Euclidean space

$$a = \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}, \ b = \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix} \in \mathbb{R}^3$$

$$\langle a, b \rangle := a^1 b^1 + a^2 b^2 + a^3 b^3 \in \mathbb{R}$$

$$|a| := \langle a, a \rangle^{1/2} \in \mathbb{R}$$

$$a \times b := \begin{bmatrix} a^2 b^3 - a^3 b^2 \\ a^3 b^1 - a^1 b^3 \\ a^1 b^2 - a^2 b^1 \end{bmatrix} \in \mathbb{R}^3$$

What is a surface ?

$$D \subset \mathbb{R}^2 := \{ \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} =: (u^i) =: u \} \text{ : a domain}$$

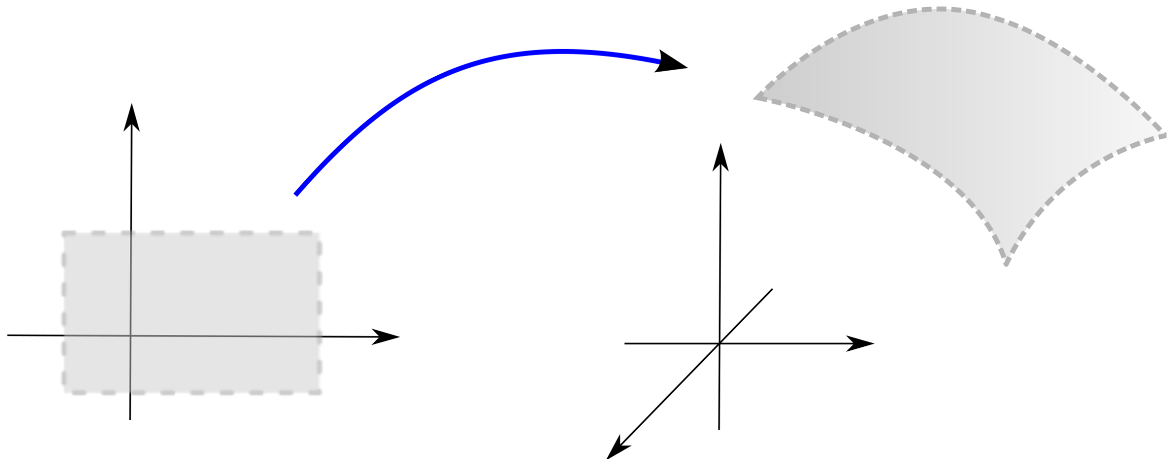
$$f : D \to \mathbb{R}^3 := \{ \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} =: (x^\alpha) =: x \} \text{ : a } C^\infty \text{ map}$$

$$\partial_i f := (\frac{\partial f^\alpha}{\partial u^i}) = \begin{bmatrix} \partial_i f^1 \\ \partial_i f^2 \\ \partial_i f^3 \end{bmatrix} : D \to \mathbb{R}^3$$

Definition 1.1.

$$f \text{ is a surface} \overset{\text{def}}{\iff} \{\partial_1 f(u), \partial_2 f(u)\} \text{ are linearly independent for } \forall u \in D$$

$$\iff \partial_1 f(u) \times \partial_2 f(u) \neq 0 \text{ for } \forall u \in D.$$

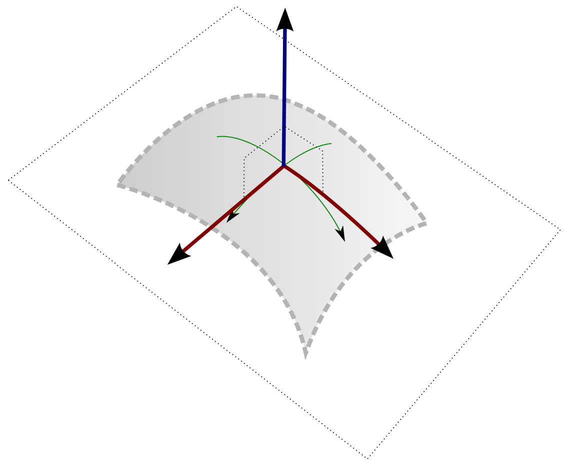


$$n(u) := |\partial_1 f(u) \times \partial_2 f(u)|^{-1} \partial_1 f(u) \times \partial_2 f(u)$$

