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# Information Geometry in Quantum Statistics


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## Plan of talk

Motivation

- Part I (Allegro moderato): Review of quantum estimation theory
- Part II (Allegro assai): Review of local asymptotic normality
- Part III (Allegro ma non troppo): Local asymptotic normality in the quantum domain

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## Part I: Review of quantum estimation theory

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## Classical estimation

- Statistical model:

$$\{P_\theta\}_{\theta\in\Theta\subset\mathbb{R}^d} \text{ on } (\Omega,\mathcal{F},\mu) \overset{P_\theta\ll\mu}{\longrightarrow} \left\{p_\theta := \frac{dP_\theta}{d\mu}\right\}_{\theta\in\Theta}$$

- Fisher information matrix:

$$J_{ij} := E_{P_\theta}\left[(\partial_i \log p_\theta)(\partial_j \log p_\theta)\right]$$

- Cramér-Rao inequality:

$$V_{P_\theta}[\hat\theta] \geq J^{-1}$$

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# Quantum estimation

1) Quantum statistical model

$$\mathcal{S} = \{\rho_\theta (>0) : \ \theta = (\theta^1, \dots, \theta^d) \in \Theta \subset \mathbb{R}^d\}$$

2) Symmetric Logarithmic Derivative (SLD):

an e-repr. of tangent vector that corresponds to the following op. mon. func.

$$f(t) = \frac{2t}{1+t}$$



Since

$$\mathbb{K}_\rho = \mathbb{L}_\rho^{-1} f\left(\frac{\mathbb{L}_\rho}{\mathbb{R}_\rho}\right) = \mathbb{L}_\rho^{-1} \frac{2(\mathbb{L}_\rho/\mathbb{R}_\rho)}{1+(\mathbb{L}_\rho/\mathbb{R}_\rho)} = \frac{2}{\mathbb{L}_\rho + \mathbb{R}_\rho}$$

the e-representation $\mathbb{K}_{\rho_\theta}(\partial_i \rho_\theta) =: L_i$ satisfies

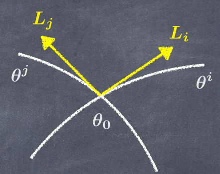
$$\partial_i \rho_\theta = \frac{1}{2}(\rho_\theta L_i + L_i \rho_\theta)$$

3) SLD Fisher information matrix $J_\theta^S = [(J_\theta^S)_{ij}]$

$$(J_\theta^S)_{ij} = \mathrm{Re}(\mathrm{Tr}\, \rho_\theta L_j L_i)$$



4) SLD Cramér-Rao inequality:




$$V_{\rho_\theta}[M] \geq (J_\theta^S)^{-1}$$

with equality iff SLDs commute and M the simultaneous spectral measure

5) It is thus customary to switch the target to

$$\min_M \ \mathrm{tr}\, W V_{\rho_\theta}[M]$$

given a weight matrix $W > 0$



6) Holevo bound

$$\mathrm{tr}\, W V_{\rho_{\theta_0}}[M] \geq C_{\theta_0}^H(\rho_\theta, W)$$

where

$$C_{\theta_0}^{\mathrm{H}}(\rho_\theta, W) := \min_{V,B}\{\mathrm{tr}\, WV \ : \ V \geq [\mathrm{Tr}\rho_{\theta_0} B_j B_i],$$

$$\mathrm{Re}\,\mathrm{Tr}\, \rho_{\theta_0} L_j B_i = \delta_{ij}\}$$

with V a real symmetric matrix and $B_1, \dots, B_d$ selfadjoint operators

# Example: qubit estimation

Qubit state model:

$$\mathcal{S} = \left\{ \rho_{(x,y,z)} = \frac{1}{2}(I + x\sigma_x + y\sigma_y + z\sigma_z) \;:\; x^2 + y^2 + z^2 < 1 \right\}$$

