Effect of microscopic pausing time distribution on evolution of macroscopic profiles in Young diagram ensembles

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Non-commutative probability and related fields

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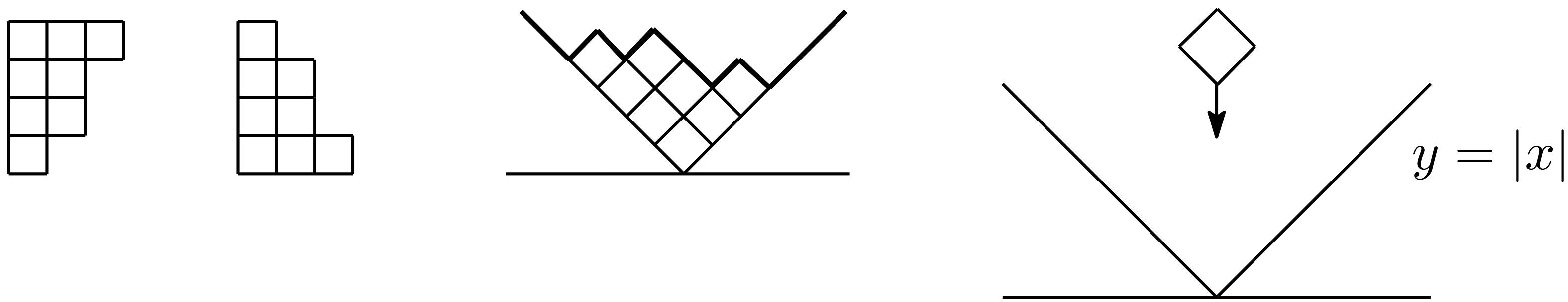
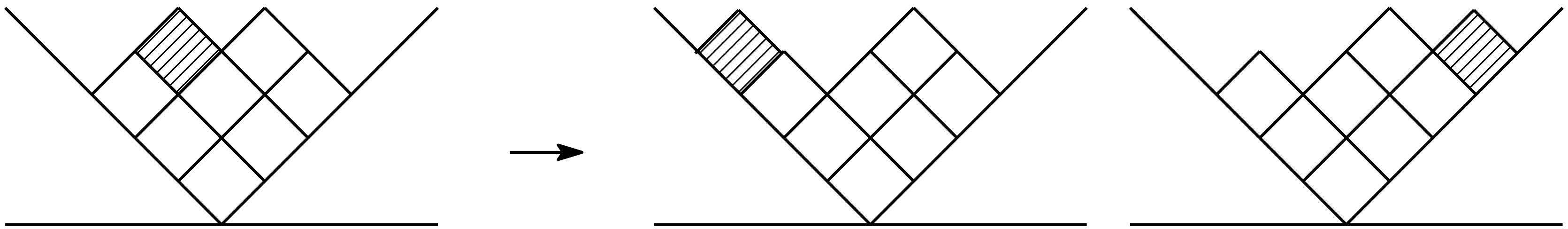


§1 Introduction

$\mathbb{Y}_n = \{$Young diagram with n boxes$\}$, $\qquad \mathbb{Y} = \bigsqcup_{n=0}^{\infty} \mathbb{Y}_n$

Markov chain on $\mathbb{Y}_n$ caused by (non-local) movement of a corner box



called Res-Ind chain $(Z_k^{(n)})_{k\in\{0,1,2,\cdots\}}$ on $\mathbb{Y}_n$

$\mathbb{Y}_n \cong \widehat{\mathfrak{S}}_n$: the equivalence classes of irreducible representations of the symmetric group of degree n

restriction $\leftrightarrow$ removing box, induction $\leftrightarrow$ adding box

Irreducible decomposition of restriction and induction of an irreducible representation of symmetric group

