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Extended Abstracts

ON MOD p RIEMANN-ROCH FORMULAE FOR MAPPING CLASS GROUPS

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1. Introduction and Definitions

Let Σ_g be a closed oriented surface of genus $g \geq 2$ and Γ_g the mapping class group of Σ_g. There are two well-known families of cohomology classes of Γ_g, Morita-Mumford classes (or Mumford-Morita-Miller classes) and Chern classes (of the homology representation).

Over $\mathbb{Q}$, these two classes are related by the Grothendieck-Riemann-Roch theorem (see (1)). The author conjectured in [1] that similar relations hold over $\mathbb{Z}$ (see Conjecture 1). The purpose of this talk is to explain some affirmative evidences for the conjecture.

Now we recall definitions of Morita-Mumford classes and Chern classes of Γ_g. Let $\pi : E \to B$ be an oriented Σ_g-bundle, $T_{E/B}$ the tangent bundle along the fiber of π, and $e \in H^2(E;\mathbb{Z})$ the Euler class of $T_{E/B}$. Then the k-th Morita-Mumford class $e_k \in H^{2k}(B;\mathbb{Z})$ of π is defined by

$$e_k := \pi_!(e^{k+1})$$

where $\pi_! : H^*(E;\mathbb{Z}) \to H^{*-2}(B;\mathbb{Z})$ is the Gysin homomorphism (or the itegration along the fiber). Passing to the universal Σ_g-bundle, one obtains the k-th Morita-Mumford class $e_k \in H^{2k}(\Gamma_g;\mathbb{Z})$ of Γ_g.

On the other hand, the natural action of Γ_g on $H_1(\Sigma_g;\mathbb{R})$ induces a homomorphism $\Gamma_g \to Sp(2g,\mathbb{R})$. The homomorphism yields a continuous map $B\Gamma_g \to BU(g)$ of classifying spaces, for the maximal compact subgroup of $Sp(2g,\mathbb{R})$ is isomorphic to $U(g)$. The k-th Chern class $c_k \in H^{2k}(\Gamma_g;\mathbb{Z})$ of Γ_g is the pull-back of the universal k-th Chern class $c_k \in H^{2k}(BU(g);\mathbb{Z})$.

2. Integral Riemann-Roch conjecture for Γ_g

Over $\mathbb{Q}$, Morita-Mumford classes and Chern classes of Γ_g are related by the Grothendieck-Riemann-Roch theorem. Let $s_k \in H^{2k}(\Gamma_g;\mathbb{Z})$ be the k-th Newton class of Γ_g, which is defined by $s_k := N_k(c_1, c_2, \dots, c_k)$ where N_k is the k-th Newton polynomial. Note that $g + \sum_{k\geq 1} s_k/k! \in H^*(\Gamma_g;\mathbb{Q})$ is nothing but the pull-back of the universal Chern character $\mathrm{ch} \in H^*(BU(g);\mathbb{Q})$. For all $k \geq 1$, $s_{2k} \in H^*(\Gamma_g;\mathbb{Q})$ vanishes, and the Grothendieck-Riemann-Roch theorem (or the Atiyah-Singer index theorem) implies

$$\frac{B_{2k}}{2k} e_{2k-1} = s_{2k-1} \in H^*(\Gamma_g;\mathbb{Q}), \tag{1}$$

where B_{2k} is the $2k$-th Bernoulli number (see [11, 12]).

Let N_{2k} and D_{2k} be coprime integers satisfying $N_{2k}/D_{2k} = B_{2k}/(2k)$ (i.e. the numerator and the denominator of $B_{2k}/(2k)$). The equation (1) implies that the difference $N_{2k}e_{2k-1} - D_{2k}s_{2k-1} \in H^*(\Gamma_g;\mathbb{Z})$ is a torsion element. Moreover, in virtue of the Harer's stability theorem [7], there is a natural number L_{2k-1}, which depends only on k and is independent of g, satisfying

$$L_{2k-1}(N_{2k}e_{2k-1} - D_{2k}s_{2k-1}) = 0 \in H^*(\Gamma_g;\mathbb{Z}).$$

What is the least value of L_{2k-1} satisfying the last equation? The author made quite an optimistic conjecture in [1] that the least value is $L_{2k-1} = 1$ for all $k \geq 1$:

Conjecture 1 (integral Riemann-Roch formulae for Γ_g)**.**

$$N_{2k}e_{2k-1} - D_{2k}s_{2k-1} = 0 \in H^*(\Gamma_g;\mathbb{Z})$$

holds for all $k \geq 1$ *and* $g \geq 2$.

The conjecture is affirmative for $k = 1$ (i.e. $e_1 = 12s_1 \in H^2(\Gamma_g;\mathbb{Z})$ for all $g \geq 2$), since $H^2(\Gamma_g,\mathbb{Z}) \cong \mathbb{Z}$ for $g \geq 3$ as was proved by Harer [6] (see [1] for the case $g = 2$).

There seems no obvious reasons for Conjecutre 1 to be affirmative for $k \geq 2$. However, the author and Kawazumi showed that the conjecture holds for any cyclic subgroup C of Γ_g. More precisely, the restriction of $N_{2k}e_{2k-1} - D_{2k}s_{2k-1}$ $(k \geq 1)$ to C vanishes in $H^*(C;\mathbb{Z})$. Moreover, Kawazumi showed that a sligtly weaker version of the conjecture holds for hyperelliptic mapping class groups [9]. The proof of the last two results involves number theoretic properties of Bernoulli numbers such as the Voronoi's congruence.

