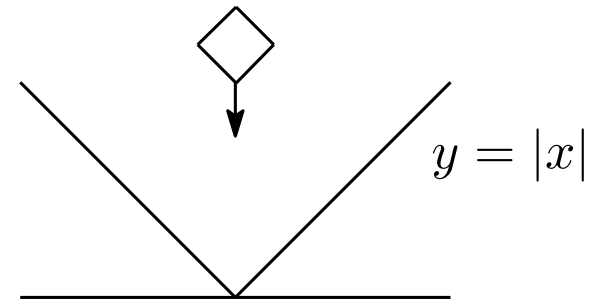
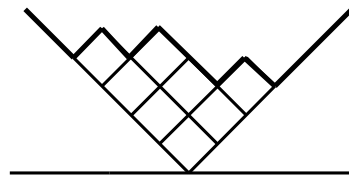
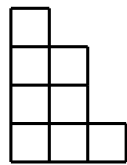
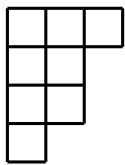


# Dynamical Limit Shapes for Random Young Diagrams

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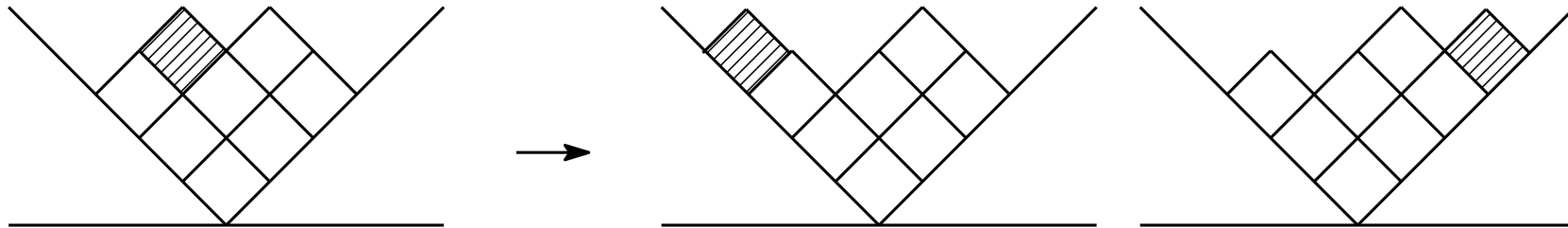
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# §1 Introduction

$$\mathbb{Y}_n = \{\text{Young diagram with } n \text{ boxes}\}, \quad \mathbb{Y} = \bigsqcup_{n=0}^{\infty} \mathbb{Y}_n$$

Markov chain on  $\mathbb{Y}_n$  caused by (non-local) movement of a corner box



called **Res-Ind chain**  $(Z_k^{(n)})_{k \in \{0,1,2,\dots\}}$  on  $\mathbb{Y}_n$

$\mathbb{Y}_n \cong \widehat{\mathfrak{S}}_n$  : the equivalence classes of irreducible representations of  
the symmetric group of degree  $n$

restriction  $\leftrightarrow$  removing box, induction  $\leftrightarrow$  adding box

Irreducible decomposition of **restriction** and **induction** of an irreducible representation of symmetric group

$$\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \pi^\lambda \cong \bigoplus_{\nu \in \mathbb{Y}_{n-1}: \lambda \searrow \nu} \pi^\nu, \quad \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \pi^\nu \cong \bigoplus_{\mu \in \mathbb{Y}_n: \nu \nearrow \mu} \pi^\mu$$

$$p^\downarrow(\lambda, \nu) = \begin{cases} \frac{\dim \nu}{\dim \lambda}, & \lambda \searrow \nu \\ 0, & \text{otherwise,} \end{cases} \quad p^\uparrow(\nu, \mu) = \begin{cases} \frac{\dim \mu}{(|\nu|+1) \dim \nu}, & \nu \nearrow \mu \\ 0, & \text{otherwise} \end{cases}$$

both proportional to dimension of terminal diagram

Set

$$p^{(n)}(\lambda, \mu) = \sum_{\nu \in \mathbb{Y}_{n-1}: \lambda \searrow \nu, \nu \nearrow \mu} p^\downarrow(\lambda, \nu) p^\uparrow(\nu, \mu), \quad \lambda, \mu \in \mathbb{Y}_n$$

