Introduction to Centroaffine Differential Geometry

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Contents

Overture: Centroaffine Geometry

Notation.

$$\mathbb{R}^n \ni x = \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix},\ y = \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix}$$

$$\begin{array}{lcl} M_n(\mathbb{R}) & := & \{n \times n\text{-real matrices }\} \\ GL(n;\mathbb{R}) & := & \left\{A \in M_n(\mathbb{R}) \mid \exists A^{-1}\right\} \\ & = & \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \\ SL(n;\mathbb{R}) & := & \{A \in M_n(\mathbb{R}) \mid \det A = 1\} \\ O(n) & := & \{A \in GL(n;\mathbb{R}) \mid {}^tAA = 1_n\} \\ & & \text{where } 1_n := \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \in M_n(\mathbb{R}) \\ SO(n) & := & O(n) \cap SL(n;\mathbb{R}) \end{array}$$

Review:

$\varphi : \mathbb{R}^n \to \mathbb{R}^n$: Euclidean motion

$\overset{\text{def}}{\Longleftrightarrow} \exists A \in SO(n),\ \exists b \in \mathbb{R}^n : \varphi(x) = Ax + b$ for $\forall x \in \mathbb{R}^n$

Euclidean Geometry

$$\varphi(\textbf{Figure}\ ) = \textbf{Figure}$$

Review : Euclidean geometry

$$\langle x, y \rangle := {}^t x y = \sum_{i=1}^{n} x^i y^i$$: Euclidean metric

$$d(x, y) := |x - y| := \sqrt{\langle x - y, x - y \rangle}$$: Euclidean distance

$\varphi : \mathbb{R}^n \to \mathbb{R}^n$: Euclidean motion

$\iff$ · $d(\varphi(x), \varphi(y)) = d(x, y),\ \forall x, y \in \mathbb{R}^n$

· orientation-preserving

i.e. $\dfrac{\det(\overrightarrow{\varphi(p_0)\varphi(p_1)} \cdots \overrightarrow{\varphi(p_0)\varphi(p_n)})}{\det(\overrightarrow{p_0 p_1} \ \cdots \ \overrightarrow{p_0 p_n})} > 0$ for $p_0, p_1, \ldots, p_n \in \mathbb{R}^n$

What is Centroaffine Geometry?

Definition.

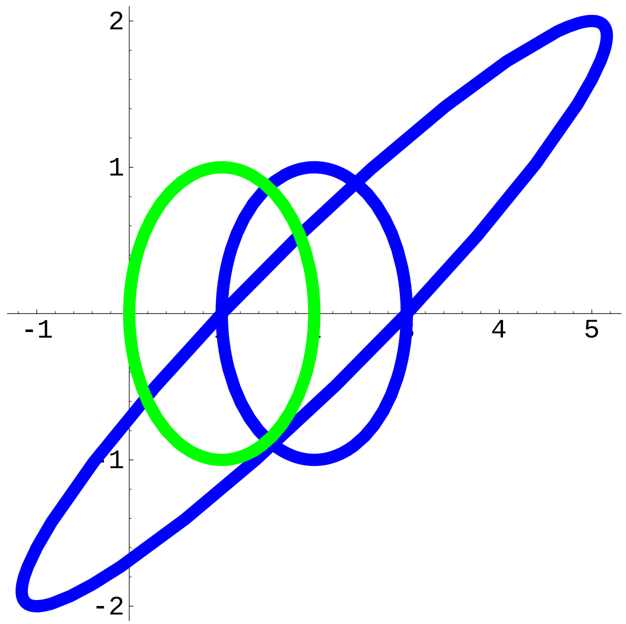
$\varphi : \mathbb{R}^n \to \mathbb{R}^n$: centroaffine transformation

$\overset{\text{def}}{\Longleftrightarrow} \exists A \in GL(n;\mathbb{R}) : \varphi(x) = Ax$ for $\forall x \in \mathbb{R}^n$

Centroaffine Geometry

$\varphi($**Figure** $) =$ **Figure**

Example.



Problem. Find geometric invariants.

difficulty: centroaffine geometry $>$ Euclidean geometry

Act 1: Centroaffine Curves on $\mathbb{R}^2$

Review : Curves in Euclidean plane

$$f(s) = \begin{bmatrix} f^1(s) \\ f^2(s) \end{bmatrix}$$

$\exists$ arclength parameter $\rightsquigarrow |f'(s)| = 1$

$\kappa(s) := \langle f''(s), Jf'(s) \rangle$: Euclidean curvature

where $J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$: $\dfrac{\pi}{2}$-rotation

Euclidean geometry has the Euclidean metric $\langle\ ,\ \rangle$.

$\langle\ ,\ \rangle$ ($\leftrightsquigarrow SO(2) \ltimes \mathbb{R}^2$) $\rightsquigarrow \kappa$

What does centroaffine geometry have ?

