２次元興奮場の障害物による自発的スパイラル形成

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「渦の特徴付け」, July, 25, 2016, 北海道大学

不整脈の原因

- 細胞電気信号として活動電位（興奮波）が心臓の中（洞房結節，心房，房室結節，ヒス束，プルキンエ繊維網から左右の心室）で順序よく伝わり，消失すれば不整脈は発生しない．
- 信号の伝わりにくい場所や異常な興奮を発生させる場所などの基質があると不整脈が発生する
  - 異常自動能，トリガードアクティビティ，リエントリ（異方性リエントリ，スパイラルリエントリなど）

梗塞巣，瘢痕，心肥大などが原因の空間的伝播異常

Ventricular fibrillation (VF)

心室細動

Heart pumps blood throughout the blood vessels by repeated, rhythmic contractions

Uncoordinated contraction of the cardiac muscle of the ventricles in the heart

Arrhythmia 不整脈

↓ cessation of effective blood circulation

Asystole, Arrest 心停止

↓ 50% within three minutes

Sudden cardiac death (SCD)

心臓性突然死

Automated external defibrillator (AED)

AED applies electrical therapy which stops the arrhythmia to reestablish an effective rhythm.

Spiral re-entry in excitable media

目標

- 心室細動を起こしやすい人を検査で見つけられるようにしたい
- 除細動のメカニズムを考えたい

- 3次元の取り扱いは分からない
- まず，1,2次元でメカニズムを理解しよう
- 障害，複数波の影響が考えられるが，ここでは，障害物が単一波に与える影響を考える
- そのための数学的準備

興奮場

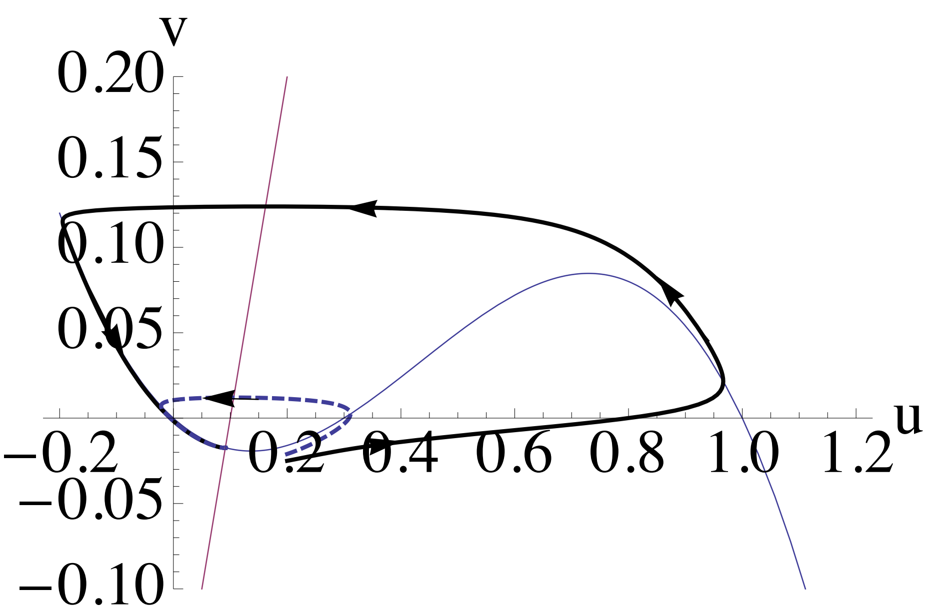
- 一定以上の刺激があると興奮する単安定系が空間的に広がったもの

- FitzHugh-Nagumo 方程式（FHN方程式）

$$\begin{cases} u_t &= \Delta u + u - u^3 - v \\ v_t &= u - \xi v + \eta \end{cases}$$

u:活性因子
v:抑制因子

刺激の大きさに依存



拡散効果を入れると進行波が現れる

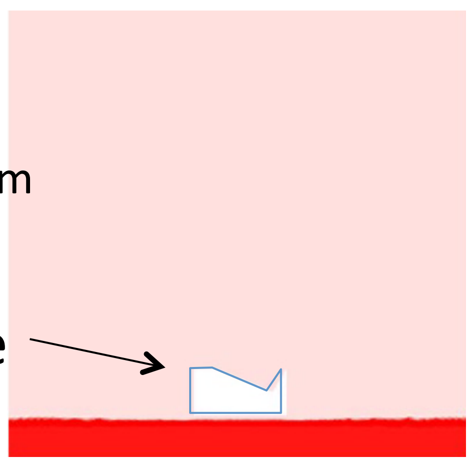
Obstacle-induced spiral

- Jalife et al (1998) : Wavebreaks may take place by an obstacle
- Shimizu, Kaihara, Suematsu and N. (2014)
- FitzHugh-Nagumo equation