Transition matrix of Res-Ind chain on  $\mathbb{Y}_n$  :

$$P^{(n)} = P^\downarrow P^\uparrow = (p^{(n)}(\lambda, \mu))_{\lambda, \mu \in \mathbb{Y}_n}$$

Plancherel measure on  $\mathbb{Y}_n$

$$M_{\text{Pl}}^{(n)}(\{\lambda\}) = \frac{(\dim \lambda)^2}{n!}, \quad \lambda \in \mathbb{Y}_n$$

**Lemma** Res-Ind chain is symmetric w.r.t. the Plancherel measure:

$$M_{\text{Pl}}^{(n)}(\{\lambda\})p^{(n)}(\lambda, \mu) = M_{\text{Pl}}^{(n)}(\{\mu\})p^{(n)}(\mu, \lambda), \quad \lambda, \mu \in \mathbb{Y}_n,$$

hence the Plancherel measure is invariant distribution for Res-Ind chain

$(Z_k^{(n)})_{k \in \{0,1,2,\dots\}}$  : Res-Ind chain on  $\mathbb{Y}_n$  with initial distribution  $M_0^{(n)}$

Construct continuous time random walk on  $\mathbb{Y}_n$  from transition matrix  $P^{(n)}$

$\{\tau_j\}_{j \in \mathbb{N}}$  : i.i.d. random variables obeying  $\psi (\neq \delta_0)$  on  $[0, \infty)$ ,

independent of  $\{(Z_k^{(n)})_{k \in \{0,1,2,\dots\}}\}_{n \in \mathbb{N}}$

$(N_s)_{s \geq 0}$  : counting process with  $\tau_j$ 's as pausing intervals

$$N_s = \sup\{j \in \mathbb{N} \mid \tau_1 + \dots + \tau_j \leq s\} < \infty \text{ a.s.}, \quad N_0 \equiv 0 \text{ a.s.}$$

Set 
$$X_s^{(n)} = Z_{N_s}^{(n)}, \quad s \geq 0$$

$$\begin{aligned} \mathbb{P}(X_s^{(n)} = \mu) &= \sum_{j=0}^{\infty} \mathbb{P}(Z_j^{(n)} = \mu) \mathbb{P}(\tau_1 + \dots + \tau_j \leq s, \tau_1 + \dots + \tau_{j+1} > s) \\ &= \sum_{j=0}^{\infty} (M_0^{(n)} P^{(n)j})_{\mu} \int_{[0,s]} \psi((s-u, \infty)) \psi^{*j}(du) \end{aligned}$$

Continuous time random walk  $(X_s^{(n)})_{s \geq 0}$  on  $\mathbb{Y}_n$  as microscopic dynamics

→ scaling limit in space and time

– macroscopic profile :  $1/\sqrt{n}$  both horizontally and vertically

$$\lambda \in \mathbb{Y}_n \quad \longrightarrow \quad [\lambda]^{\sqrt{n}}(x) = \frac{1}{\sqrt{n}} \lambda(\sqrt{n}x) \quad \text{constant "area"}$$

– macroscopic time :  $t = s/\alpha(n)$        $\alpha(n)$  : micro/macro scaling factor

Dynamic scaling limit

$$[X_{t\alpha(n)}^{(n)}]^{\sqrt{n}} \xrightarrow{n \rightarrow \infty} \text{(deterministic) macroscopic shape depending on } t$$

from LLN (concentration phenomenon)

## §2 Concentration phenomenon (static model)

LLN in probability theory

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow{n \rightarrow \infty} \text{constant}$$

if  $X_1, X_2, \dots$  : **independent** + some conditions

$$\mathbb{E}[e^{i(\xi_1 X_1 + \cdots + \xi_n X_n)}] = \prod_{k=1}^n \mathbb{E}[e^{i\xi_k X_k}] \quad (\text{factorizability of characteristic function})$$