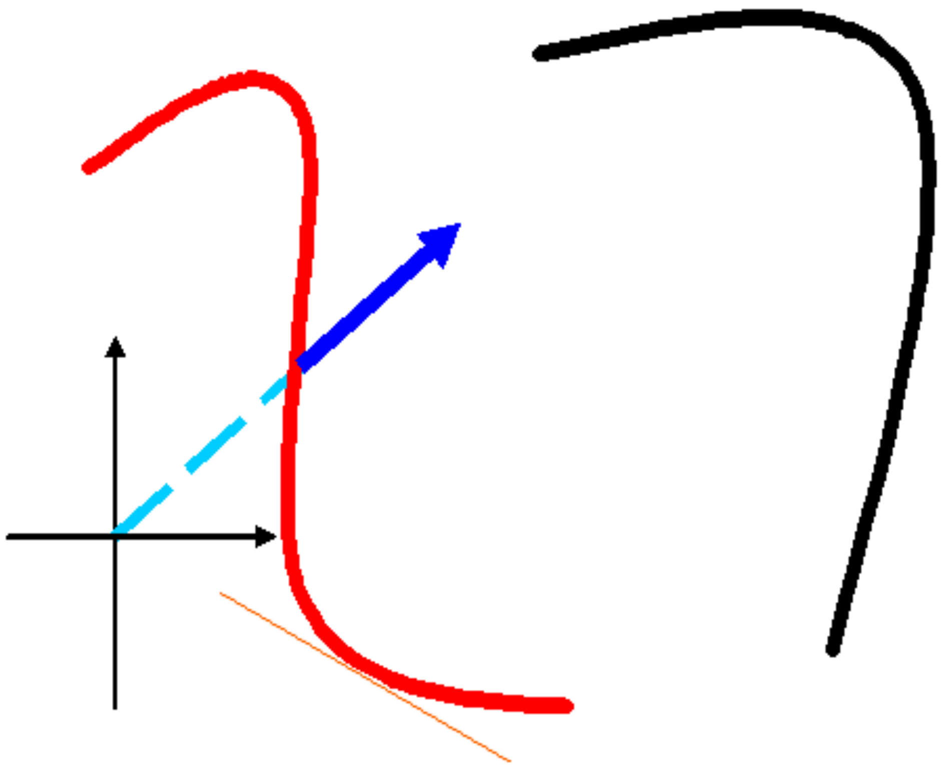
$I \subset \mathbb{R}$: interval

$f : I \to \mathbb{R}^2$: centroaffine curve

$\overset{\text{def}}{\Longleftrightarrow} \det(f(t) \quad \frac{d}{dt}f(t)) \neq 0, \quad \forall t \in I$

i.e. position vector $\pitchfork$ the tangent line

f : nondegenerate $\overset{\text{def}}{\Longleftrightarrow} \det(\frac{d}{dt}f(t) \quad \frac{d^2}{dt^2}f(t)) \neq 0, \quad \forall t \in I$



Lemma.

f : nondegenerate centroaffine curve

$$\left|\frac{\det(\frac{d}{dt}f \;\; \frac{d^2}{dt^2}f)}{\det(f \;\; \frac{d}{dt}f)}\right|^{1/2} dt \quad \text{is an invariant 1-form on } I.$$

$\because$

Claim 1.
$$\left|\frac{\det(\frac{d}{dt}f \quad \frac{d^2}{dt^2}f)}{\det(f \quad \frac{d}{dt}f)}\right|^{1/2} dt = \left|\frac{\det(\frac{d}{dt}Af \quad \frac{d^2}{dt^2}Af)}{\det(Af \quad \frac{d}{dt}Af)}\right|^{1/2} dt$$

for $\forall A \in GL(2;\mathbb{R})$.

Claim 2.
$$\left|\frac{\det(\frac{d}{dt}f \quad \frac{d^2}{dt^2}f)}{\det(f \quad \frac{d}{dt}f)}\right|^{1/2} dt = \left|\frac{\det(\frac{d}{ds}f\circ\xi \quad \frac{d^2}{ds^2}f\circ\xi)}{\det(f\circ\xi \quad \frac{d}{ds}f\circ\xi)}\right|^{1/2} ds$$

for $\forall I \ni s \mapsto \xi(s) = t \in I$: diffeo. $\square$

$$l(f;0,u) := \int_0^u \left| \frac{\det(\frac{d}{dt}f(t) \quad \frac{d^2}{dt^2}f(t))}{\det(f(t) \quad \frac{d}{dt}f(t))} \right|^{1/2} dt \qquad (0 \in I)$$

: centroaffine-arclength function

$\exists$ the inverse function of $l(f;0,\cdot)$

$\rightsquigarrow$ parameterized by centroaffine-arclength

Definition.

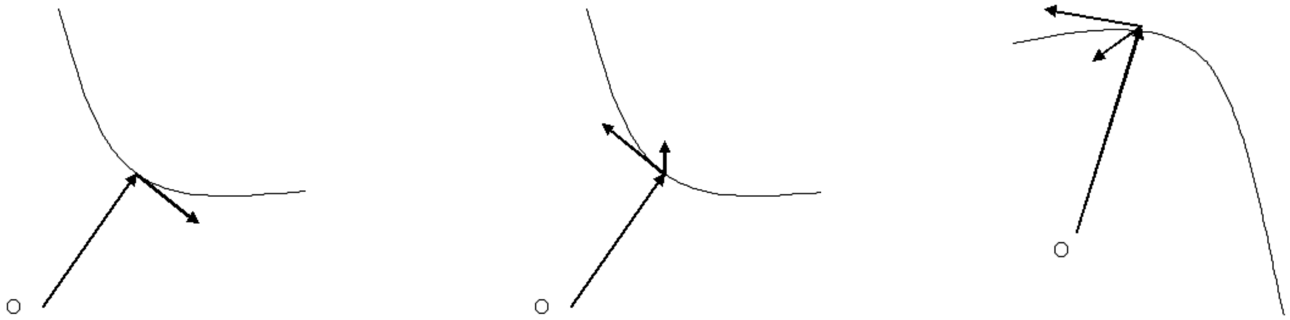
$f : I \to \mathbb{R}^2$: centroaffine curve