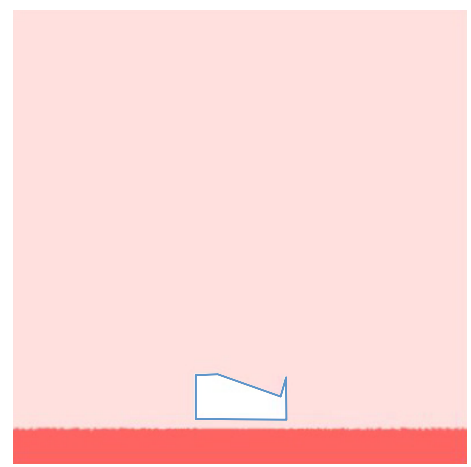
$$
\begin{cases}
u_t &= \Delta u + u - u^3 - v \\
v_t &= u - \xi v + \eta
\end{cases}
$$

Heart as an excitable medium

Obstacle



Wave moves upward



Fibrillation occurs by obstacle

cf. Berestycki, Hamel, Matano (2009): Generalized traveling wave around obstacles

どう取り扱えばいいか？

- どうやってスパイラルを作ることができるのか？
- FHN方程式の解では複雑すぎて扱えない
- 界面方程式（自由境界問題）の導出
- 界面方程式は簡単か？
- 複雑なら，簡単な状況にする必要がある．
  - 複雑な理由を理解する
  - 適度な簡単さに帰着する

FHN方程式から界面方程式へ

Chen-Kohsaka-N. 2014

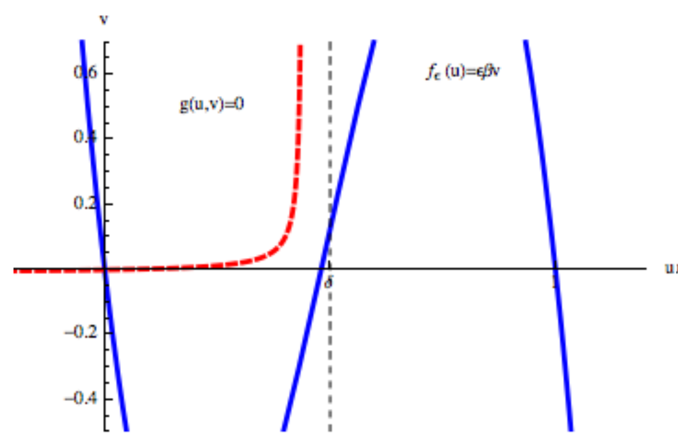
$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}(f_\varepsilon(u) - \varepsilon\beta v) \\ v_t = g(u,v) \end{cases}$$

where $f_\varepsilon(u) = u(1-u)(u - \dfrac{1}{2} + \varepsilon\alpha)$

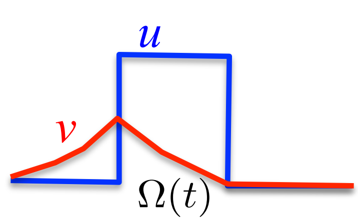
$$g(u,v) = g_1 u - \frac{g_2 v}{g_0 + g_3 v} \qquad \delta := \frac{g_2}{g_1 g_3} \leq \frac{1}{2},\ g_i > 0$$

$$\varepsilon^2 u_t = \boxed{\varepsilon^2 \Delta u} + f_\varepsilon(u) - \varepsilon\beta v$$

$$f_0(u) = 0 \longrightarrow u = 0, \frac{1}{2}, 1 \qquad v = \frac{1}{\beta\varepsilon} f_\varepsilon(u)$$



$$u \longrightarrow \chi_{\Omega(t)} := \begin{cases} 1 & \text{if } x \in \Omega(t) \\ 0 & \text{if otherwise} \end{cases}$$



FHN方程式から界面方程式へ

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(FBP)

