Pseudo-parallel CR submanifolds of a complex space form

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Dec. 2. 2011

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- $\bar{M}$: m-dimensional Riemannian manifold,
- G: Riemannian metric of $\bar{M}$,
- $\bar{\nabla}$: operator of covariant differetiation on $\bar{M}$.

- M: n-dimensional submanifold isometrically immersed in $\bar{M}$,
- g: induced metric on M,
  $(g(X,Y) := G(X,Y)$, X, Y : vector fields in $M)$
- $T_x(M)$, $T_x(M)^\perp$: tangent space and the normal space of M at x.

X, Y: vector field on M, V: normal vector field on M.

Gauss formula

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y).$$

($\nabla_X Y$: tangential component, $B(X, Y)$: normal component)

Then

- ∇ is the operator of covariant differentiation w.r.t. g,
- $B(aX, bY) = abB(X, Y)$, a, b: functions,
- $B(X, Y) = B(Y, X)$.

We call

∇: induced connection,

B: second fundamental form.

Weingarten formula

$$\bar{\nabla}_X V = -A_V X + D_X V,$$

($-A_V X$: tangential component, $D_X V$: normal component)

Then we can show that

- $A_{bV}(aX) = abA_V X$,
- $D_{aX}(bV) = a(Xb)V + abD_X V$,
- $g(B(X,Y),V) = g(A_V X, Y)$,
- D is a metric connection in the normal bundle $T(M)^\perp$ of M in $\bar{M}$ w.r.t. the induced metric on $T(M)^\perp$.

We call

A: associated second fundamental form to B

(or, second fundamental form)

D: normal connection.

$$\begin{aligned}(\nabla_X B)(Y,Z) &:= D_X B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z),\\ (\nabla_X A)_V Y &:= \nabla_X (A_V Y) - A_{D_X V} Y - A_V \nabla_X Y.\end{aligned}$$

Using the Gauss and Weingarten formulas,

$$\begin{aligned}\bar{R}(X,Y)Z &= \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z\\ &= R(X,Y)Z - A_{B(Y,Z)} X + A_{B(X,Z)} Y\\ &\quad + (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z).\end{aligned}$$

Equation of Gauss:

$$\begin{aligned}g(\bar{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) - g(B(X,W),B(Y,Z))\\ &\quad + g(B(Y,W),B(X,Z)).\end{aligned}$$

Equation of Codazzi:

$$(\bar{R}(X,Y)Z)^{\perp} = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z).$$

- curvature tensor of the normal bundle of M

$$R^{\perp}(X,Y)V := D_X D_Y V - D_Y D_X V - D_{[X,Y]}V.$$

Using the Gauss and Weingarten formulas,

$$\begin{aligned}\bar{R}(X,Y)V &= \bar{\nabla}_X \bar{\nabla}_Y V - \bar{\nabla}_Y \bar{\nabla}_X V - \bar{\nabla}_{[X,Y]}V \\ &= R^{\perp}(X,Y)V - B(X, A_V Y) + B(Y, A_V X) \\ &\quad -(\nabla_X A)_V Y + (\nabla_Y A)_V X,\end{aligned}$$

from which we obtain
Equation of Ricci:

$$g(\bar{R}(X,Y)V,U) = g(R^{\perp}(X,Y)V,U) + g([A_U, A_V]X,Y).$$

- $\bar{M}$: Kaehler manifold,
- J: complex structure of $\bar{M}$,
- G: Kaehler metric of $\bar{M}$,
    - $\forall X, Y,\ G(JX, JY) = G(X, Y)$
    - the fundamental 2-form $\Phi(X, Y) := G(X, JY)$ is closed.
- $\bar{\nabla}$: the operator of covariant differentiation in $\bar{M}$.

holomorphic sectional curvature

p: 2-dimensional subspace spanned by orthonormal basis $\{X, Y\}$,

$$K(p) := G(\bar{R}(X, Y)Y, X).$$

If p is invariant by the complex structure J, then $K(p)$ is called the holomorphic sectional curvature by p. Then,

$$K(p) = G(\bar{R}(X, JX)JX, X).$$

If $K(p)$ is constant for all planes p in $T_x(M)$ invariant by J and for all $x \in \bar{M}$, then $\bar{M}$ is called a complex space form.