(nondegenerate centroaffine curve parameterized by centroaffine-arclength)

$$\overset{\text{def}}{\Longleftrightarrow} \quad \begin{array}{l} \det(f(s)\ f'(s)) > 0, \\ \varepsilon := \dfrac{\det(f'(s)\ f''(s))}{\det(f(s)\ f'(s))} = \pm 1, \end{array} \qquad \forall s \in I.$$

ε : signature of f

Any nondegenerate centroaffine curve has a reparametrization by centroaffine-arclength.



change the orientation $\quad \varepsilon = -1 \quad \varepsilon = +1$

$$\det(f(s)\ f'(s)) > 0,$$

$$\varepsilon := \frac{\det(f'(s)\ f''(s))}{\det(f(s)\ f'(s))} = \pm 1.$$

What is centroaffine curvature?

$f : I \to \mathbb{R}^2$: centroaffine curve

$F : I \ni s \mapsto F(s) := (f(s)\ f'(s)) \in GL(2;\mathbb{R})$

$\Phi(s) \in M_2(\mathbb{R})$: $F'(s) = F(s)\Phi(s)$

Lemma and Definition.

$$\exists 1\ \kappa : I \to \mathbb{R} : \qquad \Phi(s) = \begin{bmatrix} 0 & -\varepsilon \\ 1 & \kappa(s) \end{bmatrix}.$$

$\kappa : I \to \mathbb{R}$: centroaffine curvature of f

Review : Curves in Euclidean plane

$f : I \to \mathbb{R}^2$: curve with (Euclidean) arclength parameter

$F : I \ni s \mapsto F(s) := (f'(s)\ Jf'(s)) \in SO(2)$

$\Phi(s) \in M_2(\mathbb{R})$: $F'(s) = F(s)\Phi(s)$

$$\Phi(s) = \begin{bmatrix} 0 & -\kappa(s) \\ \kappa(s) & 0 \end{bmatrix}.$$

$\kappa(s) = \langle f''(s), Jf'(s) \rangle$: Euclidean curvature of f

Remark.

$f : I \to \mathbb{R}^2$: centroaffine curve

κ : centroaffine curvature of f, $\quad \varepsilon$: signature of f

$\Longrightarrow$

(1) $f''(s) = \kappa(s) f'(s) - \varepsilon f(s)$,

(2) $\kappa(s) = \dfrac{\det(f(s)\ f''(s))}{\det(f(s)\ f'(s))}$.

Remark.

f : centroaffine curve with general parameter

$\Longrightarrow$

centroaffine curvature at $f(t)$

$$= \frac{1}{2}\left\{\varepsilon\frac{\det(f(t) \quad \frac{d}{dt}f(t))}{\det(\frac{d}{dt}f(t) \quad \frac{d^2}{dt^2}f(t))}\right\}^{1/2} \frac{d}{dt}\log\left\{\varepsilon\frac{\det(f(t) \quad \frac{d}{dt}f(t))^3}{\det(\frac{d}{dt}f(t) \quad \frac{d^2}{dt^2}f(t))}\right\}$$

What is the curve which does not curve in centroaffine geometry?

What is the curve of centroaffine curvature 0?

$f''(s) = 0f'(s) - \varepsilon f(s) \quad \rightsquigarrow$

(i) $\varepsilon = -1 \quad \Longrightarrow \; f(s) = \begin{bmatrix} \cosh s \\ \sinh s \end{bmatrix}$

(ii) $\varepsilon = +1 \quad \Longrightarrow \; f(s) = \begin{bmatrix} \cos s \\ \sin s \end{bmatrix}$

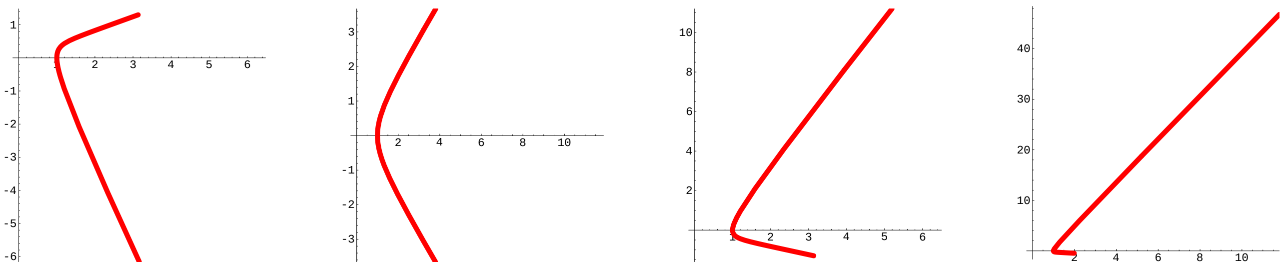
under the condition $(f(0) \; f'(0)) = 1_2$

Theorem (Classification of curves of constant centroaffine curvature).