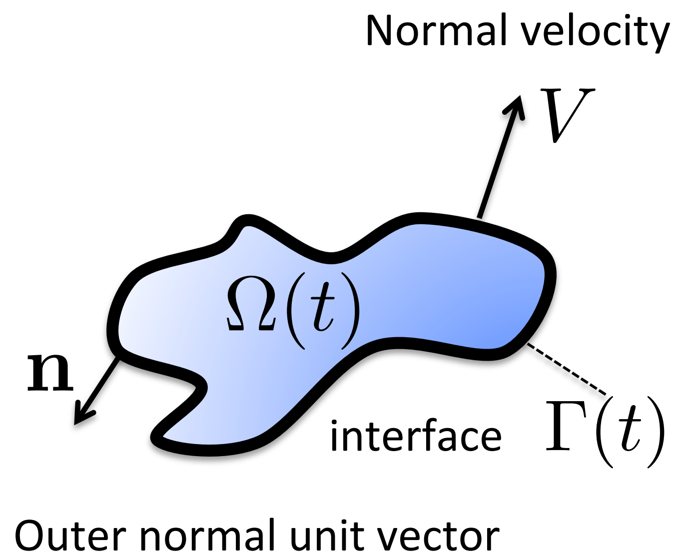
$$V = W(v) - \kappa$$

$$v_t = g(\chi_{\Omega(t)}, v)$$

where $W(v) = a - bv$

$$a = \sqrt{2}\alpha, b = 6\sqrt{2}\beta$$

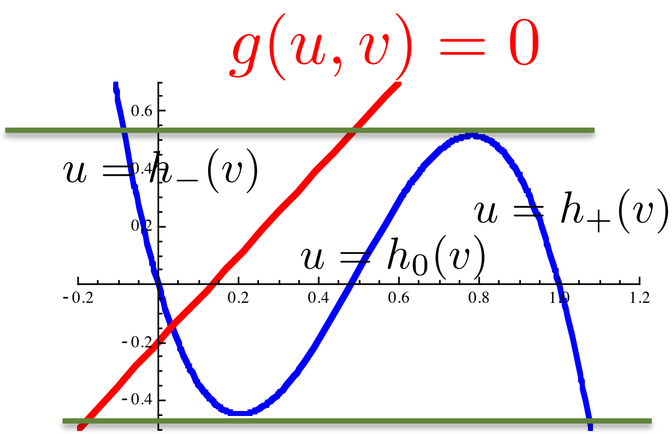
$$W(v) = 0 \Longleftrightarrow v = \frac{a}{b}$$

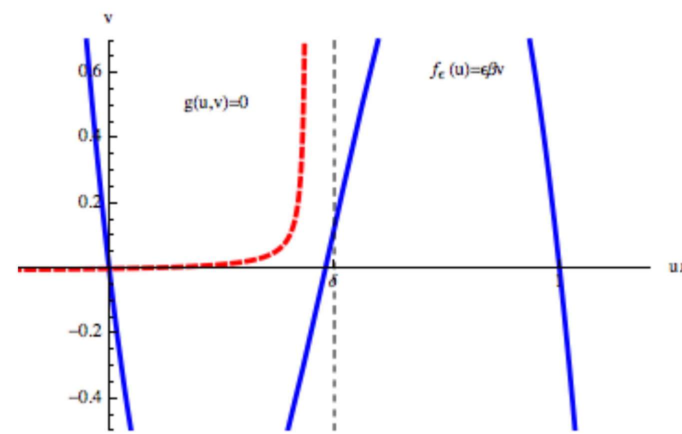


類似の極限問題との違い

$$\begin{cases} u_t &= \varepsilon\Delta u + \dfrac{1}{\varepsilon}(f(u) - \beta v) \\ v_t &= g(u,v) \ {\color{blue}+d\Delta v} \end{cases} \qquad \begin{cases} u_t &= \Delta u + \dfrac{1}{\varepsilon^2}(f_\varepsilon(u) - \varepsilon\beta v) \\ v_t &= g(u,v) \end{cases}$$

$$\boxed{\begin{aligned} V &= W(v){\color{red}-\varepsilon\kappa} \\ v_t &= g(h_\pm(v), v){\color{blue}+d\Delta v} \end{aligned}} \qquad \boxed{\begin{aligned} V &= W(v) - \kappa \\ v_t &= g(\chi_{\Omega(t)}, v) \end{aligned}}$$





自由境界問題は導出されたが，
解の挙動はあまり分からない．

一端できた波の挙動は，これで追跡できるが，
波の自己生成は，表現できない．

自由境界問題の適切性と存在

$$V = W(v) - \varepsilon\kappa$$
$$v_t = g(h_\pm(v), v) + d\Delta v$$

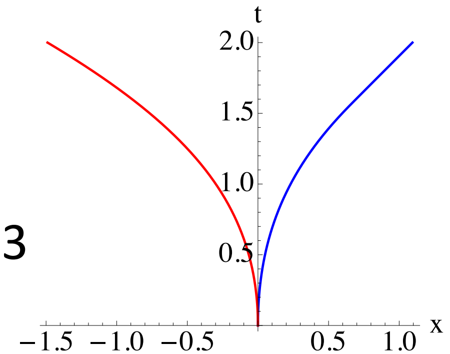
The existence of solutions of this system was studied in 1990's.