Proposition.

Let $\bar{M}$ be a real $2n$-dimensional Kaehlerian manifold. If $\bar{M}$ is of constant curvature, then $\bar{M}$ is flat provided $n > 1$.

Theorem.

A Kaehlerian manifold $\bar{M}$ is of constant holomorphic sectional curvature c if and only if

$$\begin{aligned}\bar{R}(X,Y)Z &= \frac{1}{4}\big(G(X,Z)Y - G(Y,Z)X + G(JX,Y)JY \\ &\quad -G(JY,Z)JX + 2G(JX,Y)JZ\big).\end{aligned}$$

Theorem.

A simply connected complete Kaehlerian manifold $\bar{M}$ of constant holomorpic sectional curvature c can be identified with the complex projective space $\mathbb{C}P^n$, the complex hyperbolic space $\mathbb{C}H^n$ or $\mathbb{C}^n$ according as $c > 0$, $c < 0$ or $c = 0$.

For $X \in TM,\ V \in TM^{\perp}$, we put

$$JX = PX + FX, \quad JV = tV + fV,$$

PX: tangential part of JX, FX: normal part of JX,
tV : tangential part of JV, fV: normal part of JV.

from $J^2 = -I$, we have

$$P^2 = -I - tF,\ FP + fF = 0,$$
$$Pt + tf = 0,\ f^2 = -I - Ft.$$

$$(\nabla_X P)Y := \nabla_X(PY) - P\nabla_X Y, \ (\nabla_X F)Y := D_X(FY) - F\nabla_X Y,$$
$$(\nabla_X t)V := \nabla_X(tV) - tD_X V, \quad (\nabla_X f)V := D_X(fV) - fD_X V.$$

From the Gauss and Weingarten formulas, we have

$$tB(X,Y) = fB(X,Y) = (\nabla_X P)Y - A_{FY}X + B(X,PY) + (\nabla_X F)Y.$$

Comparing the tangential and normal parts, we have

$$(\nabla_X P)Y = A_{FY}X + tB(X,Y), \ (\nabla_X F)Y = -B(X,PY) + fB(X,Y).$$

Similarly,

$$(\nabla_X t)V = -PA_V X + A_{fV}X, \ (\nabla_X f)V = -FA_V X - B(X,tV).$$

Equations of Gauss and Codazzi

When $\bar{M}$ is a complex space form,

- equation of Gauss

$$\begin{aligned} R(X,Y)Z &= \bar{R}(X,Y)Z + A_{B(Y,Z)}X - A_{B(X,Z)}Y \\ &= c\{g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX \\ &\quad - g(PX,Z)PY - 2g(PX,Y)PZ\} \\ &\quad + A_{B(Y,Z)}X - A_{B(X,Z)}Y. \end{aligned}$$

- equation of Codazzi

$$\begin{aligned} &g((\nabla_X A)_V Y, Z) - g((\nabla_Y A)_V X, Z) \\ &= (\bar{R}(X,Y)Z)^{\perp} \\ &= c\{g(Y,PZ)g(X,tV) - g(X,PZ)g(Y,tV) \\ &\quad - 2g(X,PY)g(Z,tV)\}. \end{aligned}$$

Normal curvature tensor

We define the curvature tensor $R^\perp$ of the normal bundle of M by

$$R^\perp(X,Y)V = D_X D_Y V - D_Y D_X V - D_{[X,Y]}V.$$

Then we have the *equation of Ricci*

$$\begin{aligned} &G(R^\perp(X,Y)V,U) + g([A_U,A_V]X,Y) \\ &\quad = c\{g(Y,tV)g(X,tU) - g(X,tV)g(Y,tU) \\ &\quad - 2g(X,PY)g(V,fU)\}. \end{aligned}$$

The normal connection is flat $\overset{def}{\Longleftrightarrow}$ $R^\perp = 0$,
semi-flat $\overset{def}{\Longleftrightarrow}$ $R^\perp(X,Y)V = 2cg(X,PY)fV$

Definition of a CR submanifold in a Kaehler manifold

Definition

M : CR submanifold if there exists differentiable distribution $H : x \longrightarrow H_x \subset T_x(M)$ on M s.t.