Hayashi-Gill-Massar bound:

$$\min_M \mathrm{tr} W\, V_\theta(M) = C_\theta^{\mathrm{HGM}}(\rho_\theta, W) := \left( \mathrm{tr}\sqrt{(J_\theta^S)^{-1/2}\, W\, (J_\theta^S)^{-1/2}} \right)^2$$

When $W := J_\theta^S$

$$C_\theta^{\mathrm{HGM}}(\rho_\theta, J_\theta^S) = 9 \quad \text{while} \quad C_\theta^{\mathrm{H}}(\rho_\theta, J_\theta^S) = 3 + 2\|\theta\|$$

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# Motivation

separable

$$\begin{matrix} \rho_\theta \otimes & \rho_\theta \otimes & \cdots & \otimes\, \rho_\theta \\ \uparrow & \uparrow & & \uparrow \\ M_1 \otimes & M_2 \otimes & \cdots & \otimes\, M_n \end{matrix}$$

collective

$$\underbrace{\rho_\theta \otimes \rho_\theta \otimes \cdots \otimes \rho_\theta}_{M^{(n)}}$$

- Suppose we have n copies of a quantum system each in the same state
- We are allowed to use "collective" measurement
- What is the best we can do as $n \to \infty$
- We study this problem by extending the theory of LAN to the quantum domain

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# Part II: Review of local asymptotic normality

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# Local Asymptotic Normality

A sequence $\{P_\theta^{(n)} : \theta \in \Theta \subset \mathbb{R}^d\}$ of models

is called LAN at $\theta_0 \in \Theta$ if

$$\log \frac{dP^{(n)}_{\theta_0 + h/\sqrt{n}}}{dP^{(n)}_{\theta_0}} = h^i \Delta_i^{(n)} - \frac{1}{2} h^i h^j J_{ij} + o_{P_{\theta_0}}(1)$$

where

$$\Delta^{(n)} \overset{P^{(n)}_{\theta_0}}{\rightsquigarrow} N(0, J)$$

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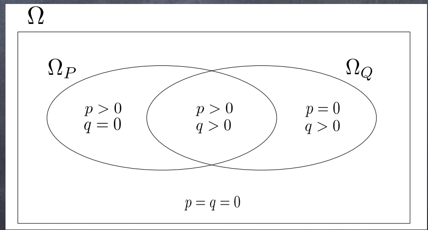
NB. Given $P$ and $Q$, let

$$Q = Q^{ac} + Q^{\perp} \qquad (Q^{ac} \ll P,\ Q^{\perp} \perp P)$$

be the Lebesgue decomposition. Then

$$\frac{dQ}{dP}(\omega) := \frac{dQ^{ac}}{dP}(\omega) = \frac{q^{ac}(\omega)}{p(\omega)}$$

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# Prototype of LAN: $p_\theta^{(n)} = p_\theta^{\otimes n}$

$$\log \frac{p^{\otimes n}_{\theta_0+h/\sqrt{n}}}{p^{\otimes n}_{\theta_0}}(X_1,\ldots,X_n)$$

$$= h^i \underbrace{\left\{\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\partial_i \log p_{\theta_0}(X_k)\right\}}_{\Delta_i^{(n)}(X_1,\ldots,X_n)} - \frac{1}{2}h^i h^j \underbrace{\left\{-\frac{1}{n}\sum_{k=1}^{n}\partial_i\partial_j \log p_{\theta_0}(X_k)\right\}}_{J_{ij}+o_{p_{\theta_0}}(1)} + o(\frac{1}{n})$$

where

$$\Delta^{(n)} \overset{p^{\otimes n}_{\theta_0}}{\rightsquigarrow} N(0,J)$$

with J being the Fisher information matrix

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# Similarity to Gaussian shift model

LAN

$$\log \frac{dP^{(n)}_{\theta_0+h/\sqrt{n}}}{dP^{(n)}_{\theta_0}} = h^i\Delta_i^{(n)} - \frac{1}{2}h^i h^j J_{ij} + o_{p_{\theta_0}}(1)$$

Gaussian shift model

$$\log \frac{dN(Jh,J)}{dN(0,J)}(X_1,\ldots,X_d) = h^i X_j - \frac{1}{2}h^i h^j J_{ij}$$