- With diffusion d>0:
  - X.Y. Chen : Hiroshima J., 21 (1991), 47-83
    Local existence of smooth solution for ε>0
  - X.F. Chen :Trans. Amer. Math. Soc. 334 (1992), 877-913
    Local existence of smooth solution for ε=0
  - Y. Giga, S. Goto, H. Ishii : SIMA, 23 (1992), 821-835
    Existence of viscosity solution for ε>0 and ε=0
  - D. Hilhorst, Y. Nishiura, M. Mimura: Proc. Royal Soc. Edin. 118A (1991), 355-378
    Interface is one point in 1D

However, the dynamics has not been studied well.

- Without diffusion i.e., d=0 :
  - X.F. Chen: Conf. on Asymptotics in NDS (1998), 9-33
    1D, weak formulation, reduction
  - X.F. Chen, C. Gao: J. PDEs 19 (2006), 48-79
    1D, local dynamics, propagation, annihilation, nucleation

Even in 1D, it is not well-posed when W(v)=0 on the interface !



Construction of TWs

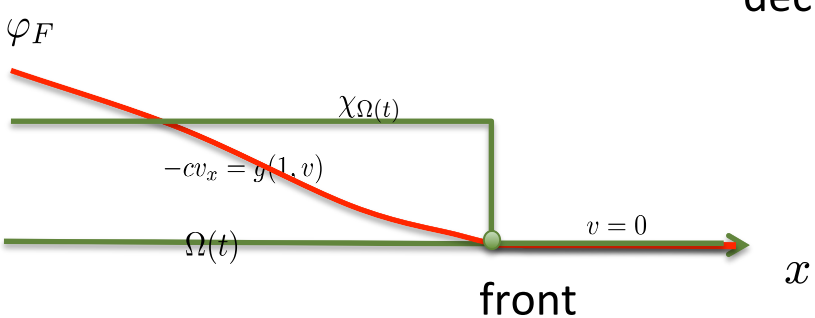
- One-dimensional traveling front and pulse

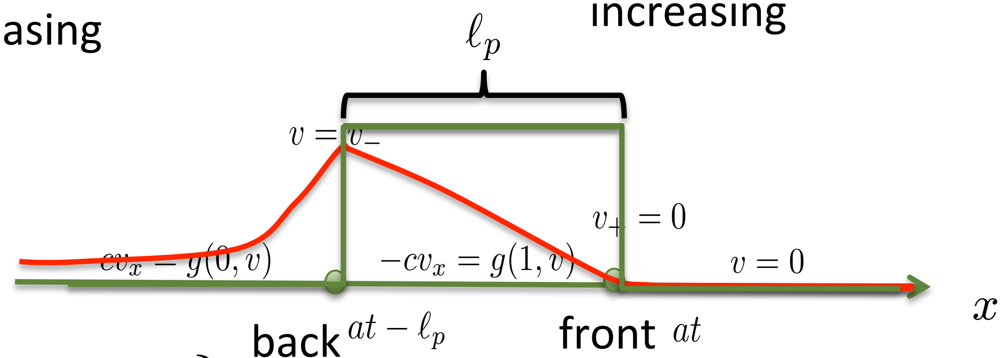
$$\Omega(t) = \{x \mid x < at\}, \quad \varphi_F(x - at) = G_1\left(\frac{(at - x)_+}{a}\right)$$

where $\quad G_0^{-1}(v) := \int_M^v \frac{d\xi}{g(0,\xi)}, G_1^{-1}(v) := \int_0^v \frac{d\xi}{g(1,\xi)}.$

decreasing

increasing





$$\Omega(t) = \{x \mid at - \ell_p < x < at\},$$

$$\varphi_P(x - at) = \begin{cases} 0, & \text{if } x \geq at, \\ G_1\left(t - \frac{x}{a}\right), & \text{if } at - \ell_p \leq x \leq at, \\ G_0\left(t - \frac{x + \ell_p}{a}\right), & \text{if } x \leq at - \ell_p, \end{cases}$$

where $\quad \ell_p := aG_1^{-1}\left(\frac{2a}{b}\right)$

[1D dynamics]

Theorem (Chen-Kohsaka-N. 2014, Chen-Taguchi-N. 2016)