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \pi^\lambda \cong \bigoplus_{\nu \in \mathbb{Y}_{n-1} : \lambda \searrow \nu} \pi^\nu, \qquad \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \pi^\nu \cong \bigoplus_{\mu \in \mathbb{Y}_n : \nu \nearrow \mu} \pi^\mu$$

$$p^\downarrow(\lambda, \nu) = \begin{cases} \frac{\dim \nu}{\dim \lambda}, & \lambda \searrow \nu \\ 0, & \text{otherwise,} \end{cases} \qquad p^\uparrow(\nu, \mu) = \begin{cases} \frac{\dim \mu}{(|\nu|+1) \dim \nu}, & \nu \nearrow \mu \\ 0, & \text{otherwise} \end{cases}$$

both proportional to dimension of terminal diagram

Set

$$p^{(n)}(\lambda, \mu) = \sum_{\nu \in \mathbb{Y}_{n-1} : \lambda \searrow \nu, \nu \nearrow \mu} p^\downarrow(\lambda, \nu) p^\uparrow(\nu, \mu), \qquad \lambda, \mu \in \mathbb{Y}_n$$

Transition matrix of Res-Ind chain on $\mathbb{Y}_n$:

$$P^{(n)} = P^\downarrow P^\uparrow = \big(p^{(n)}(\lambda, \mu)\big)_{\lambda, \mu \in \mathbb{Y}_n}$$

Planchere measure on $\mathbb{Y}_n$

$$\mathrm{M}_{\mathrm{Pl}}^{(n)}(\{\lambda\}) = \frac{(\dim\lambda)^2}{n!}, \qquad \lambda \in \mathbb{Y}_n$$

Lemma Res-Ind chain is symmetric w.r.t. the Plancherel measure:

$$\mathrm{M}_{\mathrm{Pl}}^{(n)}(\{\lambda\})p^{(n)}(\lambda,\mu) = \mathrm{M}_{\mathrm{Pl}}^{(n)}(\{\mu\})p^{(n)}(\mu,\lambda), \quad \lambda,\mu \in \mathbb{Y}_n,$$

hence the Plancherel measure is invariant distribution for Res-Ind chain

$(Z_k^{(n)})_{k\in\{0,1,2,\cdots\}}$: Res-Ind chain on $\mathbb{Y}_n$ with initial distribution $M_0^{(n)}$

Construct continuous time random walk on $\mathbb{Y}_n$ from transition matrix $P^{(n)}$

$\{\tau_j\}_{j\in\mathbb{N}}$: i.i.d. random variables obeying ψ ($\neq \delta_0$) on $[0,\infty)$,

independent of $\{(Z_k^{(n)})_{k\in\{0,1,2,\cdots\}}\}_{n\in\mathbb{N}}$

$(N_s)_{s\geqq 0}$: counting process with τ_j's as pausing intervals

$$N_s = \sup\{j \in \mathbb{N} \,|\, \tau_1 + \cdots + \tau_j \leqq s\} < \infty \text{ a.s.}, \quad N_0 \equiv 0 \text{ a.s.}$$

Set $\qquad X_s^{(n)} = Z_{N_s}^{(n)}, \qquad s \geqq 0$

$$\begin{aligned}\mathbb{P}(X_s^{(n)} = \mu) &= \sum_{j=0}^{\infty} \mathbb{P}(Z_j^{(n)} = \mu)\,\mathbb{P}(\tau_1 + \cdots + \tau_j \leqq s, \tau_1 + \cdots + \tau_{j+1} > s) \\ &= \sum_{j=0}^{\infty} (M_0^{(n)} P^{(n)\,j})_\mu \int_{[0,s]} \psi((s-u,\infty))\,\psi^{*j}(du)\end{aligned}$$

Continuous time random walk $(X_s^{(n)})_{s \geqq 0}$ on $\mathbb{Y}_n$ as microscopic dynamics

$\longrightarrow$ scaling limit in space and time

– macroscopic profile : $1/\sqrt{n}$ both horizontally and vertically

$$\lambda \in \mathbb{Y}_n \quad \longrightarrow \quad [\lambda]^{\sqrt{n}}(x) = \frac{1}{\sqrt{n}}\lambda(\sqrt{n}x) \qquad \text{constant “area”}$$

– macroscopic time : $t = s/\alpha(n)$ $\qquad \alpha(n)$: micro/macro scaling factor