(i) $\forall x \in M,\ JH_x = H_x,$

(ii) $\forall x \in M,\ JH_x^{\perp} \subset T_x(M)^{\perp}$

We call H_x a *holomorphic tangent space*.

At each point $x \in M$,

$$T_x(M) = H_x + H_x^{\perp},$$
$$T_x(M)^{\perp} = JH_x^{\perp} + N_x$$

$$T_x(M) = H_x + H_x^{\perp},$$
$$T_x(M)^{\perp} = JH_x^{\perp} + N_x$$

We put dim$H_x := h$, dim$H_x^{\perp} = q$, codim$M$=p.

- If $p = 1$, then M is a real hypersurface,
  (If $\bar{M} = M^m(c)$, $H_x^{\perp} = \{\xi\}$)
- If $q = 0$ ($JT_x(M) \subset T_x(M)$), M is called invariant submanifold,
- If $h = 0$ ($JT_x(M) \subset T_x(M)^{\perp}$), M is called anti-invariant (totally real) submanifold,
- If $p = q$, M is called a generic submanifold,
- If $p > 0$, $q > 0$, M is called non-trivial (proper).

the name of CR submanifold

M': differentiable manifold,
$T(M')^C$: complexified tangent bundle.

A CR structure on M' is a complex subbundle D of $T(M')^C$ s.t. $D_x \cap \bar{D}_x = \{0\}$ and D is involutive.
On a CR manifold, there exists a (real) distribution H and a field of endomorphism $p : H \longrightarrow H$ s.t. $p^2 = -I_H$.

$$H = Re(D \oplus \bar{D}),\ D_x = \{X - \sqrt{-1}pX \ : \ X \in H_x\}.$$

Theorem (B. Y. Chen 1973)

M: CR submanifold of a Hermitian manifold $\bar{M}$. If M is non-trivial, then M is a CR manifold.

f-structures

- M : CR submanifold $\Leftrightarrow$ $FP = 0$
  (equivalently, $fF = 0,\ tf = 0,\ Pt = 0$).
- M: CR submanifold of a Kaehlerian manifold $\bar{M}$, then P is an f-structure in M (i.e. $P^3 + P = 0$) and f is an f-structure (i.e. $f^3 + f = 0$) in the normal bundle of M.

Relation to the submanifolds in a sphere

The method of the standard fibration to push known theorems on the sphere down to $\mathbb{C}P^m$ by considering the commutative diagram (H. B. Lawson Jr., M. Okumura, K.Yano, M. Kon);

$$\begin{array}{ccc} N & \xrightarrow{i'} & S^{2m+1} \\ \pi \downarrow & & \downarrow \bar{\pi} \\ M & \xrightarrow{i} & \mathbb{C}P^m. \end{array}$$

$S^{2m+1} = \{z \in \mathbb{C}^{m+1} : |z| = 1\}$: $(2m+1)$-dimensional unit sphere,
We put
$\pi' : T_z(\mathbb{C}^{m+1}) \longrightarrow T_z(S^{2m+1})$: the orthogonal projection.

- contact metric structure (ϕ, ξ, η, G) on S^{2m+1},
  - For any point $z \in S^{2m+1}$ we put $\xi := Jz$,
    (J : almost complex structure of C^{m+1})
  - $\phi := \pi' \cdot J$,
  - η : 1-form dual to ξ,
  - G: standard metric tensor field on S^{2m+1}
    s.t. $G(\phi X, \phi Y) = G(X, Y) - \eta(X)\eta(Y)$.