For any $0<c<a$, there are b and a traveling spot (Ω, v) with speed c such that $\partial\Omega \in C^2, \quad v \in C^1(\mathbb{R}^2 \backslash \partial\Omega) \cap C^0(\mathbb{R}^2)$

$$\lim_{x^2+y^2\to\infty} v(x,y,0) = 0, \quad v(x,y,0) = 0 \text{ if } |y| \geq Y_M.$$

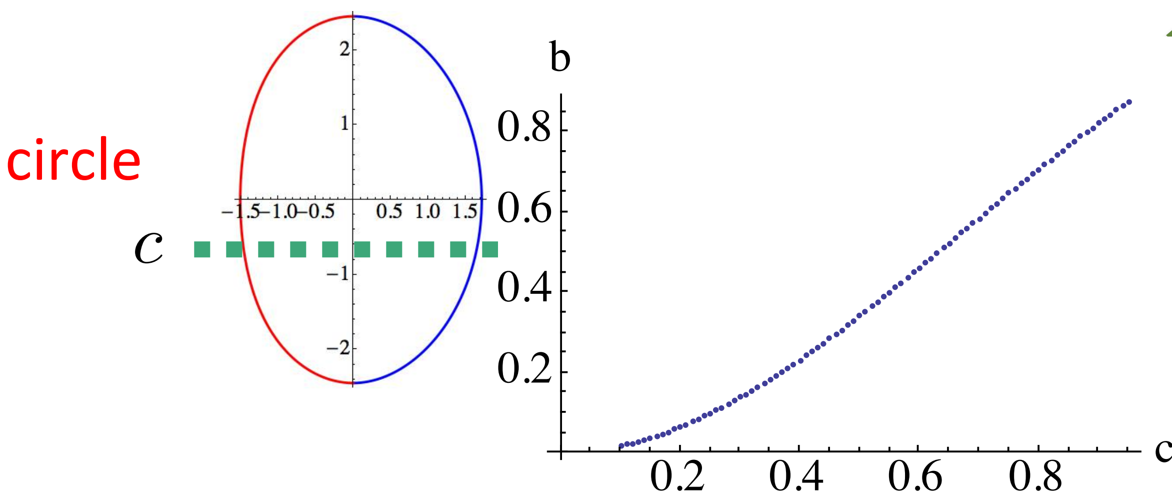
Moreover,

(i) $0 \leq c \leq c_*$

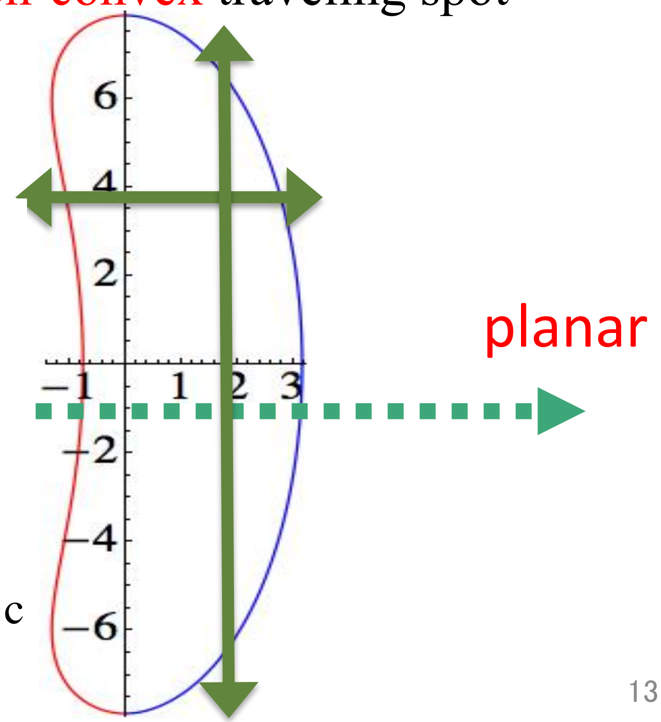
There is a convex traveling spot

(ii) $c^* < c < a$

Non-convex traveling spot

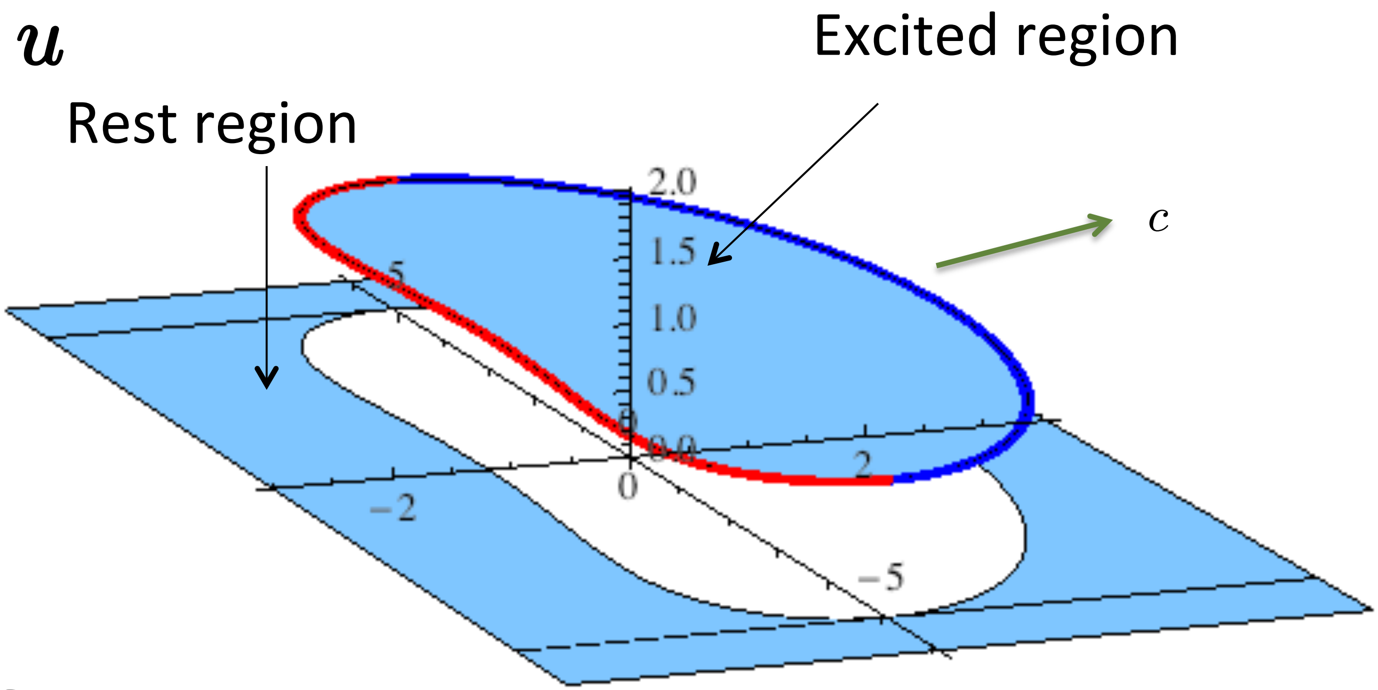


There is a traveling spot with any large size.