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# Le Cam's third lemma

Radon-Nikodym theorem:

$$Q \ll P \implies dQ = \frac{dQ}{dP}dP$$

Counterpart in weak convergence:

If $Q^{(n)} \lhd P^{(n)}$ and $\left(X^{(n)}, \frac{dQ^{(n)}}{dP^{(n)}}\right) \overset{P^{(n)}}{\rightsquigarrow} (X,V)$

then $X^{(n)} \overset{Q^{(n)}}{\rightsquigarrow} L$, where $L(B) := E[1_B(X)V]$

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# Le Cam's third lemma

(Gaussian version)

Theorem

If

$$\begin{pmatrix} X^{(n)} \\ \log \frac{dQ^{(n)}}{dP^{(n)}} \end{pmatrix} \overset{P^{(n)}}{\rightsquigarrow} N\left( \begin{pmatrix} 0 \\ -\frac{1}{2}\sigma^2 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^\top & \sigma^2 \end{pmatrix} \right)$$

then ( $Q^{(n)} \triangleleft P^{(n)}$ and )

$$X^{(n)} \overset{Q^{(n)}}{\rightsquigarrow} N(\tau, \Sigma)$$

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# Third lemma under LAN

Suppose

$$\log \frac{dP^{(n)}_{\theta_0 + h/\sqrt{n}}}{dP^{(n)}_{\theta_0}} = h^i \Delta_i^{(n)} - \frac{1}{2} h^i h^j J_{ij} + o_{p_{\theta_0}}(1)$$

and

$$\begin{pmatrix} X^{(n)} \\ \Delta^{(n)} \end{pmatrix} \overset{0}{\rightsquigarrow} N\left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^\top & J \end{pmatrix} \right)$$

Then

$$X^{(n)} \overset{h}{\rightsquigarrow} N(\tau h, \Sigma)$$

$$\Delta^{(n)} \overset{h}{\rightsquigarrow} N(Jh, J)$$

The moral:
A LAN model is locally asymptotically similar to Gaussian shift model

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# Asymptotic Representation Theorem

Assume that

1) $\{P_\theta^{(n)} : \theta \in \Theta \subset \mathbb{R}^d\}$ is LAN at $\theta_0 \in \Theta$

2) Seq. of statistics $T^{(n)}$ on $P^{(n)}_{\theta_0 + h/\sqrt{n}}$ is weakly convergent for each $h$ (i.e. $T^{(n)} \overset{h}{\rightsquigarrow} \exists \mathcal{L}_h$ )

Then there exists a statistic $T$ on $N(Jh, J)$ s.t.

$$T^{(n)} \overset{h}{\rightsquigarrow} T$$

for all $h$

$T = T(\Delta, Z)$ $(\Delta \sim N(Jh, J))$

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# Part III: LAN in the quantum domain

[ Annals of Statistics, 41, 2197-2217 (2013) ]

[ Bernoulli, 26, 2106-2142 (2020) ]

[ arXiv: 2209.00832 ]

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# History of quantum LAN

- Guta and Kahn (2006, 2009)

$$\lim_{n\to\infty} \sup_{h\in K^{(n)}} \left\| \sigma_h - \Gamma^{(n)}(\rho_{\theta_0+h/\sqrt{n}}^{\otimes n}) \right\|_1 = 0$$

$$\lim_{n\to\infty} \sup_{h\in K^{(n)}} \left\| \Lambda^{(n)}(\sigma_h) - \rho_{\theta_0+h/\sqrt{n}}^{\otimes n} \right\|_1 = 0$$

Lahiry and Nussbaum (2021)

- Guta and Jencova's (2007)

**Drawbacks**
- iid
- nondegeracy

Quantum Radon-Nikodym?