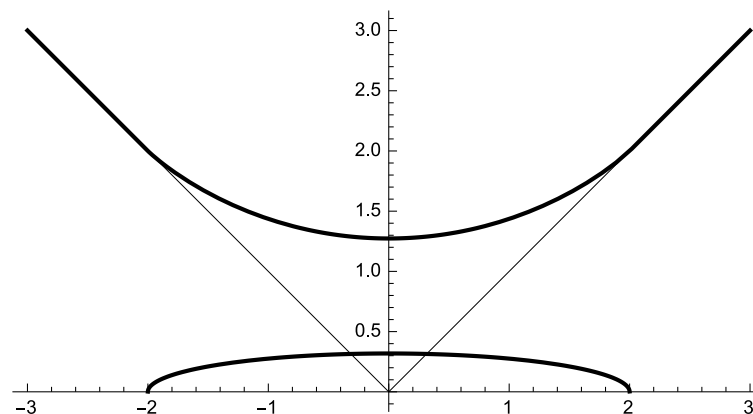
Continuous diagram

$$\mathbb{D} = \{ \omega : \mathbb{R} \longrightarrow \mathbb{R} \mid |\omega(x) - \omega(y)| \leq |x - y|, \omega(x) = |x| \text{ (for } |x| \text{ large enough)} \}$$

The following LLN holds (static scaling limit for the Plancherel measure)

Vershik – Kerov 1977, Logan – Shepp 1977

$$\Omega(x) = \begin{cases} \frac{2}{\pi} \left( x \arcsin \frac{x}{2} + \sqrt{4 - x^2} \right), & |x| \leq 2 \\ |x|, & |x| > 2 \end{cases} \quad \text{limit shape}$$



$$M_{\text{Pl}}^{(n)} \left( \left\{ \lambda \in \mathbb{Y}_n \mid \sup_{x \in \mathbb{R}} |[\lambda]^{\sqrt{n}}(x) - \Omega(x)| \geq \epsilon \right\} \right) \xrightarrow{n \rightarrow \infty} 0 \quad (\forall \epsilon > 0)$$



For a sequence of probability spaces  $\{(\mathbb{Y}_n, M^{(n)})\}_{n \in \mathbb{N}}$ , we know some sufficient condition for LLN

$$M^{(n)} \left( \left\{ \lambda \in \mathbb{Y}_n \mid \left\| [\lambda]^{\sqrt{n}} - \omega \right\|_{\text{sup}} \geq \epsilon \right\} \right) \xrightarrow[n \rightarrow \infty]{} 0 \quad (\forall \epsilon > 0)$$

to hold with some continuous diagram  $\omega \in \mathbb{D}$

$$\lambda \in \mathbb{Y}_n \cong \widehat{\mathfrak{S}}_n, \quad \rho \in \mathbb{Y}_n \cong \{\text{conjugacy classes of } \mathfrak{S}_n\}$$

$\chi_\rho^\lambda$  = value at an element in conjugacy class labeled by  $\rho$  of

irreducible character labeled by  $\lambda$

$\tilde{\chi}^\lambda = \chi^\lambda / \dim \lambda$  : normalized irreducible character

$$\mathbb{Y}^\times = \{\rho \in \mathbb{Y} \mid \rho \text{ has no one-box rows}\}$$

$$l(\rho) = \# \text{ of rows of } \rho \in \mathbb{Y}$$

**Approximate factorization property** of **Biane**: for any  $\rho, \sigma \in \mathbb{Y}^\times$

$$\begin{aligned} \mathbb{E}_{M^{(n)}} [\tilde{\chi}_{(\rho \sqcup \sigma, 1^{n-|\rho|-|\sigma|})}] &= \mathbb{E}_{M^{(n)}} [\tilde{\chi}_{(\rho, 1^{n-|\rho|})}] \mathbb{E}_{M^{(n)}} [\tilde{\chi}_{(\sigma, 1^{n-|\sigma|})}] \\ &= o\left(n^{-\frac{1}{2}(|\rho|-l(\rho)+|\sigma|-l(\sigma))}\right) \quad (n \rightarrow \infty) \end{aligned}$$

combined with some conditions implies **concentration** at some  $\psi \in \mathbb{D}$ .

