Geometry on Positive Definite Matrices Induced from V-Potentials -Foliated Structure and an Application-

Atsumi Ohara
University of Fukui
&
Shinto Eguchi
Institute of Statistical Mathematics

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1. Introduction

- $PD(n, \mathbf{R})$: the set of Positive Definite real symmetric matrices

- logarithmic characteristic func. on $PD(n, \mathbf{R})$

[Vinberg 63], [Faraut & Koranyi 94]

$$\varphi(P) = -\log\det P, \quad P \in PD(n; \mathrm{R})$$

$\phi(P) = -\log \det P$ appears in

- Semidefinite Programming (SDP)

  self-concordant barrier function

- Multivariate Analysis (Gaussian dist.)

  log-likelihood function

  (structured covariance matrix estimation)

- Symmetric cone: log characteristic function
- Information geometry on $PD(n, \mathbf{R})$

  a potential function in standard case

Information geometry on $\mathcal{M}$

Dualistic geometrical structure

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z)$$

X, Y and Z : arbitrary vector fields on $\mathcal{M}$

g :Riemannian metric

∇ , ∇^* :a pair of dual affine connections

standard IG on $PD(n, \mathbf{R})$

[O,Suda & Amari LAA96]

$\varphi(P)$ plays a role of potential function

- g : Riemannian metric is the Hesse matrix of $\varphi(P)$ (Fisher for Gaussian)
- ∇, ∇^*:related to the third derivatives of $\varphi(P)$

⇓

Nice properties: $GL(n, \mathbf{R})$-invariant (unique),

KL-divergence, Pythagorean theorem, etc

Purpose of this presentation

- The other convex potentials

V-potential functions

$$\varphi^{(V)}(P) = V(\det P)$$

- Their different and/or common geometric structures
  - Structure of submanifolds
  - Decomposition properties of divergences
  - Application to non-Gaussian pdf's

Outline

- Review
  - Dualistic geometry induced by V-potentials
- Foliated structures
  - Submanifold of constant determinant
  - Decomposition of divergence
- Applications to multivariate elliptic pdf's
  - GL(n)-invariance of geometry of q-Gaussian family induced by beta-divergence

2. Preliminaries and Notation

- $Sym(n;\mathbf{R})$: the set of *n* by *n* real symmetric matrix
  vec. sp. of dimension $N(=n(n+1)/2)$
- $\{E_i\}_{i=1}^{N}$:arbitrary set of basis matrices
- (primal) affine coordinate system

$$Sym(n;\mathbf{R}) \ni X = \sum_{i=1}^{N} x^i E_i$$

- Identification

$$T_P PD(n) \ni (\partial/\partial x^i)_P \equiv E_i \in Sym(n)$$

V-potential function

Def.

$$\varphi^{(V)}(P) = V(\det P), \qquad V(s): \mathbf{R}_+ \to \mathbf{R}$$

-The standard case:

$$V(s) = -\log s \Rightarrow \varphi(P) = -\log \det P$$

Def.

$$\nu_i(s) = \frac{d\nu_{i-1}(s)}{ds} s, \quad i = 1, 2, \cdots, \quad \text{where } \nu_0(s) = V(s)$$

Rem. The standard case *V*= -log:

$$\nu_1(s) = -1, \nu_k(s) = 0, \quad k \geq 2$$

Prop.1 (convexity conditions)

The Hessian matrix of the V-potential is positive definite on $PD(n, \mathbf{R})$ if and only if

For $\forall s > 0$,

i)$\nu_1(s) < 0$, ii)$\beta^{(V)}(s) < \frac{1}{n}$, where $\beta^{(V)}(s) = \frac{\nu_2(s)}{\nu_1(s)}$

Assumption: the convexity conditions hold.

- Riemannian metric is

$$g_P^{(V)}(X,Y)$$

$$= -\nu_1(\det P)\,\mathrm{tr}(P^{-1}XP^{-1}Y) + \nu_2(\det P)\,\mathrm{tr}(P^{-1}X)\,\mathrm{tr}(P^{-1}Y)$$

Here,

X, Y in $sym(n,\mathbf{R})$ ~ tangent vectors at P

Rem. The standard case V= -log:

$$g_P^{(V)}(X,Y) = \mathrm{tr}(P^{-1}XP^{-1}Y)$$

Prop. (affine connections)