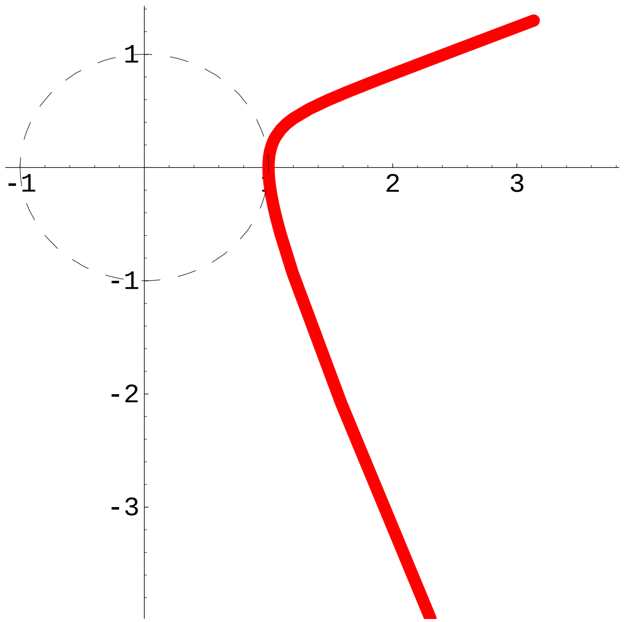
Curve of signature ε and centroaffine curvature constant κ is given as

Case (i): $\varepsilon = -1$

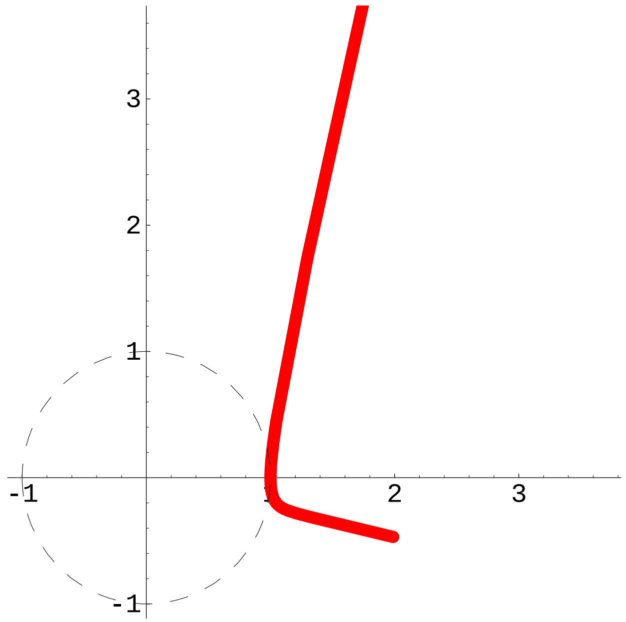
$$f(s) = \frac{1}{\lambda + \lambda^{-1}} \begin{bmatrix} \lambda \exp(-\lambda^{-1}s) + \lambda^{-1}\exp(\lambda s) \\ -\exp(-\lambda^{-1}s) + \exp(\lambda s) \end{bmatrix}, \quad \lambda := \frac{1}{2}(\kappa + \sqrt{\kappa^2 + 4}).$$



$\kappa = -2$ $\kappa = 0$ $\kappa = 2$ $\kappa = 4$



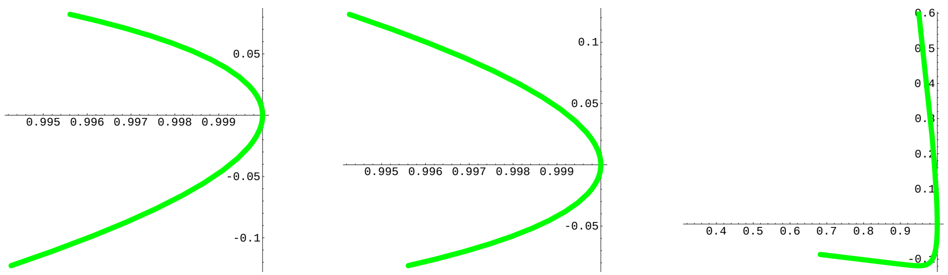
$\kappa = -2$



$\kappa = 4$

Case (ii-1): $\varepsilon = +1$, $|\kappa| > 2$

$$f(s) = \frac{1}{\lambda - \lambda^{-1}} \left[\begin{array}{r} \lambda \exp(\lambda^{-1} s) - \lambda^{-1} \exp(\lambda s) \\ - \exp(\lambda^{-1} s) + \exp(\lambda s) \end{array} \right], \quad \lambda := \frac{1}{2}(\kappa + \sqrt{\kappa^2 - 4}).$$



