化学反応系に現れるスパイラル波への数学的アプローチについて

栄 伸一郎
北海道大学理学部

Belousov-Zhabotinsky反応 (BZ反応)

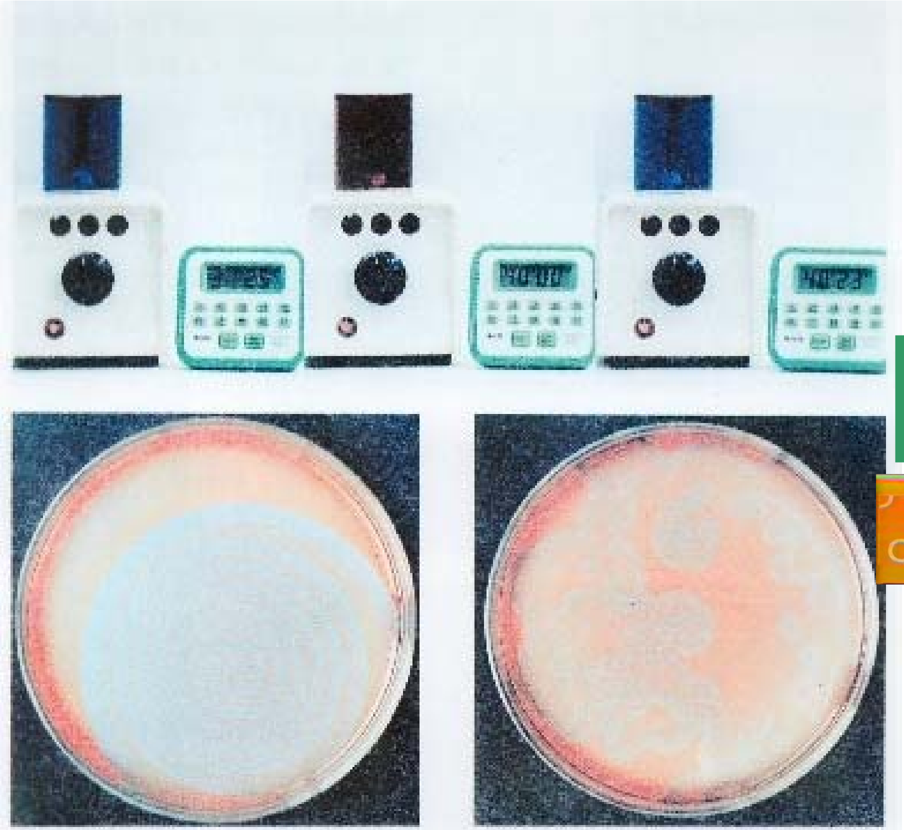
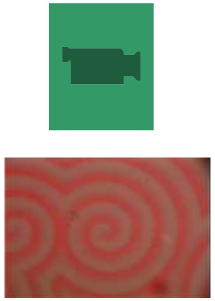
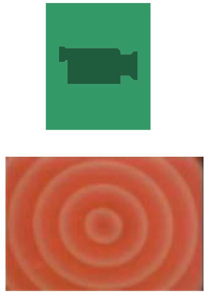
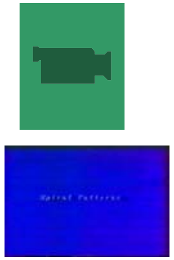
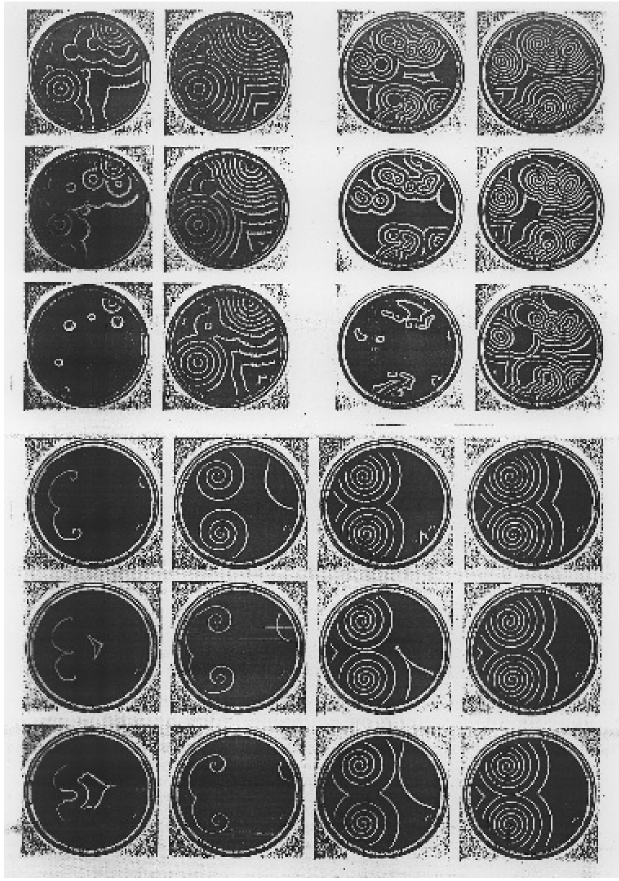


