Introduction to Centroaffine Differential Geometry

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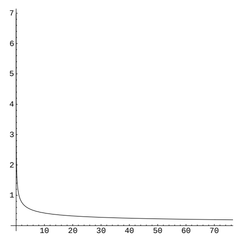
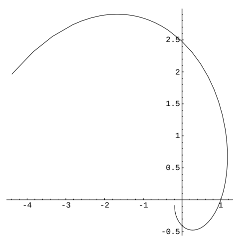
Following the ideas of Klein, presented in his famous lecture at Erlangen, several geometers in the early 20th century proposed the study of curves and surfaces with respect to different transformation groups. In 1907 Tzitzéica[*] found that for a surface in Euclidean 3-space the property that *the ratio of the Gauss curvature to the fourth power of the distance of the tangent plane from the origin is constant* is invariant under a centroaffine transformation. The surfaces with this property turn out to be what are now called Tzitzéica surfaces, or proper affine spheres with center at the origin. This work is regarded as the source of so-called Affine Differential Geometry.

A centroaffine transformation is nothing but a general linear transformation $\mathbb{R}^n \ni x \mapsto Ax \in \mathbb{R}^n$ where $A \in GL(n, \mathbb{R})$. In this talk, an introduction to centroaffine geometry of curves and surfaces will be given.

In Euclidean geometry, we investigate the properties of curves and surfaces invariant under a transformation $\mathbb{R}^n \ni x \mapsto Ax+b \in \mathbb{R}^n$ where $A \in O(n)$ [†] and $b \in \mathbb{R}^n$. You are reminded that as the first step, we should define the arc-length, the curvature and so on for a curve by using the Euclidean metric (the standard inner product). Besides, we need it to define the unit normal vector field and various curvatures for a surface as well.



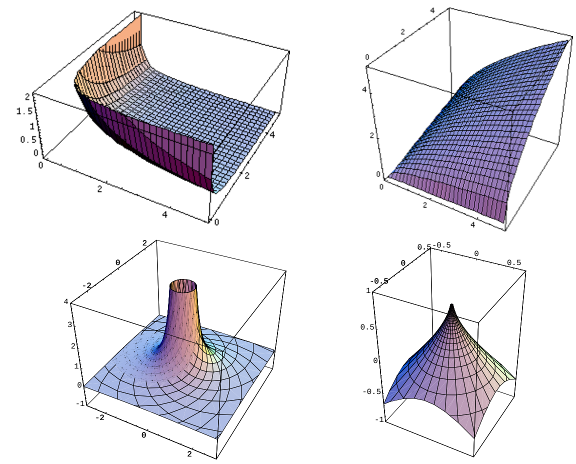


[*]Gheorghe Tzitzéica is a Romanian mathematician, and his name is spelled Tzitzeica, Titeica, Ţiţeica and so on.

[†]$O(n) := \{A \in GL(n, \mathbb{R}) \mid \langle Ax, Ax\rangle = \langle x, x\rangle, \forall x \in \mathbb{R}^n\}$ is the orthogonal group.

In centroaffine geometry, we should not expect that the ambient space has such a metric. How then do you define the arc-length of a curve, and …? As a matter of fact, we can define the centroaffine curvature, and get that the ellipse and the the hyperbola are of centroaffine curvature zero. We also obtain that the curves in the figures above are of non-zero constant centroaffine curvature.

The surfaces in the figures below are explicitly given as follows. In this talk, what these surfaces are will be clarified.



$$\begin{array}{ll}(1)(2) & f_{ab}(u,v) := (u, v, u^{-a}v^{-b}), \quad a, b \in \mathbb{R} : ab(a+b+1) \neq 0, \\ (3) & f_{\mathrm{Jon}}(u,v) := (\cos(\sqrt{3}u)\exp(-v), \sin(\sqrt{3}u)\exp(-v), \exp(2v)), \\ (4) & f_{\mathrm{Fuj}}(u,v) := (u^{-1}e^{-u}\cos v, u^{-1}e^{-u}\sin v, 1-u^{-1}).\end{array}$$

References

[1] Fujioka, A., Centroaffine minimal surfaces with constant curvature metric, Kyungpook Math. J. **46**(2006), 297–305.

[2] Furuhata, H., Surfaces in centroaffine geometry, to appear in *Differential geometry of submanifolds and related topics*, Sūrikaisekikenkyūsho Kōkyūroku (Japanese).

[3] Nomizu, K. and Sasaki, T., *Affine differential geometry –Geometry of affine immersions*, Cambridge University Press, 1994.

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