \to \infty$.

Problem Is $\theta(\omega_t)$ decreasing for sufficiently large t ?

♠ PDE describing Res-Ind model

Stieltjes transform $G(t, z)$ of $\mathfrak{m}_{\omega_t}$ satisfies

$$\frac{\partial G}{\partial t} = -G\,\frac{\partial G}{\partial z} + \frac{1}{G}\,\frac{\partial G}{\partial z} + G, \qquad t > 0,\ z \in \mathbb{C}^+$$

Problem Find PDE for $\omega(t, x) = \omega_t(x)$ itself !

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