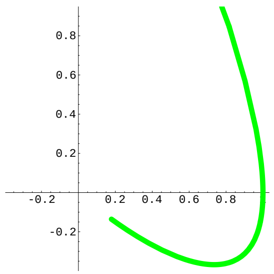
$\kappa = -4$ $\kappa = 4$ $\kappa = 8$

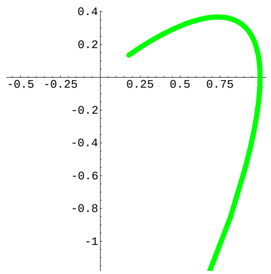
Case (ii-2): $\varepsilon = +1$, $\kappa = +2$

$$f(s) = \begin{bmatrix} \exp(s) - s\exp(s) \\ s\exp(s) \end{bmatrix}.$$

Case (ii-3) : $\varepsilon = +1$, $\kappa = -2$

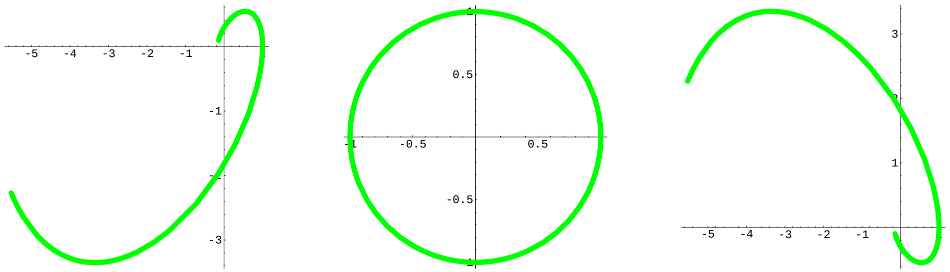
$$f(s) = \begin{bmatrix} \exp(-s) + s\exp(-s) \\ s\exp(-s) \end{bmatrix}.$$





Case (ii-4) : $\varepsilon = +1$, $|\kappa| < 2$

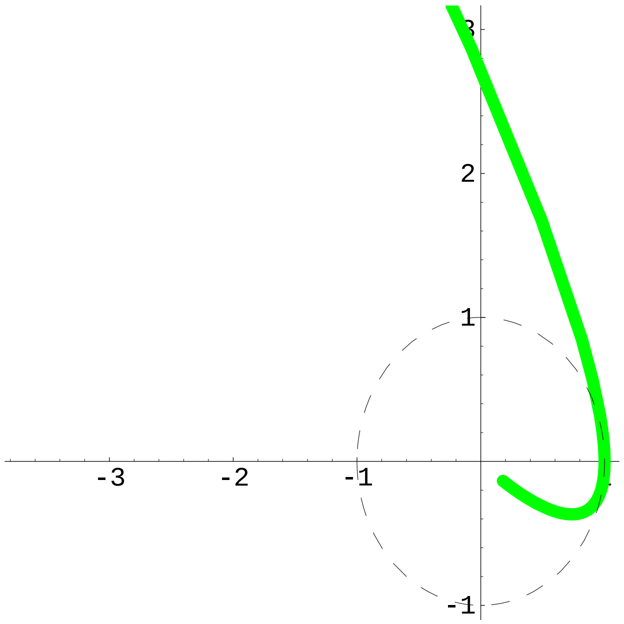
$$f(s) = \begin{bmatrix} \exp(\alpha s)\cos(\beta s) - \alpha\beta^{-1}\exp(\alpha s)\sin(\beta s) \\ -\beta^{-1}\exp(\alpha s)\sin(\beta s) \end{bmatrix}, \quad \alpha := \frac{\kappa}{2},\ \beta := \frac{1}{2}\sqrt{4-\kappa^2}.$$

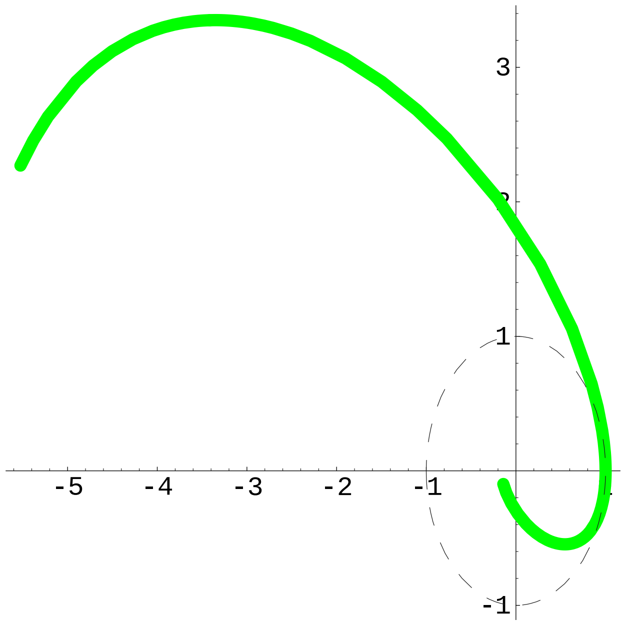


$\kappa = -1$ $\quad$ $\kappa = 0$ $\quad$ $\kappa = 1$

Question. Find a "good" representative of a given centroaffine curve!

Draw interesting pictures!





$\varepsilon = +1, \kappa = 2$ $\qquad$ $\varepsilon = +1, \kappa = 1$

Proposition (Centroaffine Clothoid).

