双対平坦空間とシンプレクティック構造 I

On dually flat spaces and symplectic structures I

野田　知宣　（大阪歯科大学）

統計多様体の幾何学とその周辺 (3) ／ 幾何学と諸科学の連携調査
北海道大学理学部 ４号館５０１
平成23年12月3日（先勝）

Introduction

In Affine differential geometry

$f : M^n \to \mathbb{R}^{n+1}$:an immersion

$\xi \in \Gamma(T\mathbb{R}^{n+1})$:transv. v.f.

$(T_{f(p)}\mathbb{R}^{n+1} = f_{*p}(T_pM) \oplus \mathrm{Span}(\xi_p))$

$$\tilde{\nabla}_{f_*X}(f_*Y) = f_*(\underline{\nabla_X Y}) + \underline{g(X,Y)\xi}$$

induced affine f.f.

Rem.

- $T^{\nabla} = 0$, g: symm.
- g: non-deg. $\Rightarrow$ (M, ∇, g): stati. mfd

$(\nabla g \in \mathrm{Sym}^3)$

In Information Geometry

$(X, \mathfrak{B}, dx)$: measurable sp.

$$S = \{(\theta^1, \ldots, \theta^n) \in \Theta;\ \Theta \subset \mathbb{R}^n\} \subset \mathcal{P}(X)$$

$$\parallel$$

$$\{p\ ;\ \int_X p dx = 1,\ p > 0\}$$

$$g = [E[\partial_i \ell_\theta\ \partial_j \ell_\theta]]$$

$$\Gamma^{(\alpha)}_{ijk} = \Gamma^0_{ijk} - \frac{\alpha}{2} T_{ijk} \longrightarrow \nabla^{(\alpha)}$$

但し
$$\ell_\theta = \log p_\theta$$
$$T_{ijk} = E[\partial_i \ell_\theta \partial_j \ell_\theta \partial_k \ell_\theta]$$

Stati. str. (g, ∇)

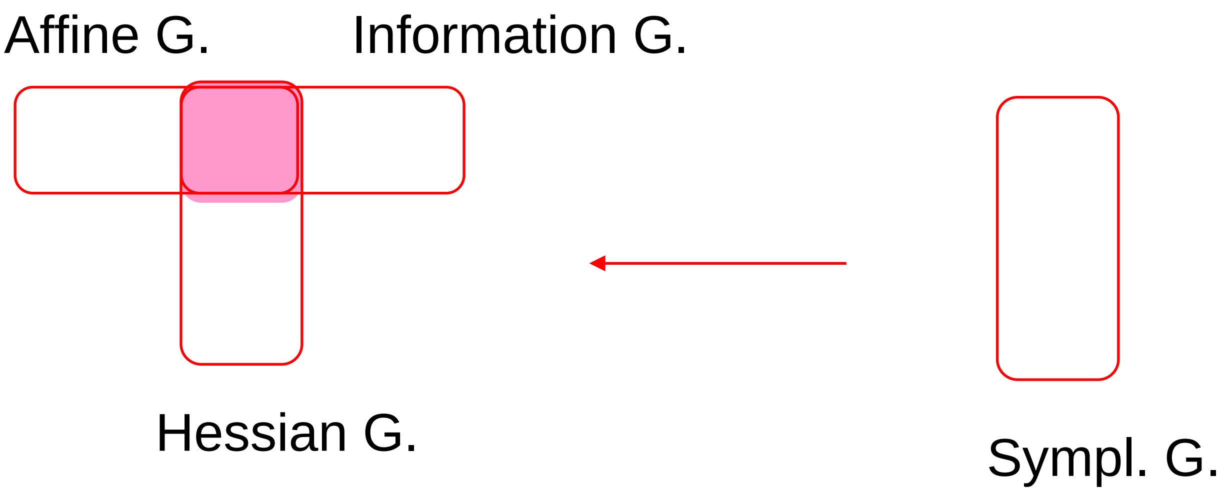
$\cdot T^{\nabla} = 0$

$\cdot \nabla g \in \mathrm{Sym}^3$ ← Codazzi eq.