## Examples

- $M^{(n)} = \delta_{\lambda^{(n)}}$  for  $[\lambda^{(n)}]^{\sqrt{n}} \rightarrow \omega_0 \in \mathbb{D}$  as  $n \rightarrow \infty$   
( $\omega_0$  then satisfies  $\int_{\mathbb{R}} (\omega_0(x) - |x|) dx = 2$ )

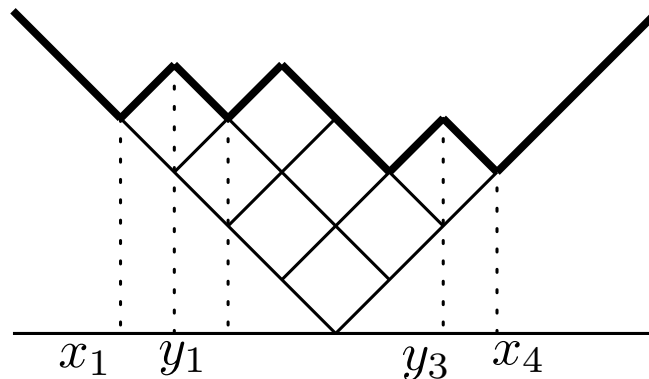
Irreducible decomposition of some representation of symmetric group

- $M^{(n)} = M_{\text{Pl}}^{(n)}$  (Plancherel measure)  $\longleftarrow$  regular representation
- Littlewood-Richardson measure  
 $\longleftarrow$  outer product of irreducible representations

### §3 Dynamical limit shape (main result)

Young diagram  $\lambda$  is characterized by

its **profile**  $y = \lambda(x)$  or **transition measure**  $\mathfrak{m}_\lambda = \sum_{i=1}^r \mu_i \delta_{x_i} \in \mathcal{P}(\mathbb{R})$



$$\frac{(z - y_1) \cdots (z - y_{r-1})}{(z - x_1) \cdots (z - x_r)} = \frac{\mu_1}{z - x_1} + \cdots + \frac{\mu_r}{z - x_r}$$

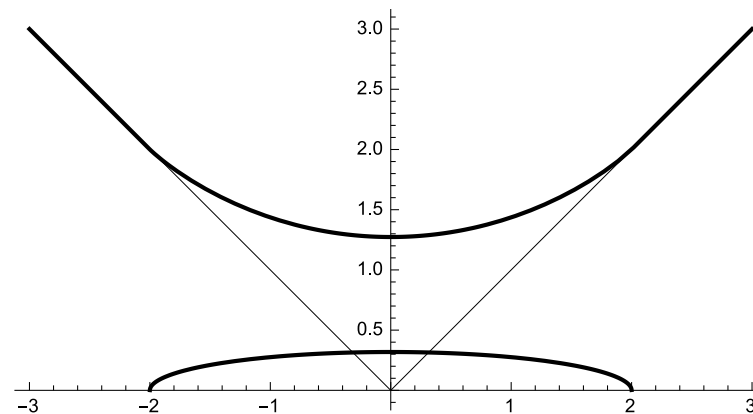
$\implies$  extended to continuous diagram:  $y = \omega(x) \longleftrightarrow \mathfrak{m}_\omega$

## Markov transform

$$\frac{1}{z} \exp \left\{ \int_{\mathbb{R}} \frac{1}{x-z} \left( \frac{\omega(x) - |x|}{2} \right)' dx \right\} = \int_{\mathbb{R}} \frac{1}{z-x} \mathfrak{m}_\omega(dx), \quad z \in \mathbb{C}^+$$

e.g.