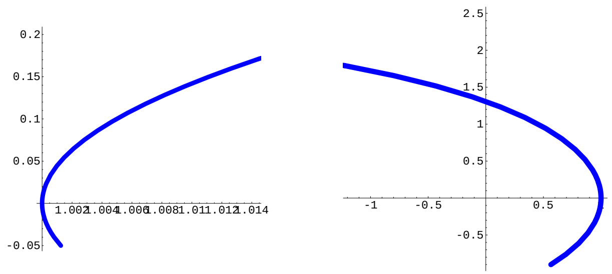
Curve of signature ε and centroaffine curvature $\kappa(s) = s$ is given as

Case (i): $\varepsilon = -1 \implies f(s) = \begin{bmatrix} \exp(\frac{s^2}{2}) \\ \sqrt{\frac{\pi}{2}}\exp(\frac{s^2}{2})\mathrm{erf}(\frac{s}{\sqrt{2}}) \end{bmatrix}.$

Case (ii): $\varepsilon = +1 \implies f(s) = \begin{bmatrix} \exp(\frac{s^2}{2}) - \sqrt{\frac{\pi}{2}}s\ \mathrm{erfi}(\frac{s}{\sqrt{2}}) \\ s \end{bmatrix},$

where erf, erfi are defined by

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2)dt, \quad \mathrm{erfi}(z) := \frac{1}{\sqrt{-1}} \mathrm{erf}(\sqrt{-1}z).$$



$\varepsilon = -1$ $\qquad$ $\varepsilon = +1$

Act 2: Centroaffine Surfaces in $\mathbb{R}^3$

M : 2-dim manifold

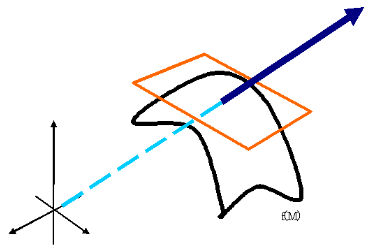
$f : M \to \mathbb{R}^3$: surface (i.e. immersion)

$f_* T_u M \subset T_{f(u)}\mathbb{R}^3$: tangent plane of the surface $f(M)$ at $f(u)$

f : centroaffine surface

$$\overset{\text{def}}{\Longleftrightarrow} T_{f(u)}\mathbb{R}^3 = f_* T_u M \oplus \mathbb{R} f(u), \quad \forall u \in M$$

i.e. position vector $\pitchfork$ the tangent plane



D : the standard flat affine connection of $\mathbb{R}^3$

Review: In general, what is affine connection ?

N : manifold

$\Gamma(TN^{(p,q)}) := \{$tensor fields on N of type $(p,q)\}$

$\Gamma(TN) := \Gamma(TN^{(1,0)}) = \{$vector fields on $N\}$

$\Gamma(TN) \ni X$

$$X_p = \sum X^i(p) \left(\frac{\partial}{\partial x^i}\right)_p \in T_pN \quad (p \in (U; x^1, \ldots, x^n) \subset N)$$

Review (cont):

$\nabla : \Gamma(TN) \ni Y \mapsto \nabla Y \in \Gamma(TN^{(1,1)})$

i.e. $\nabla : \Gamma(TN) \times \Gamma(TN) \ni (X, Y) \mapsto \nabla_X Y \in \Gamma(TN)$

: affine connection (covariant derivative) of N

$\overset{\text{def}}{\Longleftrightarrow}$

(1) $\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$, $\quad \forall f_i \in C^\infty(N)$

(2) $\nabla_X (c_1 Y_1 + c_2 Y_2) = c_1 \nabla_X Y_1 + c_2 \nabla_X Y_2$, $\quad \forall c_i \in \mathbb{R}$

(3) $\nabla_X (fY) = (Xf)Y + f \nabla_X Y$, $\quad \forall f \in C^\infty(N)$

$\forall X, X_i, Y, Y_i \in \Gamma(TN)$

Review (cont):

$N = \mathbb{R}^3$, $\quad (x^1, x^2, x^3)$: standard coordinates of $\mathbb{R}^3$

$$\Gamma(T\mathbb{R}^3) \ni X = \sum X^i \frac{\partial}{\partial x^i},\ Y = \sum Y^i \frac{\partial}{\partial x^i},$$

$$D_X Y := \sum_{i=1}^{3} (XY^i) \frac{\partial}{\partial x^i} \ \text{ where } \ XY^i(p) := \sum_{j=1}^{3} X^j(p) \frac{\partial Y^i}{\partial x^j}(p), \quad p \in \mathbb{R}^3$$

$\Longrightarrow$

D is the affine connection of $\mathbb{R}^3$ satisfying $\quad D_{\frac{\partial}{\partial x^i}} \dfrac{\partial}{\partial x^j} = 0.$

$f : M \to \mathbb{R}^3$: centroaffine surface

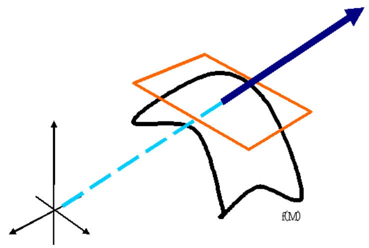
$T_{f(u)}\mathbb{R}^3 = f_* T_u M \oplus \mathbb{R} f(u), \quad \forall u \in M$

Define ∇ and h by

$$D_X f_* Y = f_* \nabla_X Y + h(X,Y) f, \quad \forall X, Y \in \Gamma(TM)$$

$\Longrightarrow$ ∇ : affine connection of M

$h \in \Gamma(TM^{(0,2)})$: symmetric $(0,2)$-tensor field on M



Proposition (Fundamental theorem for centroaffine surfaces).