Then, we have $\eta(\xi) = 1$ and $\phi^2(X) = -X + \eta(X)\xi$.

$\bar{\pi} : S^{2m+1} \longrightarrow \mathbb{C}P^m$: standard fibration,
N : $(n+1)$-dimensional submanifold immersed in S^{2m+1},
M : n-dimensional submanifold in $\mathbb{C}P^m$.

We assume that N is tangent to the vertical vector field ξ of S^{2m+1} and there exists a fibration $\pi : N \to M$ s.t. the following diagram commutes and the immersion i' is a diffeomorphism on the fibers.

$$\begin{array}{ccc} N & \xrightarrow{i'} & S^{2m+1} \\ \pi \downarrow & & \downarrow \bar{\pi} \\ M & \xrightarrow{i} & \mathbb{C}P^m, \end{array}$$

where the immersion i' is a diffeomorphism on the fibres.

$$\begin{array}{ccc} N & \xrightarrow{i'} & S^{2m+1} \\ \downarrow & & \downarrow \pi \\ M & \xrightarrow{i} & \mathbb{C}P^m, \end{array}$$

*: the horizontal lift.

$$(JX)^* = \phi X^*,\ G(X^*, Y^*) = g(X, Y)^*$$

for any vectors X and Y tangent to $\mathbb{C}P^m$.

α: second fundamental form of N in S^{2m+1},
B: second fundamental form of M in $\mathbb{C}P^m$.
Then,

$$\alpha(X^*, Y^*) = B(X, Y)^*, \quad \alpha(\xi, \xi) = 0.$$

Besides,

$$(\nabla_{X^*}\alpha)(Y^*, Z^*) = [(\nabla_X B)(Y, Z) + g(PX, Y)FZ + g(PX, Z)FY]^*,$$
$$(\nabla_{X^*}\alpha)(Y^*, \xi) = [fB(X, Y) - B(X, PY) - B(Y, PX)]^*,$$
$$(\nabla_{X^*}\alpha)(\xi, \xi) = -2(FPX)^*$$

for any vectors X, Y and Z tangent to M.
If α is parallel, then $FP = 0$ and M is a CR submanifold of $\mathbb{C}P^m$.

$K^{\perp}$: the curvature tensor of the normal bundle of N.
Then,

$$G(K^{\perp}(X^*, Y^*)V^*, U^*) = [g(R^{\perp}(X, Y)V, U) - 2g(X, PY)g(fV, U)]$$
$$G(K^{\perp}(X^*, \xi)V^*, U^*) = g((\nabla_X f)V, U)^*$$
$$(X, Y \in TM, \quad V, U \in TM^{\perp})$$

Lemma

The normal connection of N in S^{2m+1} is flat
$\Longleftrightarrow$
the normal connection of M in $\mathbb{C}P^m$ is semi-flat and $\nabla f = 0$.

Example

$$N = S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k) \longrightarrow S^{n+k}, \quad n+1 = \sum_{i=1}^{k} m_i.$$

($m_1, \cdots m_k$: odd numbers, $r_1^2 + \cdots + r_k^2 = 1$)

$M = \pi(N)$: CR submanifold of $\mathbb{C}P^m$ $(2m+1 > n+k)$.

- Since the normal connection of N is flat, the normal connection of M is semi-flat and $\nabla f = 0$.
- When $r_i = (m_i/(n+1))^{1/2}$ $(i = 1, \cdots, k)$, N is minimal and hence M is also minimal.

pseudo-parallel CR submanifold

As a generalisation of Einstein manifold,
a Riemannian manifold (M, g) has a semi-parallel Ricci operator if

$$R(X, Y)S = 0$$

for all vector fields X and Y on M. As a generalization of locally

symmetric Riemannian manifold,
(M, g) is said to be (locally) semi-symmetric if

$$R(X, Y)R = 0$$

for all vector fields X and Y on M.