Dually flat (g, ∇, ∇^*)

$\cdot R^{\nabla} = 0 \ (R^{\nabla^*} = 0)$

· Hessian



Stati. str. (g, ∇)

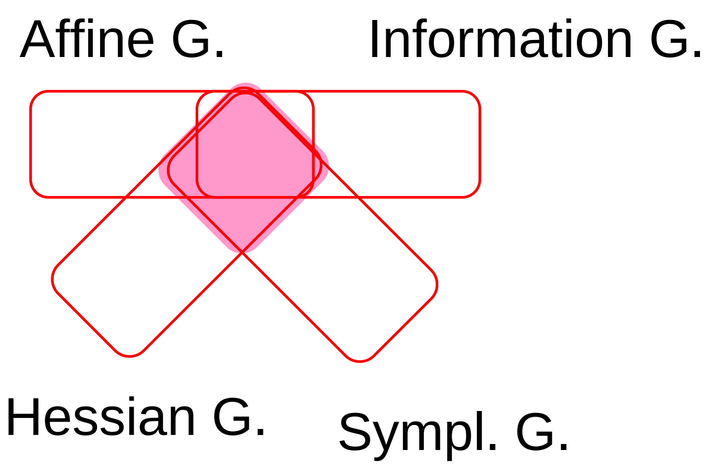
$\cdot T^{\nabla} = 0$

$\cdot \nabla g \in \mathrm{Sym}^3$ ← Codazzi eq.

Dually flat (g, ∇, ∇^*)

$\cdot R^{\nabla} = 0 \; (R^{\nabla^*} = 0)$

$\cdot$ Hessian



offering

(i) special Kähler str.

(ii) Fisher $\longleftrightarrow$ Fubini-Study

(iii) grad. flow $\longleftrightarrow$ Hamiltonian flow

History：
Friedrich（1991）, Nakamura（1993）, Hitchin（1997）,
Barndorff-Nielsen & Jupp（1997）, Shishido（2005）, Zhang … etc.

(i) special Kähler manifold

stat. str. ＋ sympl. str. ＋ flatness

Lemma.

(M, g, ∇) :statistical manifold

J: alm. cpx.str. s.t. $g(JX, JY) = g(X, Y)$

- $\nabla^* = \nabla - J(\nabla J)$
- $(\nabla_X J)Y = (\nabla_Y J)X$

$\Rightarrow \omega(X, Y) := g(JX, Y)$ then $d\omega = 0,\ \nabla\omega = 0.$

(M, ω) : symplectic manifold

∇ : symplectic connection (i.e., torsionfree and preserve ω)

J: alm. cpx. str. compatible with ω

$\Rightarrow$ (i) $\nabla^* = \nabla - J(\nabla J)$

(ii) $(\nabla_X J)Y = (\nabla_Y J)X$ $\Leftrightarrow$ (S, g, ∇) is statistical manifold

Lemma.

$(M, g, J, \omega, \nabla, \nabla^*)$: stati. & sympl.

（ $\Leftarrow$ special symplectic manifold）

(i) $\nabla^* \omega = 0$ （i.e. ∇^* is a symplectic connection）

(ii) $\nabla^0 := \frac{1}{2}(\nabla + \nabla^*)$

then $\nabla^0 g = 0$, $T^{\nabla^0} = 0$ and $\nabla^0 J = 0$.

Levi-Civita

Integrable

c.f. [Furuhata]
holomorphic statistical

Definition.

(M, J):cpx.mfd

∇:torsionfree flat conn.

ω:sympl.

(i) (M, J, ∇):special complex if $(\nabla_X J)Y = (\nabla_Y J)X$.

(ii) (M, J, ∇, ω):special symplectic if $\nabla\omega = 0$.

(iii) (M, J, ∇, ω):special Kähler if ω is J-invariant.

cf. $d_\nabla J = 0 \in \Omega^2(M, TM)$

$d_\nabla : \Omega^p(M, TM) \to \Omega^{p+1}(M, TM)$, $J \in \Omega^1(M, TM)$

Fact. （Cortés）

(M, J, g, ∇) simply conn. sp. K.

$\Rightarrow {}^{\exists} f : M \to \mathbb{R}^{2n+1}$ s.t. ∇:induced, g: affine f.f.

Fact. （Alekseevsky-Cortés-Devchand）

(i) $U \subset \mathbb{C}^n$: open, conn.