Let ∇ be the canonical flat connection on $PD(n, \mathbf{R})$. Then the V-potential defines the following dual connection ${}^*\nabla^{(V)}$ with respect to $g^{(V)}$:

$$\left({}^*\nabla^{(V)}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right)_P = -E_i P^{-1} E_j - E_j P^{-1} E_i - \Phi(E_i, E_j, P) - \Phi^{\perp}(E_i, E_j, P),$$

$$\Phi(X, Y, P) = \frac{\nu_2(s)\,\mathrm{tr}(P^{-1}X)}{\nu_1(s)} Y + \frac{\nu_2(s)\,\mathrm{tr}(P^{-1}Y)}{\nu_1(s)} X,$$

$$\Phi^{\perp}(X, Y, P)$$

$$= \frac{(\nu_3(s)\nu_1(s) - 2\nu_2^2(s))\,\mathrm{tr}(P^{-1}X)\,\mathrm{tr}(P^{-1}Y) + \nu_2(s)\nu_1(s)\mathrm{tr}(P^{-1}XP^{-1}Y)}{\nu_1(s)(\nu_1(s) - n\nu_2(s))} P$$

$(g^{(V)}, \nabla, {}^*\nabla^{(V)})$: Dually flat structure on $PD(n, \mathbf{R})$ induced by the V-potential

divergence function

$$D^{(V)}(P,Q) = \varphi^{(V)}(P) + \varphi^{(V)*}(Q^*) - \langle Q^*, P\rangle$$
$$= V(\det P) - V(\det Q) + \langle Q^*, Q - P\rangle.$$

$$P^* = \mathrm{grad}\varphi^{(V)}(P) = \nu_1(\det P)P^{-1}$$
$$\varphi^{(V)*}(P^*) = n\nu_1(\det P) - \varphi^{(V)}(P).$$

- a variant of relative entropy,
- Pythagorean type decomposition

3. Group Invariance of the structure $(g^{(V)}, \nabla, {}^*\nabla^{(V)})$ on $PD(n, \mathbf{R})$

- Linear transformation on $PD(n, \mathbf{R})$

congruent transformation: $\tau_G P = GPG^T, G \in GL(n, \mathbf{R}),$

the differential: $(\tau_G)_* X = GXG^T$

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- Invariance
  - metric: $g_{\widetilde{P}}(\widetilde{X}, \widetilde{Y}) = g_P(X, Y)$
  - connections: $(\tau_G)_*(\nabla_X Y)_P = (\nabla_{\widetilde{X}} \widetilde{Y})_{\widetilde{P}}$

    and the same for ${}^*\nabla^{(V)}$

where

$$\widetilde{P} = \tau_G P, \widetilde{X} = (\tau_G)_* X, \widetilde{Y} = (\tau_G)_* Y$$

Prop.

The largest group that preserves the dualistic structure $(g^{(V)}, \nabla, {}^*\nabla^{(V)})$ invariant is

$$\tau_G \quad \text{with} \quad G \in SL(n, \mathbf{R})$$

except in the standard case.

Rem. the standard case: τ_G with $G \in GL(n, \mathbf{R})$

Rem. The power potential of the form:

$$V(s) = c_1 + c_2 s^\beta$$

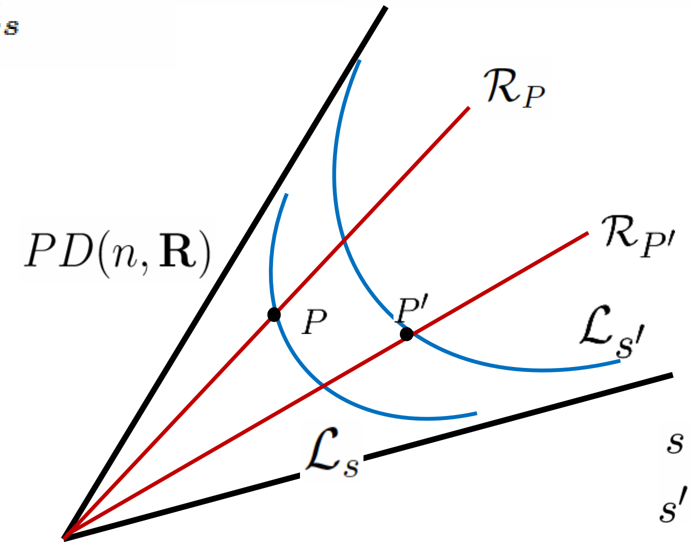
has a special property.