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# Difficulties in extending LAN to the quantum domain

LAN: $$\log \frac{dP^{(n)}_{\theta_0+h/\sqrt{n}}}{dP^{(n)}_{\theta_0}} = h^i \Delta_i^{(n)} - \frac{1}{2} h^i h^j J_{ij} + o_{P_{\theta_0}}(1)$$

$$\left(\Delta^{(n)} \overset{\theta_0}{\rightsquigarrow} N(0,J)\right)$$

What are the quantum counterparts of

1) Radon-Nikodym density?
2) infinitesimal in probability?
3) weak convergence?

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# A breakthrough : quantum info. geometry

SLD repr.: $L^S : T_\rho \mathcal{S}(\mathcal{H}) \longrightarrow \mathcal{L}_h(\mathcal{H}) \,:\, d\rho = \frac{1}{2}(L^S \rho + \rho L^S)$

Teleparallel translation:

$$\Pi^{(e)} : T_\rho^{(e)} \mathcal{S}(\mathcal{H}) \longrightarrow T_\sigma^{(e)} \mathcal{S}(\mathcal{H})$$

$$L^S(X_\rho) \longmapsto L^S(X_\rho) - (\mathrm{Tr}\,\sigma L^S(X_\rho)) I$$

e-connection: $\nabla^{(e)}$

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# SLD geodesic

$$\rho_\theta = e^{\frac{1}{2}[\theta F - \psi(\theta)]}\, \rho_0\, e^{\frac{1}{2}[\theta F - \psi(\theta)]}$$

Nice properties:

1) quantum Cramér-Rao lower bound is uniformly achievable
2) can be extended to non-faithful states

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# Mutual absolute continuity

[ Annals of Statistics, 41, 2197-2217 (2013) ]

Quantum states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ are said to be mutually absolutely continuous, $\rho \sim \sigma$ in symbols, if there is a selfadjoint operator $\mathcal{L} := \mathcal{L}(\sigma|\rho)$ that satisfies

$$\sigma = e^{\frac{1}{2}\mathcal{L}} \rho \, e^{\frac{1}{2}\mathcal{L}}$$

$\mathcal{L} = \log(\sigma/\rho)$



# Examples

1) Faithful states $\rho, \sigma$ are mutually a.c., and

$$e^{\frac{1}{2}\mathcal{L}(\sigma|\rho)} = \sigma \# \rho^{-1} \quad \left( = \sqrt{\rho^{-1}}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\sqrt{\rho^{-1}} \right)$$

2) Pure states $\rho = |\xi\rangle\langle\xi|$ and $\sigma = |\eta\rangle\langle\eta|$ are mutually a.c. if and only if

$$\langle\xi|\eta\rangle \neq 0$$



# Singularity & a.c.

[ Bernoulli, 26, 2105-2142 (2020) ]

1) $\rho$ is singular to $\sigma$, $\rho \perp \sigma$ in symbols, if

$$\operatorname{supp}\rho \perp \operatorname{supp}\sigma$$

2) $\rho$ is absolutely continuous to $\sigma$, $\rho \ll \sigma$ in symbols, if there is an $R (\geq 0)$ s.t.

$$\rho = R\sigma R$$



# Quantum Lebesgue decomposition

Theorem. Given quantum states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, there is a unique decomposition:

$$\sigma = R\rho R + \tau \qquad (R \geq 0,\ \tau \geq 0,\ \tau \perp \rho)$$

$$(=: \sigma^{ac} + \sigma^{\perp})$$

In fact, let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ with

$\mathcal{H}_1 := \ker(\sigma|_{\mathrm{supp}\,\rho})$, $\mathcal{H}_2 := \mathrm{supp}\,(\sigma|_{\mathrm{supp}\,\rho})$, $\mathcal{H}_3 := \ker\rho$

$(\sigma|_{\mathrm{supp}\,\rho} := \iota_\rho^* \, \sigma \, \iota_\rho)$

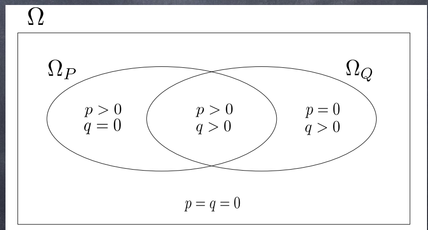
and let

$$\rho = \begin{pmatrix} \rho_2 & \rho_1 & 0 \\ \rho_1^* & \rho_0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ (\text{block} > 0), \qquad \sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_0 & \alpha \\ 0 & \alpha^* & \beta \end{pmatrix} \ (\sigma|_{\mathrm{supp}\,\rho})$$