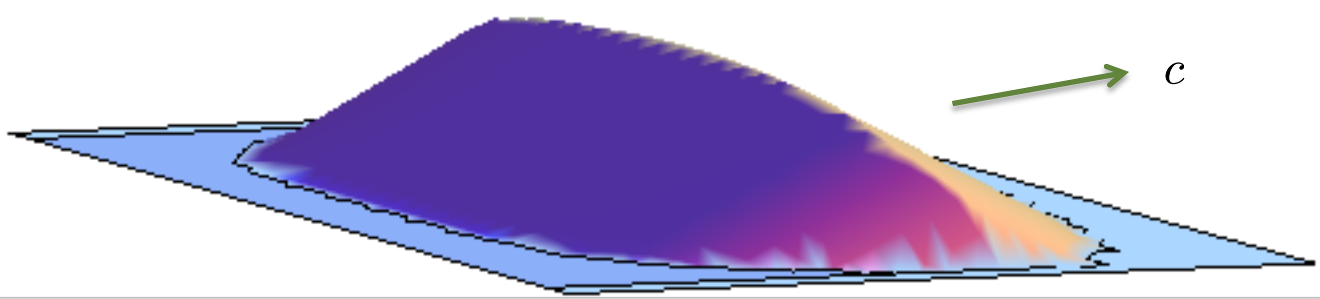


Profile of u and v

Profile of u

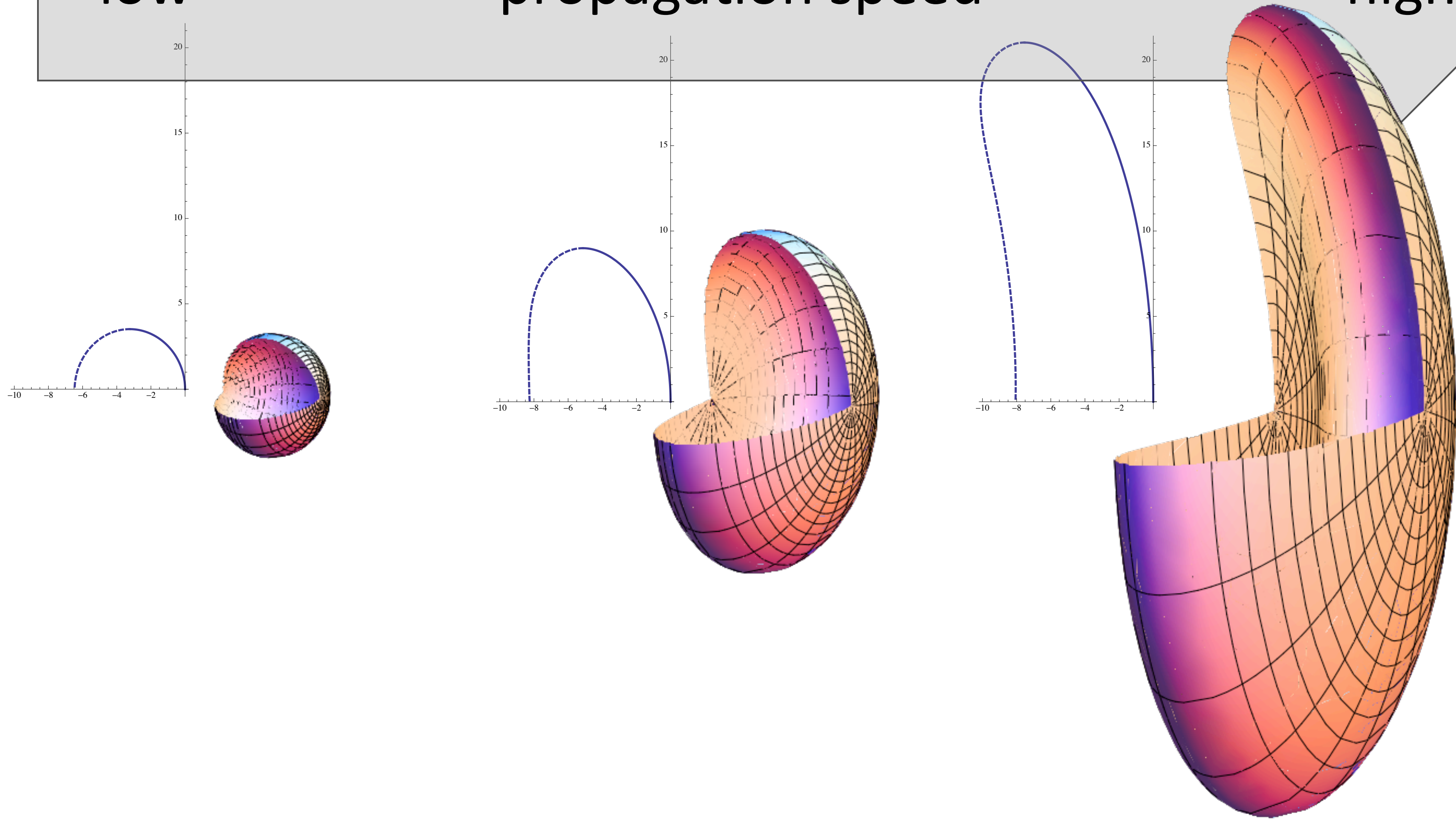


Profile of v



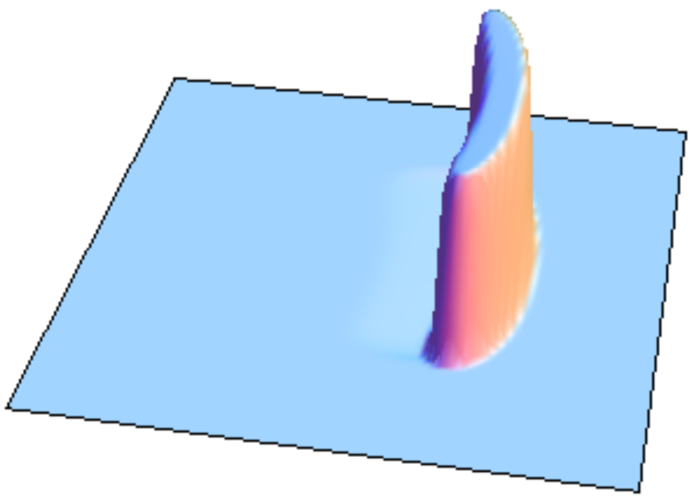
The shape of numerical 3D traveling spot