Dynamic scaling limit

$[X_{t\alpha(n)}{}^{(n)}]^{\sqrt{n}} \xrightarrow{n\to\infty}$ (deterministic) macroscopic shape depending on t

from LLN (concentration phenomenon)

§2 Concentration phenomenon (static model)

LLN in probability theory

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow{n \to \infty} \text{constant}$$

if $X_1, X_2, \cdots$: independent + some conditions

$$\mathbb{E}[e^{i(\xi_1 X_1 + \cdots + \xi_n X_n)}] = \prod_{k=1}^{n} \mathbb{E}[e^{i\xi_k X_k}] \quad \text{(factorizability of characteristic function)}$$

Continuous diagram

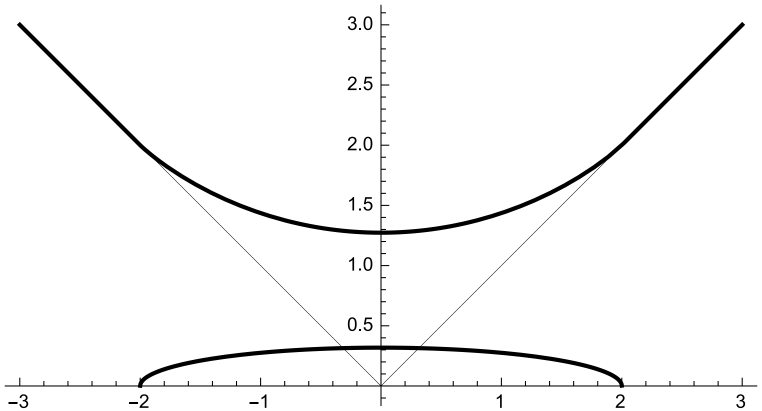
$$\mathbb{D} = \Big\{ \omega : \mathbb{R} \longrightarrow \mathbb{R} \,\Big|\, |\omega(x) - \omega(y)| \leqq |x - y|,\ \omega(x) = |x| \ \text{(for } |x| \text{ large enough)} \Big\}$$

The following LLN holds (static scaling limit for the Plancherel measure)

Vershik – Kerov 1977, Logan – Shepp 1977

$$\Omega(x) = \begin{cases} \frac{2}{\pi}\left(x \arcsin\frac{x}{2} + \sqrt{4-x^2}\right), & |x| \leqq 2 \\ |x|, & |x| > 2 \end{cases}$$

limit shape



$$\mathrm{M}_{\mathrm{Pl}}^{(n)}\left(\left\{\lambda \in \mathbb{Y}_n \;\middle|\; \sup_{x\in\mathbb{R}}\left|[\lambda]^{\sqrt{n}}(x) - \Omega(x)\right| \geqq \epsilon\right\}\right) \xrightarrow{n\to\infty} 0 \qquad (\forall \epsilon > 0)$$

For a sequence of probability spaces $\{(\mathbb{Y}_n, M^{(n)})\}_{n\in\mathbb{N}}$, we know some sufficient condition for LLN

$$M^{(n)}\Big(\Big\{\lambda \in \mathbb{Y}_n \;\Big|\; \big\|[\lambda]^{\sqrt{n}} - \psi\big\|_{\sup} \geqq \epsilon\Big\}\Big) \xrightarrow[n\to\infty]{} 0 \quad (\forall \epsilon > 0)$$

to hold with some continuous diagram $\psi \in \mathbb{D}$

$\lambda \in \mathbb{Y}_n \;\cong\; \widehat{\mathfrak{S}}_n, \qquad \rho \in \mathbb{Y}_n \;\cong\; \{$conjugacy classes of $\mathfrak{S}_n\}$

$\chi^\lambda_\rho =$ value at an element in conjugacy class labeled by ρ of irreducible character labeled by λ

$\widetilde{\chi}^\lambda = \chi^\lambda / \dim\lambda$: normalized irreducible character