- In 1972, P. J, Ryan proposed a question:
  " Euclidean hypersurfaces with semi-parallel Ricci operator $(R \cdot S = 0)$ are semisymmetric $(R \cdot R = 0)$? "
- F. Defever constructed hypersurface in Euclidean 6-space E^6 with semi-parallel Ricci operator which are not semi-symmetric.
- B. E. Abdalla and F. Dillen generalized Defevers construction for hypersurfaces of dimension greater than 5.

One of the generalization of Ricci semi-parallelity is the Ricci pseudo-parallelity:

$$R(X,Y)\cdot S=\alpha(X\wedge Y)S,$$

α: function.

$$(X\wedge Y)Z:=g(Y,Z)X-g(X,Z)Y.$$

ex)

every Cartan's isoparametric hypersurface in spheres has pseudo-parallel Ricci operator.

Theorem (K, 2011)

M : an n-dimensional proper CR submanifold of a complex space form $M^m(c)$, $c \neq 0$, with semi-flat normal connection. We suppose $h > 2$.
If $(R(X,Y)A)_V = \alpha((X \wedge Y)A)_V$ for any V normal to M and X and Y tangent to M, α being a function, then M is a hypersurface of totally geodesic $M^{\frac{n+1}{2}}(c)$ in $M^m(c)$.

Theorem (G. A. Lobos and M. Ortega, 2004)

Let M be a connected pseudo-parallel real hypersurface in $M^n(c),\ n \geq 2$.
$R(X,Y)A = \alpha(X \wedge Y)A$
$\Longrightarrow$ M is one of the following:

(a) $(c > 0)$ $1 < \alpha = \cot^2(r),\ 0 < r < \pi/2$, M is an open subset of a geodesic hypersphere of radius r,
(b) $(c < 0)$ $\alpha = 1$ and M is an open subset of a horosphere,
(c) $(c < 0)$ $0 < \alpha = \tanh^2(r) < 1,\ r > 0$, and M is an open subset of a tube of radius r over a totally geodesic $\mathbb{C}H^{n-1}$.

Outline of the proof

- Using the condition of $R \cdot A$ and the equation of Gauss, get some equations for $g(A_V X, Y)$
  - $V \in N_x \implies A_V = 0$,
  - $X \in H_x$ and $Y \in H_x^\perp \implies g(A_V X, Y) = 0$,
  - $X, Y \in H_x \implies g(A_V X, Y) = \alpha_V g(X, Y)$, α_V: function
  - $X, Y \in H_x^\perp \implies A_V X = \beta X$, β:function.
- We prove dim$H_x^\perp = 1$.
- Check that the first normal space has constant dimension and is parallel with respect to the normal connection.

Proof of theorem

Lemma 1

$R(X,Y)A = \alpha(X \wedge Y)A$, α: function, then,

$$A_{fV}X = 0 \quad for\ X \in T_x(M), \tag{1}$$

$$g(A_V X, Y) = 0 \quad for\ X \in H_x, Y \in H_x^{\perp}. \tag{2}$$

Moreover, if the dimension of the holomorphic tangent space $h > 2$,

$$g(A_V X, Y) = -\frac{1}{h}\mathrm{tr}(A_V P^2)g(X,Y) \quad for\ X, Y \in H_x, \tag{3}$$

$$PA_V = A_V P, \tag{4}$$

where tr denotes the trace of an operator.

Since $(R(X,Y)A)_V = \alpha((X \wedge Y)A)_V$, we have

$$\begin{aligned} R(X,Y)A_V Z &= A_{R^{\perp}(X,Y)V} Z + A_V R(X,Y)Z \\ &= 2cg(X,PY)A_{fV} Z + A_V R(X,Y)Z + \alpha g(X, A_V Z)Y \\ &\quad - \alpha g(Y, A_V Z)X - \alpha g(X,Z)A_V Y + \alpha g(Y,Z)A_V X. \end{aligned}$$

Thus we have

$$\begin{aligned} &\mathrm{tr} R(X,Y)A_V A_{fV} \\ &= 2cg(X,PY)\mathrm{tr} A_{fV}^2 + \mathrm{tr} R(X,Y)A_{fV} A_V \\ &\quad + 2\alpha g(X, A_V A_{fV} Y) - 2\alpha g(Y, A_V A_{fV} X). \end{aligned}$$

By the equation of Ricci, we have $A_{fV} A_V = A_V A_{fV}$. Thus we obtain $\mathrm{tr} A_{fV}^2 = 0$, which proves (1).