$F : U \to \mathbb{C}$ hol. s.t. $(\frac{\partial^2 F}{\partial z_i \partial z_j})$: invertible

$M_F := dF(U)$: image

J stand. cpx. str. on $T^*\mathbb{C}^n = \mathbb{C}^{2n}$

$g := \mathrm{Re}(\sqrt{-1}\Omega(\cdot, \bar{\cdot})|_{M_F})$, $\Omega = \sum dz_i \wedge dw_i$

∇:torsionfree affine s.t. $(x_i = \mathrm{Re} z_i, y_j = \mathrm{Re} w_j)$

affine coord.

$\Rightarrow (M_F, J, g, \nabla)$: sp. K.

(ii) inverse is also true.

$(V, \langle , \rangle_V)$:real symplectic v. sp.

(i) sympl. str. on $V_1 \times V_2$:

Ω_1:the can. sympl. str. on $V_1 \times V_2 \cong V_1 \times V_1^*$.

Ω_2:sympl. str. $\pi_1^* \langle , \rangle_1 - \pi_2^* \langle , \rangle_2$ on $(V_1, \langle , \rangle_1) \times (V_2, \langle , \rangle_2)$.

(ii) metric on $V_1 \times V_2$:

$$g((a,b),(a,b)) := \frac{1}{2} \langle a, b \rangle_V .$$

Fact. (Hitchin)

$M \subset V_1 \times V_2$:Lagr. w.r.t. Ω_1 and Ω_2 s.t. transv. to π_1 and π_2

$\Rightarrow g|_M$ is special Kähler.

${}^\forall g_M$:special Kähler on a mfd M

$\Rightarrow {}^\exists V$ s.t. $g_M = g|_M$ for $M \subset V \times V$.

$(V, \langle , \rangle_V)$:real symplectic v. sp.

$(x^1, \ldots, x^{2n})$: coord.

$\langle \cdot, \cdot \rangle_V = \omega_{ij} dx^i \wedge dx^j$

$V_1 \times V_2 = \{(x^1, \ldots, x^{2n}, y^1, \ldots, y^{2n})\}$

$d\xi_i := \omega_{ij} dy^j$

$\Rightarrow \quad \Omega_1 = 2\omega_{ij} dx^i \wedge dy^j \longleftarrow V_1 \times V_2 \cong V_1 \times V_1^*$

$$\Omega_2 = \omega_{ij} dx^i \wedge dx^j - \omega_{ij} dy^i \wedge dy^j$$
$$= \omega_{ij} dx^i \wedge dx^j + \omega^{ij} d\xi_i \wedge d\xi_j \longleftarrow \pi_1^* \langle , \rangle_1 - \pi_2^* \langle , \rangle_2$$

outline of proof

◎ ∇ and ω

identify

$$\mathrm{pr}_1 : V_1 \times V_2 \to V_1 \quad \text{で} \quad M \cong V_1$$

∇ : flat
$< , >_V$: symplectic

$\longrightarrow$

∇ : flat
ω : symplectic
$\nabla \omega = 0$

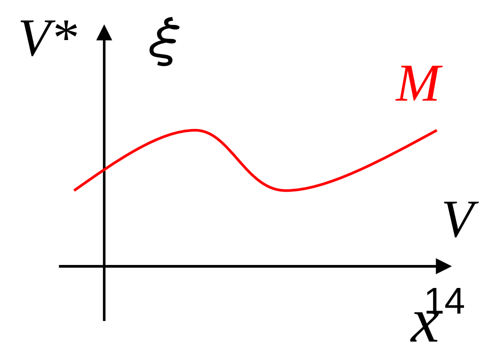
} on M

◎ g

$g = g|_M$ （正定値とは限らない）

$M \subset (T^*V, \Omega_1)$:Lagr. & $\pi_2 \pitchfork M$

$\Rightarrow {}^\exists \phi$ s.t. $\xi_j = \dfrac{\partial \phi}{\partial x_j}$



- $\frac{\partial}{\partial x^i} \longleftrightarrow X_i = \frac{\partial}{\partial x^i} + \frac{\partial \xi_k}{\partial x^i}\frac{\partial}{\partial \xi_k} = \frac{\partial}{\partial x^i} + \frac{\partial^2 \phi}{\partial x^i \partial x^k}\frac{\partial}{\partial \xi_k}$