the unit normal vector of f at u

$$\begin{aligned} f_* T_u D &:= \text{span}\{\partial_1 f(u), \partial_2 f(u)\} \\ &= \{v^1 \partial_1 f(u) + v^2 \partial_2 f(u) \in \mathbb{R}^3 \mid v^1, v^2 \in \mathbb{R}\} \\ &= \{X \in \mathbb{R}^3 \mid \langle X, n(u) \rangle = 0\} \end{aligned}$$

the tangent space of f at u



What is a minimal surface ?

$g_{ij}(u) := \langle \partial_i f(u), \partial_j f(u) \rangle$

(the coefficients of) the first fundamental form of f at u

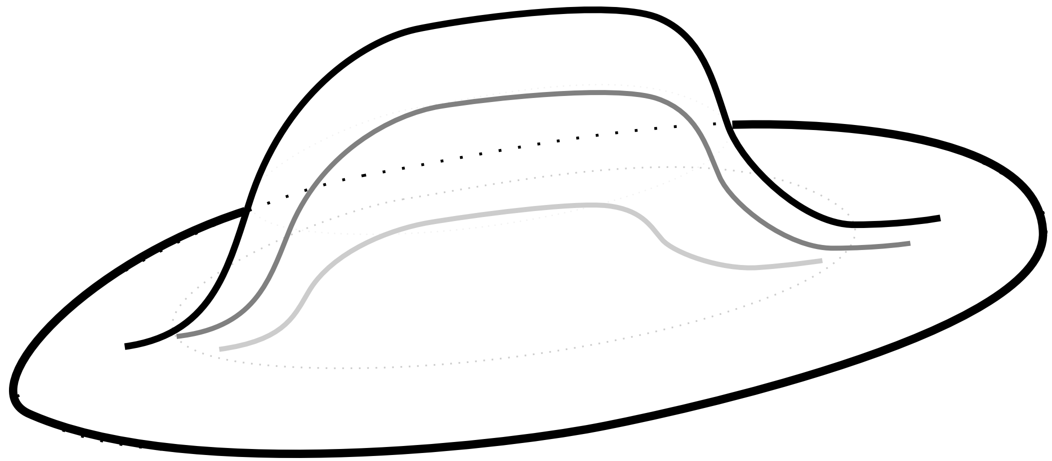
$$\begin{aligned}\text{Area}(f) &:= \int_D |\partial_1 f(u) \times \partial_2 f(u)| du^1 du^2 \\ &= \int_D \sqrt{\det(g_{ij}(u))} du^1 du^2\end{aligned}$$

the area of the surface $f(D)$

$f : D \to \mathbb{R}^3$ a surface

$\{f_t\}$ a variation of f, i.e.

$$f_t : D \to \mathbb{R}^3 \text{ is a surface, } \; t \in (-\varepsilon, \varepsilon), \;\; f_0 = f, \;\; f_t|_{\partial D} = f|_{\partial D}.$$



$f : D \longrightarrow \mathbb{R}^3$ a surface

$\{f_t\}$ a variation of f, i.e.

$$f_t : D \longrightarrow \mathbb{R}^3 \text{ is a surface}, \ \ t \in (-\varepsilon, \varepsilon), \ \ f_0 = f, \ \ f_t|_{\partial D} = f|_{\partial D}.$$

Definition 1.2.

f is a **minimal surface**

$$\overset{\text{def}}{\Longleftrightarrow} \ \ \left.\frac{d}{dt}\right|_{t=0} \text{Area}(f_t) = 0 \ \ \text{for any variation } \{f_t\} \text{ of } f$$

Exercise 1.3.

Choose a picture of a minimal surface and explain its properties.

Exercise 1.4.

Find various properties of the following family of surfaces:

$$f_t(u^1, u^2) = \begin{bmatrix} |1-t^2|^{-1/2}(\, tu^1 + \ \sin u^1 \cosh u^2) \\ |1-t^2|^{-1/2}(\ \ u^2 + t\cos u^1 \sinh u^2) \\ \cosh u^1 \sinh u^2 \end{bmatrix}$$

See the animation!