$\mathbb{Y}^\times = \{\rho \in \mathbb{Y} \,|\, \rho$ has no one-box rows$\}$

$l(\rho) = \sharp$ of rows of $\rho \in \mathbb{Y}$

Approximate factorization property of Biane: for any $\rho, \sigma \in \mathbb{Y}^\times$

$$\mathbb{E}_{M^{(n)}}\left[\widetilde{\chi}^{\cdot}_{(\rho\sqcup\sigma,1^{n-|\rho|-|\sigma|})}\right]-\mathbb{E}_{M^{(n)}}\left[\widetilde{\chi}^{\cdot}_{(\rho,1^{n-|\rho|})}\right]\mathbb{E}_{M^{(n)}}\left[\widetilde{\chi}^{\cdot}_{(\sigma,1^{n-|\sigma|})}\right]$$
$$= o\big(n^{-\frac{1}{2}(|\rho|-l(\rho)+|\sigma|-l(\sigma))}\big) \qquad (n\to\infty)$$

combined with some conditions implies concentration at some $\psi \in \mathbb{D}$.

Examples

- $M^{(n)} = \delta_{\lambda^{(n)}}$ for $[\lambda^{(n)}]^{\sqrt{n}} \to \omega_0 \in \mathbb{D}$ as $n \to \infty$
  (ω_0 then satisfies $\int_{\mathbb{R}}(\omega_0(x)-|x|)dx = 2$)

Irreducible decomposition of some representation of symmetric group

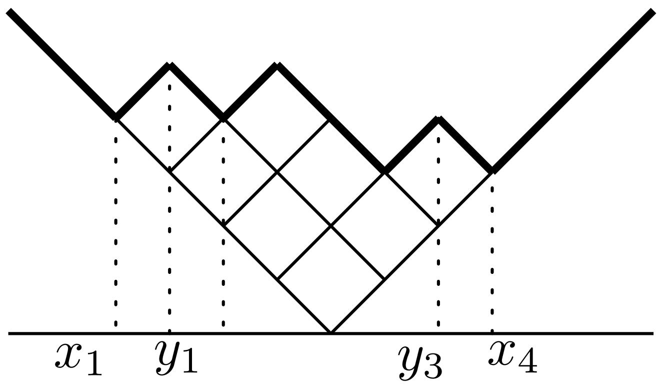
- $M^{(n)} = \mathrm{M}^{(n)}_{\mathrm{Pl}}$ (Plancherel measure) $\longleftarrow$ regular representation
- Littlewood-Richardson measure

$\longleftarrow$ outer product of irreducible representations

§3 Dynamical limit shape (main result)

Young diagram λ is characterized by

its profile $y = \lambda(x)$ or transition measure $\mathfrak{m}_\lambda = \sum_{i=1}^{r} \mu_i \delta_{x_i} \in \mathcal{P}(\mathbb{R})$



$$\frac{(z-y_1)\cdots(z-y_{r-1})}{(z-x_1)\cdots(z-x_r)} = \frac{\mu_1}{z-x_1} + \cdots + \frac{\mu_r}{z-x_r}$$

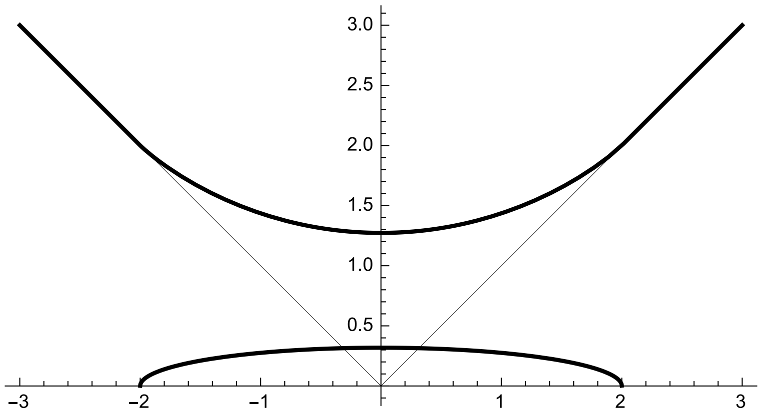
$\Longrightarrow$ extended to continuous diagram: $y = \omega(x) \;\longleftrightarrow\; \mathfrak{m}_\omega$