Then

$$\sigma^{ac} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_0 & \alpha \\ 0 & \alpha^* & \alpha^*\sigma_0^{-1}\alpha \end{pmatrix}$$

$$\sigma^{\perp} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta - \alpha^*\sigma_0^{-1}\alpha \end{pmatrix}$$

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# Quantum likelihood ratio

The positive operator R that satisfies

$$\sigma = R\rho R + \tau \qquad (R \geq 0,\ \tau \geq 0,\ \tau \perp \rho)$$

is called the square-root likelihood ratio, and is denoted as

$$\mathcal{R}(\sigma|\rho)$$

This is our solution to the problem of finding the noncommutative Radon-Nikodym dens.

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# Difficulties in extending LAN to the quantum domain

LAN: $\log \dfrac{dP^{(n)}_{\theta_0+h/\sqrt{n}}}{dP^{(n)}_{\theta_0}} = h^i\Delta_i^{(n)} - \dfrac{1}{2}h^ih^jJ_{ij} + o_{P_{\theta_0}}(1)$

$$\left(\Delta^{(n)} \overset{\theta_0}{\rightsquigarrow} N(0,J)\right)$$

What are the quantum counterparts of

1) Radon-Nikodym density → done
2) infinitesimal in probability?
3) weak convergence?

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# Infinitesimal observables

A sequence $O^{(n)}$ of observables is called

- infinitesimal in $L^2$ w.r.t. $\rho^{(n)}$ if $\lim_{n\to\infty} \mathrm{Tr}\,\rho^{(n)}{O^{(n)}}^2 = 0$
- infinitesimal in distribution w.r.t. $(Z^{(n)}, \rho^{(n)})$ if

$$\lim_{n\to\infty} \mathrm{Tr}\,\rho^{(n)}\left\{\prod_{t=1}^{r} e^{\sqrt{-1}(\xi_t Z^{(n)} + \eta_t O^{(n)})}\right\} = \lim_{n\to\infty} \mathrm{Tr}\,\rho^{(n)}\left\{\prod_{t=1}^{r} e^{\sqrt{-1}\xi_t Z^{(n)}}\right\}$$

They are denoted as

$$O^{(n)} = o_{L^2}(\rho^{(n)}) \quad \text{and} \quad O^{(n)} = o_D\left(Z^{(n)}, \rho^{(n)}\right)$$

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# Quantum weak convergence

$\rho^{(n)}$ : state, $X^{(n)} = \left(X_1^{(n)}, \dots, X_d^{(n)}\right)$ : obs. on $\mathcal{H}^{(n)}$

$\phi$ : state, $X^{(\infty)} = \left(X_1^{(\infty)}, \dots, X_d^{(\infty)}\right)$ : obs. on $\mathcal{H}^{(\infty)}$

We say $(X^{(n)}, \rho^{(n)}) \rightsquigarrow \left(X^{(\infty)}, \phi\right)$ if

$$\lim_{n\to\infty} \mathrm{Tr}\, \rho^{(n)} \left( \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right) = \phi\left( \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(\infty)}} \right)$$

NB. noncommutative Lévy-Cramér continuity theorem holds (Jaksic et al., 2010)



# Sandwiched weak convergence

If

$$\lim_{n\to\infty} \mathrm{Tr}\, \rho^{(n)} e^{\sqrt{-1}\eta_1 Y^{(n)}} \left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} e^{\sqrt{-1}\eta_2 Y^{(n)}}$$

$$= \phi\left( e^{\sqrt{-1}\eta_1 Y^{(\infty)}} \left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(\infty)}} \right\} e^{\sqrt{-1}\eta_2 Y^{(\infty)}} \right)$$

we denote

$$\left\langle Y^{(n)}, X^{(n)}, Y^{(n)} \right\rangle_{\rho^{(n)}} \rightsquigarrow \left\langle Y^{(\infty)}, X^{(\infty)}, Y^{(\infty)} \right\rangle_{\phi}$$