If we consider the "mod p reduction" of Conjecture 1, there are yet other affirmative evidences which will be discussed in the next section.

3. MOD p RIEMANN-ROCH CONJECTURE FOR Γ_g

Let p be a prime number and $\mathbb{F}_p$ the field of p elements. In what follows, we collect affirmative evidences for the "mod p reduction" of Conjecture 1:

Conjecture 2 (mod p Riemann-Roch formulae for Γ_g)**.**

$$N_{2k}e_{2k-1} - D_{2k}s_{2k-1} = 0 \in H^*(\Gamma_g;\mathbb{F}_p) \tag{2}$$

holds for all $k \geq 1$ *and* $g \geq 2$.

3.1. Steenrod operations.

For an odd prime p, let

$$P^i : H^k(-;\mathbb{F}_p) \to H^{k+2i(p-1)}(-;\mathbb{F}_p)$$

be the i-th reduced power operation. Applying the generalized Riemann-Roch theorem (see [2]) to the total reduced power operation, one can prove the following equation:

$$P^i(e_k) = \binom{k}{i} e_{k+i(p-1)} \in H^*(\Gamma_g;\mathbb{F}_p).$$

The last formula, together with the Kummer's congruence on Bernoulli numbers, implies the following result:

Theorem 3.1. *Let p be an odd prime. If*

$$N_{2k}e_{2k-1} - D_{2k}s_{2k-1} = 0 \in H^*(\Gamma_g;\mathbb{F}_p)$$

holds for some $g \geq 2$ and $k \geq 1$, and if $\binom{2k-1}{i}$ is prime to p, then

$$N_{2k+i(p-1)}e_{2k-1+i(p-1)} - D_{2k+i(p-1)}s_{2k-1+i(p-1)} = 0 \in H^*(\Gamma_g;\mathbb{F}_p).$$

In other words, the affirmative solution of Conjecture 2 for some k implies that for $k+i(p-1)/2$, provided $\binom{2k-1}{i}$ is prime to p. In particular, since Conjecture 2 is affirmative for $k=1$, one has:

Proposition 3.2. *Let p be an odd prime. Then*

$$N_{p^n+1}e_{p^n} - D_{p^n+1}s_{p^n} = 0 \in H^*(\Gamma_g;\mathbb{F}_p)$$

for all $n \geq 0$.

Similar considerations are possible for $p=2$ by using squaring operations in place of reduced power operations.

3.2. **Elementary abelian p-subgroups.** In view of the last subsection, it is reasonable to consider Conjecture 2 for small k relative to p. Recall that $E = (\mathbb{Z}/p\mathbb{Z})^n$ is called an elementary abelian p-group (of rank n). We have proved:

Proposition 3.3 ([1]). *Let $E \subset \Gamma_g$ be an elementary abelian p-subgroup of rank $n \geq 2$. Then $e_k = 0 \in H^*(E;\mathbb{F}_p)$ for all $k \geq 1$.*

The proposition is an easy consequence of the Kawazumi-Uemura formula for Morita-Mumford classes on finite subgroups of Γ_g [10]. Recently, we made the following calculation:

Proposition 3.4. *Let $E \subset \Gamma_g$ be an elementary abelian p-subgroup of rank $n \geq 2$. Then $D_{2k}s_{2k-1} = 0 \in H^*(E;\mathbb{F}_p)$, provided $2k-1 \leq p$.*

The calculation relies on (i) properties of the G-equivariant coboridim group of surfaces along the lines of [5, 14] (ii) the G-signature theorem (iii) the Evens-Kahn-Roush formulae for induced representations [3].

Applying Quillen's F-isomorphism theorem [13] to the last two propositions, we obtain the following result:

Theorem 3.5. *For all $g \geq 2$, $N_{2k}e_{2k-1} - D_{2k}s_{2k-1} \in H^*(\Gamma_g;\mathbb{F}_p)$ is nilpotent, provided $2k-1 \leq p$.*

Here $u \in H^*(\Gamma;\mathbb{F}_p)$ is said to be nilpotent if $u^r = 0$ for some $r \geq 1$. The assumption of the theorem can be relaxed by using Theorem 3.1. But we omit the detail for simplicity.

3.3. **von Staudt's theorem.** Recall that von Staudt's theorem asserts that a prime p divides D_{2k} if and only if $p-1$ divides $2k$ (see [8] for instance). Hence Conjecture 2 implies the following conjecture which was established in [1], provided k is odd.

Conjecture 3. *If $k \equiv -1 \pmod{p-1}$ then $e_k \in H^*(\Gamma_g;\mathbb{F}_p)$ vanishes.*

Let $H^*(\Gamma_\bullet;\mathbb{F}_p)=\lim_{g\to\infty}H^*(\Gamma_g;\mathbb{F}_p)$ be the stable mod p cohomology and $e_k\in H^*(\Gamma_\bullet;\mathbb{F}_p)$ the stable mod p Morita-Mumford class. Among other things, Galatius, Madsen, and Tillmann [4] proved the following:

Theorem 3.6. *$e_k\in H^*(\Gamma_\bullet;\mathbb{F}_p)$ vanishes if and only if $k\equiv -1 \pmod{p-1}$.*

(”only if” part was also proved in [1].) This proves Conjecture 3 in the *stable range*. Moreover, Conjecture 3 is affirmative for hyperelliptic mapping class groups [9]. Note that Theorem 3.6 implies Conjecture 2 for $p=2,3$ in the stable range, for D_{2k} is divisible by 6 for all k.

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