Markov transform

$$\frac{1}{z}\exp\Big\{\int_{\mathbb{R}} \frac{1}{x-z}\Big(\frac{\omega(x)-|x|}{2}\Big)' dx\Big\} = \int_{\mathbb{R}} \frac{1}{z-x}\mathfrak{m}_\omega(dx), \qquad z \in \mathbb{C}^+$$

e.g.

limit shape $\Omega \;\longleftrightarrow\; \mathfrak{m}_\Omega$: standard semi-circle distribution



Free convolution, free compression

(A, ϕ) : probability space

self-adjoint $a \in A$, $\quad \mu \in \mathcal{P}_c(\mathbb{R})$ with compact support

$a \sim \mu \iff \phi(a^k) = M_k(\mu)$ for any $k \in \{0, 1, 2, \cdots\}$

Given $\mu, \nu \in \mathcal{P}_c(\mathbb{R})$, $a \sim \mu$, $b \sim \nu$, a and b are free, then

$a + b \sim \mu \boxplus \nu \in \mathcal{P}_c(\mathbb{R})$: free convolution of μ and ν

$p \in A$: projection free from a, $\phi(p) = \alpha > 0$,

pap in $(pAp, \alpha^{-1}\phi|_{pAp}) \sim \mu_\alpha \in \mathcal{P}_c(\mathbb{R})$: free compression of μ

- $R_k(\mu \boxplus \nu) = R_k(\mu) + R_k(\nu)$, $\qquad k \in \mathbb{N}$
- $R_k(\mu_\alpha) = \alpha^{k-1} R_k(\mu)$, $\qquad k \in \mathbb{N}$

Theorem 1

The continuous time random walk $(X_s^{(n)})_{s \geqq 0}$ with distribution at time $s \geqq 0$:

$$M_s^{(n)}(\{\lambda\}) = \mathbb{P}(X_s^{(n)} = \lambda), \qquad \lambda \in \mathbb{Y}_n$$

Assume

- the sequence of initial distributions $\{(\mathbb{Y}_n, M_0^{(n)})\}_{n \in \mathbb{N}}$ has concentration property at $\omega_0 \in \mathbb{D}$
- the pausing time distribution ψ with characteristic function φ satisfies:
  - φ is differentiable at 0 (ψ having mean $m > 0$)
  - integrability condition for φ : for some $\delta > 0$

$$\int_{\{|\xi| \geqq \delta\}} \left| \frac{\varphi(\xi)}{\xi} \right| d\xi < \infty.$$

Then

for macroscopic time $t > 0$ and microscopic time $s = tn$

i.e. scaling factor $\alpha(n) = n$,

concentration property is propagated along t, hence the sequence $\{(\mathbb{Y}_n, M_{tn}^{(n)})\}_{n\in\mathbb{N}}$ has concentration at some $\omega_t \in \mathbb{D}$. The macroscopic profile ω_t is characterized by

$$\mathfrak{m}_{\omega_t} = (\mathfrak{m}_{\omega_0})_{e^{-t/m}} \boxplus (\mathfrak{m}_{\Omega})_{1-e^{-t/m}}$$

and recovered by using the Markov transform.

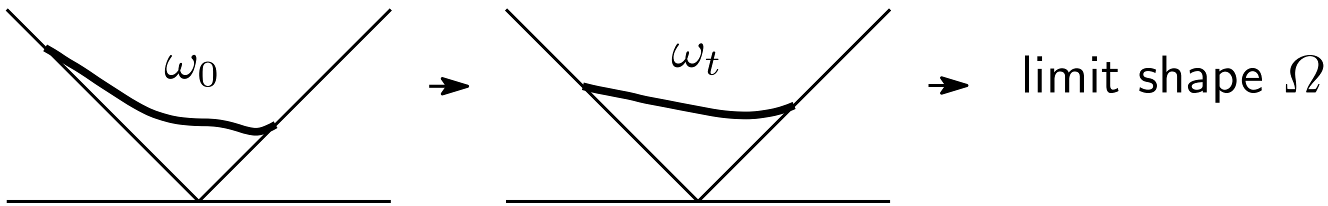
initial distribution $M_0^{(n)}$ $\longrightarrow$ ω_0