2. Foliated structures

The following foliated structure features the dualistic geometry $(g^{(V)}, \nabla, {}^*\nabla^{(V)})$ induced from every V-potential.

$$PD(n, \mathbf{R}) = \bigcup_{s>0} \mathcal{L}_s, \quad \mathcal{L}_s = \{P | P > 0, \det P = s\}.$$ leaf

$$PD(n, \mathbf{R}) = \bigcup_{P \in \mathcal{L}_s} \mathcal{R}_P. \ \mathcal{R}_P = \{Q | Q = \lambda P, 0 < \lambda \in \mathbf{R}\}$$ ray



$s = \det P,$
$s' = \det P'$

Prop.
Each leaf $\mathcal{L}_s$ and ray $\mathcal{R}_P$ are orthogonal to each other with respect to $g^{(V)}$.

Prop.
Every $\mathcal{R}_P$ is simultaneously a ∇ - and ${}^*\nabla^{(V)}$-geodesic for an arbitrary V-potential.

Submanifolds of const det

Induced geometry on $\mathcal{L}_s$ from $(g^{(V)}, \nabla, {}^*\nabla^{(V)})$

<u>Prop</u>. For any V, the followings hold:

i)Riemannian metric: $\tilde{g}^{(V)} = -\nu_1(s)\tilde{g}^{(-\log)}$

ii)Divergence: $D^{(V)}(P,Q) = -\nu_1(s)D^{(-\log)}(P,Q)$

iii)Dual connection: ${}^*\tilde{\nabla}^{(V)} = \tilde{\nabla}^{(-\log)}$

Submanifolds of const det

Prop.

Each leaf $(\mathcal{L}_s, \tilde{g}^{(V)})$ is a Riemannian symmetric space

$$\mathcal{L}_s \simeq SL(n,\mathbf{R})/SO(n)$$

$\iota_s : \mathcal{L}_s \longrightarrow \mathcal{L}_s$: Involutive isometry of $(\mathcal{L}_s, \tilde{g}^{(V)})$

$$\iota_s P = -\left(\frac{s}{s^*}\right)^{\frac{1}{n}} \mathrm{grad}\varphi^{(V)}(P) = s^{\frac{2}{n}} P^{-1}, \quad s^* = \frac{(-\nu_1(s))^n}{s}.$$

Submanifolds of const det

- Level surface of both $\varphi^{(V)}$ and $\varphi^{(V)*}$ ⇨ ADG
- Normal vector field N

$$N = -\frac{1}{d\varphi^{(V)}(E)} E.$$

$$g^{(V)}(X, E) = d\varphi^{(V)}(X), \quad \forall X \in \mathcal{X}(PD(n, \mathbf{R}))$$

- Centro-affine immersion

$$\nabla_X Y = \tilde{\nabla}_X Y + h(X, Y) N,$$
$$\nabla_X N = -A(X) + \tau(X) N.$$

$$h = \tilde{g}^{(V)}, \quad \tau = 0$$

Submanifolds of const det

Lemma [UOF 00]

- The submfd $(\mathcal{L}_s, \tilde{\nabla}, \tilde{g}^{(V)})$ is 1-conformally flat.
- Assume

$$P, Q \in \mathcal{L}_s, R \in \mathcal{R}_Q, R = \lambda Q, \lambda > 0.$$

Then

$$D^{(V)}(P, R) = \mu D^{(V)}(P, Q) + D^{(V)}(Q, R),$$

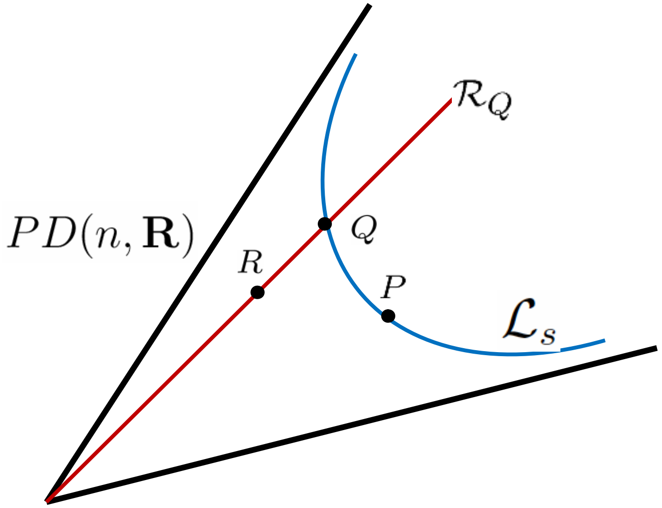
where

$$R^* = \mu Q^*, \ i.e., \ \mu = \lambda^{-1}\nu_1(\det R)/\nu_1(\det Q) > 0.$$

Submanifolds of const det

+ Illustration: decomposition of divergence (1)

If $P, Q \in \mathcal{L}_s$ and $R \in \mathcal{R}_Q$ with $R = \lambda Q,\ \lambda > 0$,



then $D^{(V)}(P, R) = \mu D^{(V)}(P, Q) + D^{(V)}(Q, R)$

where $\mu = \lambda^{-1}\nu_1(\det R)/\nu_1(\det Q) > 0$

Submanifolds of const det

Prop.