low propagation speed high

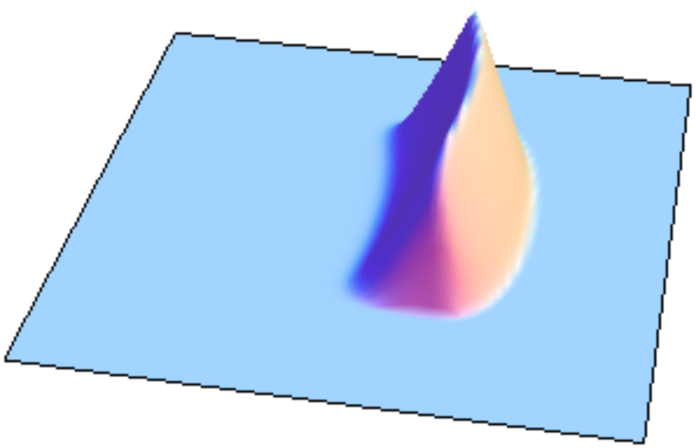


Numerical solutions of FitzHugh-Nagumo equation

$$\begin{cases} u_t &= \Delta u + \dfrac{1}{\varepsilon^2}(f_\varepsilon(u) - \varepsilon\beta v) \\ v_t &= g(u, v) \end{cases}$$

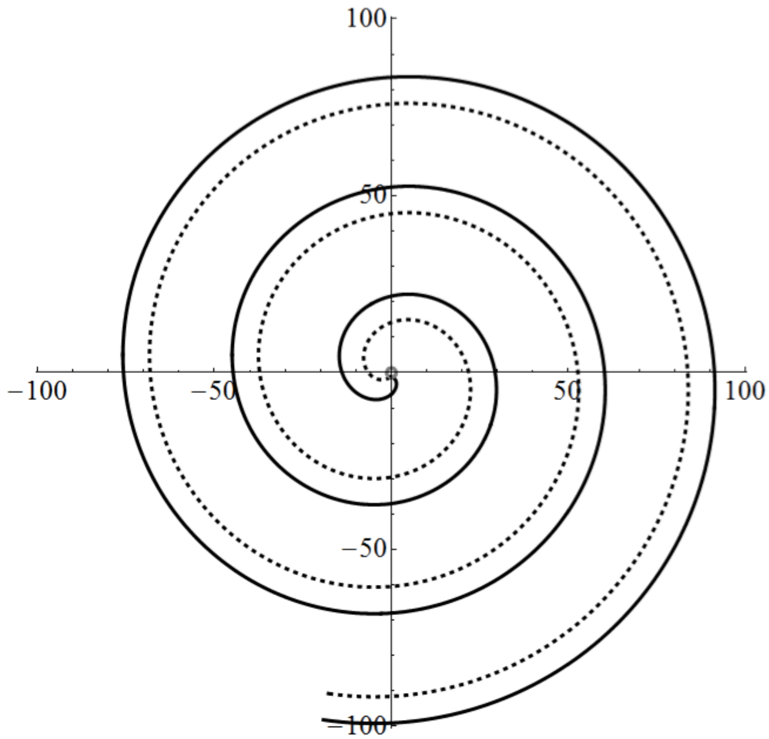


Profile of u

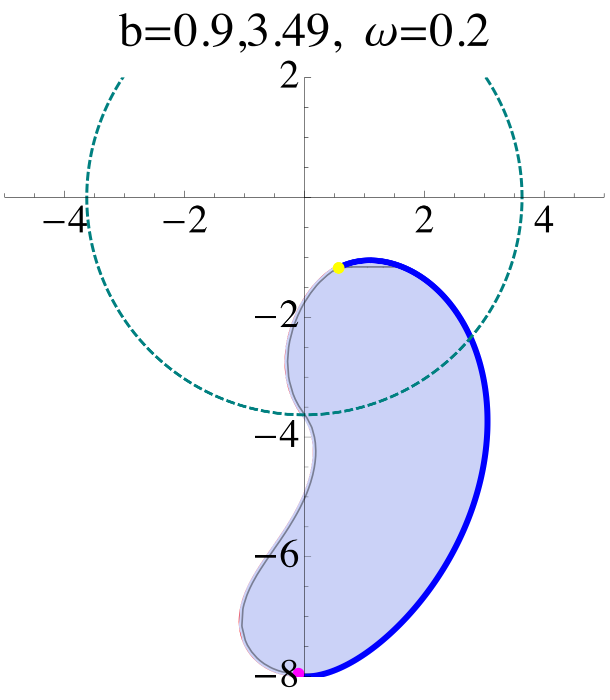


Profile of v

もう少し簡単化したモデルには以下のような解も構成できる



Chen-Guo-N. 2012



Chen-Guo-N. 2016