Since the normal connection of M is semi-flat, the equation of Ricci gives

$$A_a A_b X = A_b A_a X$$

for any $X \in H_x$. Using these equations, we can shows that

$$\begin{aligned} &-(h+1)cg(PA_V X, Y) - c\mathrm{tr}(P^2 A_V)g(PX, Y) \\ &\quad + cg(P^2 A_V PX, Y) - 2cg(PA_V P^2 X, Y) = 0, \end{aligned} \tag{5}$$

Using this, we have $g(A_V X, Y) = 0$ for $X \in H_x^\perp$ and $Y \in H_x$.

Next we consider the case that $X, Y \in H_x$. We have

$$-(h-1)cg(PA_V X, Y) - cg(A_V PX, Y) - c\mathrm{tr}(P^2 A_V)g(PX, Y) = 0,$$
$$-(h-1)cg(A_V PX, Y) + cg(A_V X, PY) - c\mathrm{tr}(P^2 A_V)g(PX, Y) = 0.$$

From these equations and the assumption that $h > 2$, we get

$$g(PA_V X, Y) - g(A_V Y, PX) = 0.$$

From this, we have $PA_V = A_V P$ for any V normal to M.
Thus we obtain

$$g(A_V X, Y) = -\frac{1}{h}\mathrm{tr}(A_V P^2)g(X, Y)$$

for $X, Y \in H_x$.

Lemma 2

$R(X,Y)A = \alpha(X \wedge Y)A$, α : function $\Longrightarrow$ dim${H_x}^\perp = 1$.

We take an o.n.b. $\{v_1, \cdots, v_q, v_{q+1}, \cdots, v_p\}$ of $T_x^\perp(M)$,
$v_1, \cdots, v_q \in FH_x^\perp$
$v_{q+1}, \cdots, v_p \in N_x$.
Since A_{v_1} is symmetric, taking a suitable basis $\{e_1, \cdots .e_n\}$ of $T_x(M)$,

$$A_{v_1} = \left(\begin{array}{ccc|ccc} a_1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & a_1 & & & \\ \hline & & & b_1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & b_q \end{array}\right),$$

where $a_1 = -\frac{1}{h}\mathrm{tr}(A_{v_1}P^2)$.

We compute the sectional curvature $g(R(e_{h+j}, e_{h+i})e_{h+i}, e_{h+j}$ by two ways;

- using (5),

$$(b_i - b_j)(g(R(e_{h+j}, e_{h+i})e_{h+i}, e_{h+j}) + \alpha) = 0.$$

- using the equation of Gauss and Lemma 1,

$$g(R(e_{h+j}, e_{h+i})e_{h+i}, e_{h+j}) = 0.$$

So we have $b_i = b_j$ for any $i \neq j$, that is, $A_{v_1}X = b_1 X$ for any $X \in H_x^{\perp}$.

By the similar computation, we see that $A_{v_i}X = b_i X$ $(i = 2, \cdots, q)$ for $X \in H_x^{\perp}$, where $b_2, \cdots, b_q$ are functions. Thus we have

$$A_{v_i} t v_j = b_i t v_j.$$

On the other hand, since $A_V tU = A_U tV$ for any $U, V \in FH_x^{\perp}$, we have

$$A_{v_i} t v_j = A_{v_j} t v_i = b_j t v_i.$$

Since tv_i and tv_j are linearly independent, we see that $q = \dim H_x^{\perp} = 1$ or $b_1 = \cdots = b_q = 0$.