Each leaf $\mathcal{L}_s$ is a homogeneous space of constant negative curvature $k_s = 1/(\nu_1(s)n)$.

$$R(X,Y)Z = k_s\{g(Y,Z)X - g(X,Z)Y\}.$$

Shown by Gauss eq. and $A = k_s I$

Lemma (modified Pythagorean)[Kurose94]

$P, Q, R \in \mathcal{L}_s \quad \tilde{\gamma}_{PQ} \perp \tilde{\gamma}^*_{QR}$ at $Q \Rightarrow$

$$D^{(V)}(P,R) = D^{(V)}(P,Q) + D^{(V)}(Q,R) - k_s D^{(V)}(P,Q) D^{(V)}(Q,R).$$

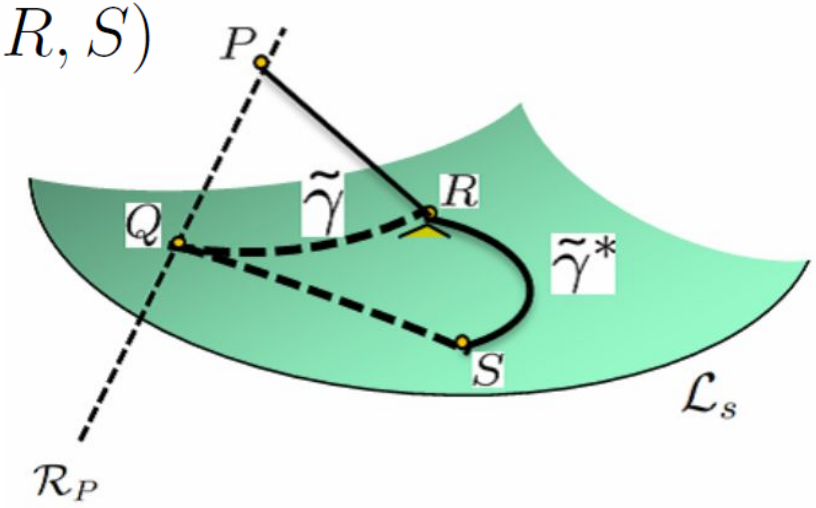
Submanifolds of const det

+ Combining the two decomposition results

<u>Prop.</u>

If $\tilde{\nabla}$-*geodesic* $\tilde{\gamma}$ and ${}^{*}\tilde{\nabla}$-*geodesic* $\tilde{\gamma}^{*}$ are orthogonal at R, then

$$D^{(V)}(P,S) = D^{(V)}(P,R) + \kappa D^{(V)}(R,S)$$



$$\kappa = \lambda\{1 - k_s D^{(V)}(Q,R)\} > 0$$

$$Q \in \mathcal{L}_s \cap \mathcal{R}_P$$

3. Application to multivariate statistics

- Non Gaussian distribution (generalized exponential family)
  - Robust statistics
    - beta-divergence,
    - Machine learning, and so on
  - Nonextensive statistical physics
    - Power distribution,
    - generalized (Tsallis) entropy, and so on

U-model and U-divergence

- U-model

Def.

Given a convex function U on $\mathbf{R}$ and set $u=U'$, U-model is a family of elliptic pdf's specified by P:

$$\mathcal{M}_U = \left\{ f(x,P) = u\left(-\frac{1}{2}x^T P x - c_U(\det P)\right) : P \in PD(n,\mathbf{R}) \right\}$$

$c_U(\det P)$:normalizing const.

Rem. When U=exp, the U-model is the family of Gaussian distributions.

U-divergence:

Natural closeness measure on the U-model

$$D_U(f,g) = \int U(\xi(g)) - U(\xi(f)) - \{\xi(g) - \xi(f)\} f dx \ ,$$

$$f, g \in \mathcal{M}_U$$

where ξ is the inverse function of u.