# Quantum LAN

We say $\mathcal{S}^{(n)} = \left\{ \rho_\theta^{(n)} : \theta \in \Theta \subset \mathbb{R}^d \right\}$ is q-LAN at $\theta_0 \in \Theta$ if

$R_h^{(n)} := \mathcal{R}\left( \rho^{(n)}_{\theta_0 + h/\sqrt{n}} \middle| \rho^{(n)}_{\theta_0} \right)$ is expanded in h as

$$\log\left( R_h^{(n)} + o_{L^2}(\rho^{(n)}_{\theta_0}) \right)^2 = h^i \Delta_i^{(n)} - \frac{1}{2}\left(J_{ij} h^i h^j\right) I^{(n)} + o_D(h^i \Delta_i^{(n)}, \rho^{(n)}_{\theta_0})$$

where

$$\Delta^{(n)} \overset{\rho^{(n)}_{\theta_0}}{\rightsquigarrow} N(0, J)$$



# q/c-hybrid Gaussian state

A state $\phi$ on a CCR(S) with

$$e^{\sqrt{-1}\xi^i X_i} e^{\sqrt{-1}\eta^j X_j} = e^{\sqrt{-1}\xi^\top S \eta} e^{\sqrt{-1}(\xi+\eta)^i X_i}$$

Heisenberg: $\frac{\sqrt{-1}}{2}[X_i, X_j] = S_{ij}$

is called a q/c-hybrid Gaussian state,

$\phi \sim N(h, J)$ in symbols, if

$$\phi(e^{\sqrt{-1}\xi^i X_i}) = e^{\sqrt{-1}\xi^i h^j \delta_{ij} - \frac{1}{2}\xi^i \xi^j V_{ij}}$$

where

$$J := V + \sqrt{-1}S \geq 0$$

Noncommutative Fisher information

# Quantum contiguity

Let $\rho^{(n)}, \sigma^{(n)} \in \mathcal{S}(\mathcal{H}^{(n)})$ and let $R^{(n)} := \mathcal{R}\left(\sigma^{(n)} \middle| \rho^{(n)}\right)$

We say $\sigma^{(n)}$ is contiguous to $\rho^{(n)}$, $\sigma^{(n)} \triangleleft_{O^{(n)}} \rho^{(n)}$

in symbols, if

i) $\lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} {R^{(n)}}^2 = 1$

ii) there is a sequence $O^{(n)} = o_{L^2}(\rho^{(n)})$ s.t.

$\overline{R}^{(n)} := R^{(n)} + O^{(n)} \geq 0$ and ${\overline{R}^{(n)}}^2$ is UI, i.e.

$$(\forall \varepsilon > 0)(\exists M > 0)\left[\sup_n \operatorname{Tr} \rho^{(n)} {\overline{R}^{(n)}}^2 \left(I - 1\!\!1_M(\overline{R}^{(n)})\right) < \varepsilon\right]$$



# Quantum Le Cam third lemma

If $\sigma^{(n)} \triangleleft_{O^{(n)}} \rho^{(n)}$, and $R^{(n)} := \mathcal{R}\left(\sigma^{(n)} \middle| \rho^{(n)}\right)$ enjoys

$$\left\langle R^{(n)} + O^{(n)}, X^{(n)}, R^{(n)} + O^{(n)} \right\rangle_{\rho^{(n)}} \rightsquigarrow \left\langle R^{(\infty)}, X^{(\infty)}, R^{(\infty)} \right\rangle_{\phi}$$

Then

$$\left(X^{(n)}, \sigma^{(n)}\right) \rightsquigarrow \left(X^{(\infty)}, \psi\right)$$

where

$$\psi(A) := \phi\left(R^{(\infty)} A R^{(\infty)}\right)$$

This gives a complete characterization of an alternative state ψ in terms of the reference state Φ and the limiting likelihood ratio