limit shape  $\Omega \longleftrightarrow \mathfrak{m}_\Omega$  : standard semi-circle distribution



Free convolution, free compression

$(A, \phi)$  : probability space

self-adjoint  $a \in A$ ,  $\mu \in \mathcal{P}_c(\mathbb{R})$  with compact support

$$a \sim \mu \iff \phi(a^k) = M_k(\mu) \text{ for any } k \in \{0, 1, 2, \dots\}$$

Given  $\mu, \nu \in \mathcal{P}_c(\mathbb{R})$ ,  $a \sim \mu$ ,  $b \sim \nu$ ,  $a$  and  $b$  are **free**, then

$$a + b \sim \mu \boxplus \nu \in \mathcal{P}_c(\mathbb{R}) : \text{free convolution of } \mu \text{ and } \nu$$

$p \in A$  : projection free from  $a$ ,  $\phi(p) = \alpha > 0$ ,

$pap$  in  $(pAp, \alpha^{-1}\phi|_{pAp}) \sim \mu_\alpha \in \mathcal{P}_c(\mathbb{R})$  : **free compression** of  $\mu$

$$\blacktriangleright R_k(\mu \boxplus \nu) = R_k(\mu) + R_k(\nu), \quad k \in \mathbb{N}$$

$$\blacktriangleright R_k(\mu_\alpha) = \alpha^{k-1} R_k(\mu), \quad k \in \mathbb{N}$$

## Theorem 1

The continuous time random walk  $(X_s^{(n)})_{s \geq 0}$  with distribution at time  $s \geq 0$ :

$$M_s^{(n)}(\{\lambda\}) = \mathbb{P}(X_s^{(n)} = \lambda), \quad \lambda \in \mathbb{Y}_n$$

Assume

- the sequence of initial distributions  $\{(\mathbb{Y}_n, M_0^{(n)})\}_{n \in \mathbb{N}}$  has concentration property at  $\omega_0 \in \mathbb{D}$
- the pausing time distribution  $\psi$  with characteristic function  $\varphi$  satisfies:
  - $\varphi$  is differentiable at 0 ( $\psi$  having mean  $m > 0$ )
  - integrability condition for  $\varphi$  : for some  $\delta > 0$

$$\int_{\{|\xi| \geq \delta\}} \left| \frac{\varphi(\xi)}{\xi} \right| d\xi < \infty.$$

Then

for macroscopic time  $t > 0$  and microscopic time  $s = tn$

i.e. scaling factor  $\alpha(n) = n$ ,

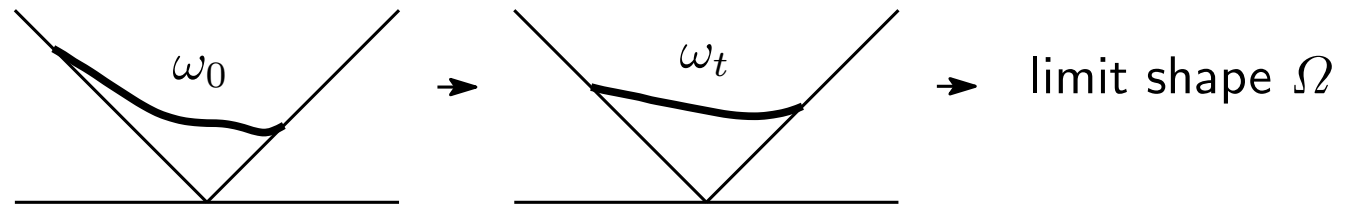
concentration property is propagated along  $t$ , hence the sequence

$\{(\mathbb{Y}_n, M_{tn}^{(n)})\}_{n \in \mathbb{N}}$  has concentration at some  $\omega_t \in \mathbb{D}$ . The macroscopic profile  $\omega_t$  is characterized by

$$\mathbf{m}_{\omega_t} = (\mathbf{m}_{\omega_0})_{e^{-t/m}} \boxplus (\mathbf{m}_{\Omega})_{1-e^{-t/m}}$$

and recovered by using the Markov transform.