Rem. When U=exp, the U-divergence is the Kullback-Leibler divergence (relative entropy).

Example: beta-model and beta-divergence (1)

- Beta-model $\mathcal{M}_\beta$
  - For $\beta \neq 0$ and $\beta \neq -1$

$$U(s) = \begin{cases} \dfrac{1}{\beta+1}(1+\beta s)^{(\beta+1)/\beta}, & s \in I_\beta = \{s \in \mathbf{R} | 1+\beta s > 0\} \\ 0, & \text{otherwise} \end{cases}$$

$$u(s) = \begin{cases} \dfrac{dU(s)}{ds} = (1+\beta s)^{1/\beta} & s \in I_\beta = \{s \in \mathbf{R} | 1+\beta s > 0\} \\ 0, & \text{otherwise} \end{cases}$$

$$\xi(t) = \frac{t^\beta - 1}{\beta}, \qquad t > 0$$

  - q-exponential and q-logarithmic functions

Example: beta-model and beta-divergence (2)

- Beta-divergence

$$D_\beta(f,g) = \int \frac{g(x)^{\beta+1} - f(x)^{\beta+1}}{\beta+1} - \frac{f(x)\{g(x)^\beta - f(x)^\beta\}}{\beta} dx$$

IG induced from divergences

- Divergence induces stat mfd structure.

$$g^{(D)}(X,Y) = -D(X|Y),$$

$$g^{(D)}(\nabla^{(D)}_X Y, Z) = -D(XY|Z),$$

$$g^{(D)}({}^*\nabla^{(D)}_X Y, Z) = -D(Z|XY),$$

where

$$D(X_1 \cdots X_n | Y_1 \cdots Y_m)(p) = (X_1)_p \cdots (X_n)_p (Y_1)_q \cdots (Y_m)_q D(p,q)|_{p=q}$$

Relation between the U- and V-geometries

Prop.

IG on $\mathcal{M}_U$ induced from D_U coincides with $(g^{(V)}, \nabla, {}^*\nabla^{(V)})$ derived from the following V-potential function:

$$V(s) = s^{-\frac{1}{2}} \int U\left(-\frac{1}{2}x^T x - c_U(s)\right) dx + c_U(s), \quad s > 0.$$

Group invariance for the power potentials $V(s) = c_1 + c_2 s^\beta$

Prop.

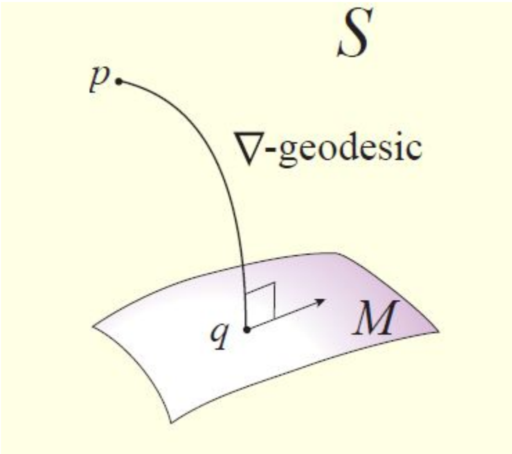
V is of the power form ⟺

1) Orthogonality is $GL(n)$-invariant.

2) The dual affine connections derived from the power potentials are $GL(n)$-invariant.

Hence,

- Both ∇ - and ${}^*\nabla^{(V)}$ -projections are $GL(n)$ -invariant.



Thm [O & Eguchi 13]

IG on $\mathcal{M}_\beta$ induced from D_β coincides with $(g^{(V)}, \nabla, {}^*\nabla^{(V)})$ on $PD(n, \mathbf{R})$ induced from

$$V(s) = \begin{cases} \dfrac{1}{\beta} + c^+ s^{1/(2n_\beta)}, & \beta > 0 \\ \dfrac{1}{\beta} + c^- s^{1/(2n_\beta)}, & -\dfrac{2}{n+2} < \beta < 0 \end{cases}$$

Implication: statistical inference on $\mathcal{M}_\beta$ using D_β is GL(n)-invariant.

Conclusions

- Derived dualistic geometry is invariant under the SL(n,R)-group actions.
  - For power funcion, dual connections and orthogonality are GL(n,R)-invariant.
- Each leaf is a homogeneous manifold with a negative constant curvature.
- Decomposition of the divergence function
- Correspondence between the U- and V-geometries
  - Statistical inference on Beta-model using dual projections are GL(n,R)-invariant.

Main References

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