We suppose that $b_1 = \cdots = b_q = 0$.
Since $[A_U, A_V] = 0$ for any U and V normal to M, by the equation of Ricci, we have

$$0 = c\{g(Y, tV)g(X, tU) - g(X, tV)g(Y, tU)\}$$

for any $X, Y \in T_x(M)$.

If $\dim H_x > 2$, we can take U and V orthogonal to each other. Putting $X = tU$ and $Y = tV$, we have $c = 0$. This is a contradiction.
So we see that $q = 1$.
Consequently, we obtain $\dim H_x^{\perp} = 1$.

Lemma

$R(X,Y)A = \alpha(X \wedge Y)A$, α : function
$\Longrightarrow$
M is a hypersurface of totally geodesic $M^{\frac{n+1}{2}}(c)$ in $M^m(c)$.

$N_0(x) = \{V \in T_x(M)^\perp \mid A_V = 0\}$,
$N_1(x)$: orthogonal complement of $N_0(x)$ in $T_x(M)^\perp$
(first normal space)

We prove that the first normal space has constant dimension and it is parallel with respect to the normal connection.

If $A_V = 0$ for $V \in FH_x^\perp$, then $N_1(x) = \emptyset$.
If $A_V \neq 0$, then $N_0(x) = N_x$ and $N_1(x) = FH_x^\perp$ is of dimension 1.
For $V \in FH_x^\perp$ and $U \in N_x$,

$$g(D_X V, fU) = -g(V, (\nabla_X f)U) = -g(V, -FA_U X - B(X, tU)) = 0.$$

Thus,

$$D_X V \in FH_x^\perp.$$

So the first normal space is parallel with respect to the normal connection.
Thus we see that M is a hypersurface of totally geodesic $M^{\frac{n+1}{2}}(c)$ in $M^m(c)$. q.e.d

For the case that M is real hypersurface

If the curvature tensor R and the Ricci tensor S of M satisfy

$$g((R(X,Y)S)Z,W) = \alpha g(((X \wedge Y)S)Z,W), \;\; X,Y,Z,W \perp \xi,$$

α being a function, we call S the pseudo η-parallel Ricci tensor.

c.f.

- $(R(X,Y)S)Z = 0$ (Kimura and Maeda, 1989),
- $(R(X,Y)S)Z + (R(Y,Z)S)X + (R(Z,X)S)Y = 0$
  (Ki, Nakagawa and Suh, 1990)
- $g((R(X,Y)S)Z,W) = 0, \;\; X,Y,Z,W \perp \xi$ (K, 2007),
- $(R(X,Y)S)Z = \alpha g((X \wedge Y)S)Z$ (Inoguchi).

Theorem (Inoguchi-K, 2011).

Let M be a real hypersurface of a complex space form $M^n(c)$, $c \neq 0$, $n \geq 3$. Then S is pseudo η-parallel if and only if M is pseudo-Einstein.

Theorem (Inoguchi-K, 2011).

Let M be a real hypersurface of a complex projective space $\mathbb{C}P^n$, $n \geq 3$. If S is pseudo η-parallel, then M is locally congruent to one of the following:

(i) a geodesic hypersphere of radius r $(0 < r < \pi/2)$,

(ii) a minimal tube of radius $\pi/4$ over a complex projective space $\mathbb{C}P^{\frac{n-1}{2}}$ with principal curvatures 1, -1 and 0 whose multiplicities are $n-1$, $n-1$ and 1, respectively.

Theorem (Inoguchi-K, 2011).

Let M be a real hypersurface of a complex hyperbolic space $\mathbb{C}H^n$, $n \geq 3$. If S is pseudo η-parallel, then M is locally congruent to one of the following:

(i) a geodesic hypersphere,

(ii) a tube over a complex hyperbolic hyperplane,

(iii) a horosphere.

Theorem (Inoguchi-K, 2011).

Let M be a real hypersurface of a complex space form $M^n(c)$, $c \neq 0$, $n \geq 3$. The shape operator A is pseudo η-parallel if and only if M is either η-umbilical or it is locally a ruled real hypersurface.

Thank you for your attention!