$f \rightsquigarrow \nabla, h,\ \widetilde{f} \rightsquigarrow \widetilde{\nabla}, \widetilde{h}$

$\widetilde{f}$ is centroaffinely congruent to f

$\iff \widetilde{\nabla} = \nabla$ and $\widetilde{h} = h$

$\iff \widetilde{\nabla} = \nabla$

$\rightsquigarrow$ Tensor fields defined from ∇ and h are also centroaffine invariant.

Definition.

$f : M \to \mathbb{R}^3$: (definite centroaffine surface) or centroaffine surface

$\overset{\text{def}}{\iff}$ $T_{f(u)}\mathbb{R}^3 = f_* T_u M \oplus \mathbb{R} f(u), \quad \forall u \in M$

h : $\pm$ Riemannian metric on M

∇ : centroaffine induced connection of f

h : centroaffine metric of f

Definition.

∇^h : Levi-Civita connection of h

$K := \nabla - \nabla^h \in \Gamma(TM^{(1,2)})$: difference tensor field of f

$T := \mathrm{tr}_h K \in \Gamma(TM)$: Tchebychev vector field of f

$\mathcal{T} := \nabla^h T \in \Gamma(TM^{(1,1)})$: Tchebychev operator of f

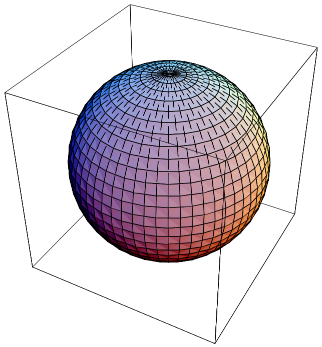
Remark.

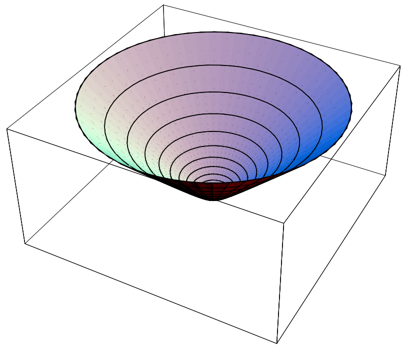
$\{K = 0\} \implies \{T = 0\} \implies \{\mathcal{T} = 0\} \implies \{\mathrm{tr}\,\mathcal{T} = 0\}$

$K = 0$?

Theorem (G.Pick, 1917 ?).

$K = 0 \Longrightarrow$ ellipsoid or hyperboloid of two-sheets





$T = 0$?

G. Tzitzéica, in 1908, without the discussion above,

found a property invariant under centroaffine transformations

for surfaces in Euclidean 3-space.

($=$ the source of Affine Differential Geometry)

What is the property?

Notation.

$(\mathbb{R}^3, \langle\cdot,\cdot\rangle)$: Euclidean 3-space

$f : M \to \mathbb{R}^3$: centroaffine immersion

$n \in \Gamma(f^{-1}T\mathbb{R}^3)$: unit normal vector field of f

$I := f^*\langle\cdot,\cdot\rangle \in \Gamma(TM^{(0,2)})$: first fundamental form of f

(Riemannian metric on M induced by f)

$I\!I \in \Gamma(TM^{(0,2)})$: second fundamental form of f

∇^I : affine connection induced by f (Levi-Civita connection of I)

defined by

$$D_X f_* Y = f_* \nabla^I_X Y + I\!I(X,Y)n, \quad X, Y \in \Gamma(TM)$$

$\rho : M \to \mathbb{R}$: Euclidean support function of f w.r.t. 0

defined by $\rho(u) := \langle f(u), n(u) \rangle, \quad u \in M$

the distance of the tangent plane $f_* T_u M$ from the origin 0

$\mathcal{K} = \det(I^{-1} I\!I)$: Gauss (-Kronecker) curvature of f

Theorem (Modern version of Tzitzéica's theorem).

$$T = 0 \iff \mathcal{K}\rho^{-4} \text{ is const.}$$

In particular, the property that
the ratio of the Gauss curvature to the fourth power of the distance of the tangent plane from the origin is constant
is invariant under a centroaffine transformation.

$f : M \to \mathbb{R}^3$: centroaffine surface

f : Tzitzéica surface $\overset{\text{def}}{\iff}$ $T = 0$

(proper affine sphere with center at the origin)

$\because$ Claim. $$T = -\frac{1}{2}\operatorname{grad}_h \log \mathcal{K}\rho^{-4}. \qquad \square$$

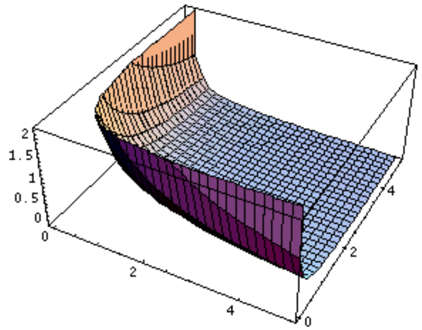
Remark.