# Quantum Le Cam third lemma

(q-Gaussian version)

If $\begin{pmatrix} X^{(n)} \\ \log(R^{(n)} + O^{(n)})^2 - \tilde{O}^{(n)} \end{pmatrix} \overset{\rho^{(n)}}{\rightsquigarrow} N\left(\begin{pmatrix} \mu \\ -\frac{1}{2}s^2 \end{pmatrix}, \begin{pmatrix} \Sigma & \kappa \\ \kappa^* & s^2 \end{pmatrix}\right)$

(where $O^{(n)} = o_{L^2}(\rho^{(n)})$ s.t. $R^{(n)} + O^{(n)} > 0$ and $\tilde{O}^{(n)} = o_D(\log(R^{(n)} + O^{(n)}), \rho^{(n)})$)

then ( $\sigma^{(n)} \triangleleft_{O^{(n)}} \rho^{(n)}$ and )

$$X^{(n)} \overset{\sigma^{(n)}}{\rightsquigarrow} N(\mu + \mathrm{Re}(\kappa), \Sigma)$$



# Third lemma under q-LAN

Suppose $\mathcal{S}^{(n)} = \left\{\rho_\theta^{(n)} : \theta \in \Theta \subset \mathbb{R}^d\right\}$ is q-LAN at $\theta_0 \in \Theta$ and

$$\begin{pmatrix} X^{(n)} \\ \Delta^{(n)} \end{pmatrix} \overset{\rho_{\theta_0}^{(n)}}{\rightsquigarrow} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^* & J \end{pmatrix}\right)$$

Then

$$X^{(n)} \overset{\rho_{\theta_0 + h/\sqrt{n}}^{(n)}}{\rightsquigarrow} N((\mathrm{Re}\,\tau)h, \Sigma)$$

The moral:
A q-LAN model is locally asymptotically similar to q-Gaussian shift model

## Asymptotic q-Repr. Theorem

Assume that 【 arXiv: 2209.00832 】

$\begin{pmatrix} X^{(n)} \\ \Delta^{(n)} \end{pmatrix} \overset{\rho_{\theta_0}^{(n)}}{\rightsquigarrow} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^* & J \end{pmatrix}\right)$

1) $\left\{\rho_\theta^{(n)}\right\}$ is q-LAN and D-extendible at $\theta_0$

2) Seq. of POVMs $M^{(n)} = \{M^{(n)}(B)\}_{B \in \mathcal{B}(\mathbb{R}^s)}$ enjoys

$$\mathrm{Tr}\,\rho^{(n)}_{\theta_0 + h/\sqrt{n}} M^{(n)} \overset{h}{\rightsquigarrow} \exists\, \mathcal{L}_h$$

Then $\exists\, M^{(\infty)} = \{M^{(\infty)}(B)\}_{B \in \mathcal{B}(\mathbb{R}^s)}$ on $\mathrm{CCR}(\mathrm{Im}\,\Sigma)$ s.t.

$$\phi_h(M^{(\infty)}(B)) = \mathcal{L}_h(B) \qquad (\forall h)$$

where $\phi_h \sim N((\mathrm{Re}\,\tau)\,h, \Sigma)$



## Applications

- asymptotic quantum representation theorem converts a statistical problem for the local parameter h into another one for the limiting q-Gaussian shift model
- asymptotic representation bound beyond iid
- asymptotic regularity and asymptotic minimax theorems that exclude quantum superefficiency



## Summary

- Quantum information geometry naturally led us to the quantum Lebesgue decomposition
- With additional notions (such as quantum weak convergence and quantum contiguity), we derived quantum LeCam third lemma and asymptotic quantum representation theorem
- They establish solid foundations of the theory of quantum local asymptotic normality, providing a powerful tool to cope with asymptotics in the quantum domain



"Thank you for your attention"

– Akio Fujiwara