  $\Cap \qquad\qquad \Cap$

  $\Gamma(TM) \qquad \Gamma(V \times V)$

- $g((x,\xi),(x,\xi)) = \frac{1}{2}\langle x, \xi\rangle_V$ on $V \times V$

  $\Rightarrow \quad g_{kj}dx^k dx^j = g(X_k.X_j)dx^k dx^j = \frac{\partial^2 \phi}{\partial x^k \partial x^j}dx^k dx^j$

  on M

◎ *I*

$$\Omega_2|_M \equiv 0$$

$$\Leftrightarrow \quad 0 = \Omega_2(X_k, X_j) = \omega_{kj} + g_{ka}g_{jb}\omega^{ab}$$

$$\Leftrightarrow \quad I^2 = -id, \text{ where } I^j_k = \omega^{ja}g_{ak}$$

$$X_\phi = \omega^{ij}\frac{\partial \phi}{\partial x^i}\frac{\partial}{\partial x^j} =: a_j\frac{\partial}{\partial x^j}$$ ← Hamiltonian v.f.

$$\Rightarrow \quad \frac{\partial a_j}{\partial x^k} = \omega^{ij}g_{ik} = I^j_k.$$

$$\Rightarrow \quad I = d_\nabla X_\phi$$

∇ : flat $\quad \Rightarrow \quad d_\nabla I = d^2_\nabla X_\phi = 0.$

◎ Integrability of I

$$z_j := x^j - \sqrt{-1}\sum \omega^{jk}\frac{\partial\phi}{\partial x^k}$$

$\Rightarrow$ $$dz_j = dx_j - \sqrt{-1}\omega^{jk}g_{ka}dx^a = dx_j - \sqrt{-1}I_a^j dx^a$$

: type (1,0)

$$\Lambda^{1,0} \supset E := \mathrm{Span}\{dz_1, \ldots, dz_{2n}\}$$

$\Rightarrow$ $$2dx^j = dz_j + d\bar{z}_j$$

$dz^1 \sim dx^{2n}$ is basis for Λ^1

$\Rightarrow$ rank $E = n$ □

Ex. (Hitchin) M: moduli sp. of cpt. cpx. Lagr.

(X, ω^c):cpx.sympl., $\omega^c = \omega_1 + \sqrt{-1}\omega_2$

$Y \subset X$:cpt. cpx. Lagr.

$$\Rightarrow \quad T_{[Y]}M \cong H^0(Y, N) \cong H^0(Y, T^*)$$

$V := H^1(Y, \mathbb{R})$

$\langle a, b \rangle := \int_Y \alpha \wedge \beta \wedge h_Y^{n-1}$

$\left(\begin{array}{l} a = [\alpha],\ b = [\beta] \in H^1, \\ h_Y \ :\ \text{Kähler form} \end{array} \right)$

$\Rightarrow$

$$\left. \begin{array}{lll} \omega_1 & \rightsquigarrow & \mu : M \to V \\ \omega_2 & \rightsquigarrow & \nu : M \to V \end{array} \right\} \Rightarrow \quad (\mu, \nu) : M \to V \times V$$

sp. K. str. on M

文献

DV. Alekseevsky, V. Cortés, C. Devchand :
Special Complex Manifolds, J. Geom. Phys. 42 (2002), 85-105.

O.E. Barndorff-Nielsen and P.E. Jupp :
Statistics, yokes and symplectic geometry, Annales de la facultè des sciences de Toulouse Sèr. 6, 6 no. 3 (1997), p. 389-427.

V. Cortés :
A holomorphic representation formula for parabolic hyperspheres. Proceedings of the international conference ”PDEs, Submanifolds and Affine Differential Geometry” (math.DG/0107037)

T. Friedrich :
Die Fisher-Information und symplektische Strukturen, Math. Nachr. 153 (1991), 273--296.

H. Furuhata :
Hypersurfaces in statistical manifolds, Differential Geom. Appl. 27(2009), 420--429

N. Hitchin:
The moduli space of special Lagrangian submanifolds,
Asian J. Math. 3 (1999), no. 1, 77--91

Y. Nakamura :
Completely integrable gradient systems on the manifolds of Gaussian and multinomial distributions, Japan J. Indust. Appl. Math. 10 (1993), 179--189.

T. Noda :
Symplectic structures on Statistical manifolds, J. Aust. Math. Soc. 90 (2011), 371--384.

Y. Shishido :
Strong symplectic structures on spaces of probability measures with oisitive density function, Proc. Japan Acad., 81, Ser. A (2005), 134--136.