initial distribution $M_0^{(n)}$	$\longrightarrow$	$\omega_0$
$\downarrow$	$1/\sqrt{n}, n \rightarrow \infty$	
distribution at time $tn$ $M_{tn}^{(n)}$	$\longrightarrow$	$\omega_t$
$\downarrow$		
invariant distribution $M_{PI}^{(n)}$	$\longrightarrow$	$\Omega$



Initial profile  $\omega_0$  is arbitrarily taken to satisfy  $\int_{\mathbb{R}} (\omega_0(x) - |x|) dx = 2$ .



Let the pausing time obey a heavy-tailed distribution without the mean.  
 As an example of such distribution on  $[0, \infty)$ , let  $\psi$  be the **one-sided stable** distribution of exponent  $1/2$  :

$$\psi(x) = \frac{1}{\sqrt{2\pi}} x^{-3/2} e^{-1/(2x)} 1_{[0, \infty)}(x),$$

with characteristic function

$$\varphi(\xi) = e^{-\sqrt{|\xi|} (1 - i \operatorname{sign}(\xi))}, \quad \xi \in \mathbb{R}.$$

## Theorem 2

The continuous time random walk  $(X_s^{(n)})_{s \geq 0}$  with distribution at time  $s \geq 0$ :

$$M_s^{(n)}(\{\lambda\}) = \mathbb{P}(X_s^{(n)} = \lambda), \quad \lambda \in \mathbb{Y}_n$$

Assume

- the sequence of initial distributions  $\{(\mathbb{Y}_n, M_0^{(n)})\}_{n \in \mathbb{N}}$  has concentration property at  $\omega_0 \in \mathbb{D}$
- pausing time obeys the one-sided stable distribution of exponent  $1/2$ .

Then

for macroscopic time  $t > 0$  and microscopic time  $s = t\alpha(n)$

1. if  $\alpha(n)/n^2 \rightarrow 0$  as  $n \rightarrow \infty$ ,  $\{(\mathbb{Y}_n, M_{t\alpha(n)}^{(n)})\}_{n \in \mathbb{N}}$  inherits concentration property, however, at  $\omega_t = \omega_0$
2. if  $\alpha(n)/n^2 \rightarrow \infty$  as  $n \rightarrow \infty$ ,  $\{(\mathbb{Y}_n, M_{t\alpha(n)}^{(n)})\}_{n \in \mathbb{N}}$  inherits concentration property, however, at  $\omega_t = \Omega$
3. if  $\alpha(n) = n^2$ ,  $\{(\mathbb{Y}_n, M_{tn^2}^{(n)})\}_{n \in \mathbb{N}}$  inherits concentration property if and only if  $\omega_0 = \Omega$   
(essentially **no propagation** of concentration property)

As an essential step of computation

- $\mathbb{E}_{M_s^{(n)}} [M_k(\mathbf{m}_{\lambda\sqrt{n}})] \quad (M_k : k\text{th moment})$

$\{M_k\} : \text{moment sequence} \iff \{R_k\} : \text{free cumulant sequence}$

- $\mathbb{E}_{M_s^{(n)}} [R_k(\mathbf{m}_{\lambda\sqrt{n}})] = n^{-k/2} \mathbb{E}_{M_s^{(n)}} [R_k(\mathbf{m}_\lambda)]$

$$\{R_{k+1}(\mathbf{m}_\lambda)\} \iff \{\chi_{(k,1^{n-k})}^\lambda\} \quad \text{Kerov polynomial}$$

$(\chi_{(k,1^{n-k})}^\lambda)_{\lambda \in \mathbb{Y}_n} : \text{eigenvector of transition matrix } P^{(n)} \text{ of Res-Ind chain}$

- $\mathbb{E}_{M_s^{(n)}} [\chi_{(k,1^{n-k})}^\lambda] \longrightarrow \mathbb{E}_{M_0^{(n)}} [\chi_{(k,1^{n-k})}^\lambda]$

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