$\downarrow$ $1/\sqrt{n}$, $n \to \infty$

distribution at time tn $M_{tn}^{(n)}$ $\longrightarrow$ ω_t

$\downarrow$

invariant distribution $\mathrm{M}_{\mathrm{Pl}}^{(n)}$ $\longrightarrow$ Ω



Initial profile ω_0 is arbitrarily taken to satisfy $\int_{\mathbb{R}}(\omega_0(x) - |x|)dx = 2$.

Let the pausing time obey a heavy-tailed distribution without the mean.
As an example of such distribution on $[0, \infty)$, let ψ be the one-sided stable distribution of exponent $1/2$:

$$\psi(x) = \frac{1}{\sqrt{2\pi}} x^{-3/2} e^{-1/(2x)} \, 1_{[0,\infty)}(x),$$

with characteristic function

$$\varphi(\xi) = e^{-\sqrt{|\xi|}\,(1 - i\,\mathrm{sign}(\xi))}, \qquad \xi \in \mathbb{R}.$$

Theorem 2

The continuous time random walk $(X_s^{(n)})_{s \geqq 0}$ with distribution at time $s \geqq 0$:

$$M_s^{(n)}(\{\lambda\}) = \mathbb{P}(X_s^{(n)} = \lambda), \qquad \lambda \in \mathbb{Y}_n$$

Assume

- the sequence of initial distributions $\{(\mathbb{Y}_n, M_0^{(n)})\}_{n\in\mathbb{N}}$ has concentration property at $\omega_0 \in \mathbb{D}$
- pausing time obeys the one-sided stable distribution of exponent $1/2$.

Then

for macroscopic time $t > 0$ and microscopic time $s = t\alpha(n)$

1. if $\alpha(n)/n^2 \to 0$ as $n \to \infty$, $\{(\mathbb{Y}_n, M_{t\alpha(n)}^{(n)})\}_{n\in\mathbb{N}}$ inherits concentration property, however, at $\omega_t = \omega_0$
2. if $\alpha(n)/n^2 \to \infty$ as $n \to \infty$, $\{(\mathbb{Y}_n, M_{t\alpha(n)}^{(n)})\}_{n\in\mathbb{N}}$ inherits concentration property, however, at $\omega_t = \Omega$
3. if $\alpha(n) = n^2$, $\{(\mathbb{Y}_n, M_{tn^2}^{(n)})\}_{n\in\mathbb{N}}$ inherits concentration property if and only if $\omega_0 = \Omega$

   (essentially no propagation of concentration property)

As an essential step of computation

- $\mathbb{E}_{M_s^{(n)}}[M_k(\mathfrak{m}_{\lambda\sqrt{n}})]$ $\qquad (M_k$: kth moment)

$$\{M_k\} : \text{ moment sequence } \iff \{R_k\} : \text{ free cumulant sequence}$$

- $\mathbb{E}_{M_s^{(n)}}[R_k(\mathfrak{m}_{\lambda\sqrt{n}})] = n^{-k/2}\mathbb{E}_{M_s^{(n)}}[R_k(\mathfrak{m}_\lambda)]$

$$\{R_{k+1}(\mathfrak{m}_\lambda)\} \iff \{\chi^\lambda_{(k,1^{n-k})}\} \qquad \text{Kerov polynomial}$$

$(\chi^\lambda_{(k,1^{n-k})})_{\lambda\in\mathbb{Y}_n}$: eigenvector of transition matrix $P^{(n)}$ of Res-Ind chain

- $\mathbb{E}_{M_s^{(n)}}[\chi^\lambda_{(k,1^{n-k})}] \longrightarrow \mathbb{E}_{M_0^{(n)}}[\chi^\lambda_{(k,1^{n-k})}]$

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