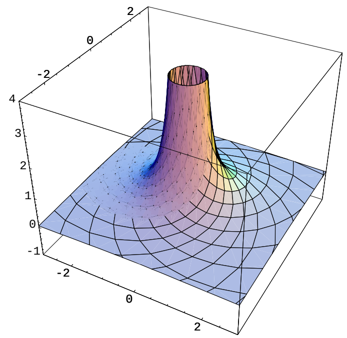
$f : M^n \to \mathbb{R}^{n+1}$: centroaffine hypersurface

$$\Longrightarrow \ T = -\frac{1}{2}\operatorname{grad}_h \log \mathcal{K}\rho^{-(n+2)}.$$

Reference.
Furuhata, H., Surfaces in centroaffine geometry, to appear in *Differential geometry of submanifolds and related topics*, Sūrikaisekikenkyūsho Kōkyūroku (Japanese).

Example.





$$(1)\ f(u,v) := (u, v, u^{-1}v^{-1}),$$
$$(2)\ f_{\mathrm{Jon}}(u,v) := (\cos(\sqrt{3}u)\exp(-v), \sin(\sqrt{3}u)\exp(-v), \exp(2v))$$

$\exists$ many Tzitzéica surfaces

$\{K = 0\} \implies \{T = 0\} \implies \{\mathcal{T} = 0\} \implies \{\mathrm{tr}\,\mathcal{T} = 0\}$

$\mathcal{T} = 0$?

Local Classification Theorem (Liu,H.- Wang,C.P., 1995).

$f : M \to \mathbb{R}^3$: centroaffine surface with $\mathcal{T} = 0$

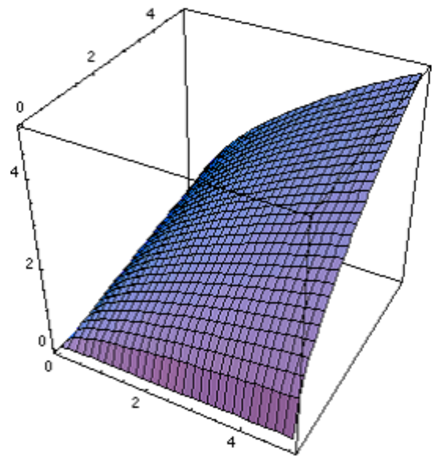
$\implies$

(1) Tzitzéica surface

(2) $f_{ab}(u,v) := (u, v, u^{-a}v^{-b}), \quad a, b \in \mathbb{R} : ab(a+b+1) \neq 0$

$\vdots$

Example. (2) $f_{ab}(u,v) := (u, v, u^{-a}v^{-b}), \quad a, b \in \mathbb{R} : ab(a+b+1) \neq 0$



$a = -1/2, \; b = -1/2 + 1/100$

$\{K = 0\} \implies \{T = 0\} \implies \{\mathcal{T} = 0\} \implies \{\operatorname{tr} \mathcal{T} = 0\}$

$\operatorname{tr} \mathcal{T} = 0$?

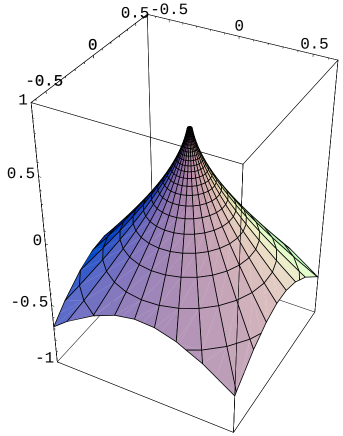
Theorem (Wang,C.P., 1994).

For $\forall f_t : M \to \mathbb{R}^3$ centroaffine variation of f

$$\left.\frac{d}{dt}\right|_{t=0} \int_M \mathrm{vol}_{h_t} = 0 \iff \operatorname{tr} \mathcal{T} = 0$$

$\rightsquigarrow$ centroaffine minimal surfaces

Example.



$$f_{\mathrm{Fuj}}(u,v) := (u^{-1}e^{-u}\cos v, u^{-1}e^{-u}\sin v, 1-u^{-1})$$

$f_{\mathrm{Fuj}} \in \{\operatorname{tr}\mathcal{T} = 0\} \setminus \{\mathcal{T} = 0\}$

Reference.

Fujioka, A., Centroaffine minimal surfaces with constant curvature metric, Kyungpook Math. J. **46**(2006), 297–305.

$\exists$ many centroaffine minimal surfaces

$\rightsquigarrow$ Centroaffine Bernstein Problem ?

Reference.
Li, A.M., Li, H. and Simon, U., Centroaffine Bernstein problems, Differential Geom. Appl. **20**(2004), 331–356.

Problem. Characterize the surfaces with $\mathcal{T} = 0$ geometrically.

Candidate: surfaces with self-congruent center map

Reference.
Furuhata, H. and Vrancken, L., The center map of an affine immersion, Results Math. **49**(2006), 201–217.

Problem. ? $\exists$ Application of Centroaffine Geometry in technology