図２：振動する化学溶液









千葉大学 北畑氏より

BZ 反応

表2.1 FKN メカニズム

(R1)	$Br^- + HOBr + H^+ \rightleftharpoons Br_2 + H_2O$
(R2)	$Br^- + HBrO_2 + H^+ \rightleftharpoons 2HOBr$
(R3)	$Br^- + BrO_3^- + 2H^+ \rightleftharpoons HOBr + HBrO_2$
(R4)	$2HBrO_2 \rightleftharpoons HOBr + BrO_3^- + H^+$
(R5)	$HBrO_2 + BrO_3^- + H^+ \rightleftharpoons 2BrO_2\cdot + H_2O$
(R6)	$BrO_2\cdot + Ce^{3+} + H^+ \rightleftharpoons HBrO_2 + Ce^{4+}$
(R7)	$BrO_2\cdot + Ce^{4+} + H_2O \rightleftharpoons BrO_3^- + Ce^{3+} + 2H^+$
(R8)	$Br_2 + CH_2(COOH)_2 \longrightarrow BrCH(COOH)_2 + Br^- + H^+$
(R9)	$6Ce^{4+} + CH_2(COOH)_2 + 2H_2O \longrightarrow 6Ce^{3+} + HCOOH + 2CO2 + 6H^+$
(R10)	$4Ce^{4+} + BrCH(COOH)_2 + 2H_2O \longrightarrow 4Ce^{3+} + HCOOH + Br^- + 2CO2 + 5H^+$

* MA, BrMAはそれぞれ$CH_2(COOH)_2$, $BrCH(COOH)_2$を表す．反応中間体はBr^-, HOBr, Br_2, $HBrO_2$, $BrO_2\cdot$, Ce^{3+}, Ce^{4+}の7つであるが，定常状態近似により$BrO_2\cdot$を除き，6つとすることもできる．

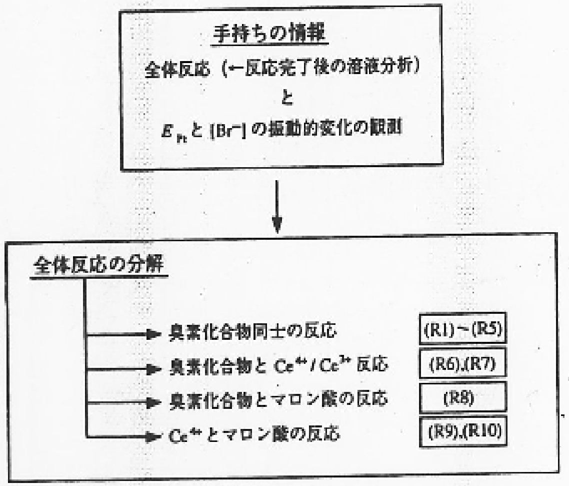


図2.6 FKNメカニズム構築の道筋．

表2.2

(R3)	$BrO_3^- + Br^- + 2H^+ \longrightarrow HBrO_2 + HOBr$
(R2)	$HBrO_2 + Br^- + H^+ \longrightarrow 2HOBr$
(R5+R6)	$2Ce^{3+} + BrO_3^- + HBrO_2 + 3H^+ \longrightarrow 2Ce^{4+} + 2HBrO_2 + H_2O$
(R4)	$2HBrO_2 \longrightarrow BrO_3^- + HOBr + H^+$
(R10)	$4Ce^{4+} + BrCH(COOH)_2 + 2H_2O \longrightarrow 4Ce^{3+} + Br^- + HCOOH + 2CO2 + 5H^+$
(O1)	$A + Y \longrightarrow X + P$
(O2)	$X + Y \longrightarrow 2P$
(O3)	$A + X \longrightarrow 2X + 2Z$
(O4)	$2X \longrightarrow A + P$
(O5)	$Z + B \longrightarrow hY$

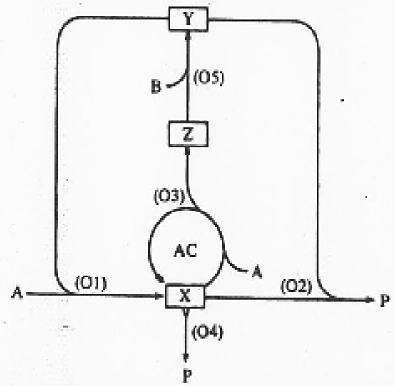
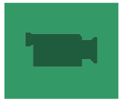
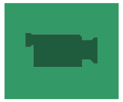
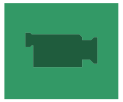


図2.10 オレゴネータにおける化学反応の流れ（岡崎紀明博士提供）．この図を利用して1サイクルの振動を説明すると次のようになる．まず，高濃度のBr^-が存在する状況から考える．つまり出発時の系は還元状態にある．この条件下では，(O2)による$HBrO_2$がBr^-と反応し，$HBrO_2$が消費される過程が優勢である（速度定数が(O1)よりはるかに大きい）．その反応が進行して[Br^-]が低下してくると，(O1)が逆転して[$HBrO_2$]がしだいに上昇してくる．[$HBrO_2$]がある程度大きくなると，(O3)つまり自触媒反応が有効に働きだし，$HBrO_2$を爆発的に生産する．この爆発的生産は(O4)の$HBrO_2$に関する2次の消費反応により拮抗される．一方，(O3)により生産されたCe^{4+}は臭化マロン酸と反応してBr^-を生成する．これによる[Br^-]の上昇により再び(O2)が優位となり，ACは抑制され，元の状態に戻ったことになる．

BZ 反応の数理モデル

Keener-Tyson model

$$\begin{cases} u_t = \varepsilon d_1 \Delta u + u(1-u) - \dfrac{bv(u-a)}{u+a}, \\ v_t = \varepsilon(d_2 \Delta v + u - v), \end{cases}$$


反応拡散モデル (2次元)

$$\boldsymbol{u}_t = D\Delta \boldsymbol{u} + \boldsymbol{F}(\boldsymbol{u}),\ t > 0,\ x \in \boldsymbol{R}^2$$

$$D = \begin{pmatrix} d_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & d_n \end{pmatrix},\ \ \boldsymbol{u} \in \boldsymbol{R}^n,\ \ \boldsymbol{F}:\ \boldsymbol{R}^n \to \boldsymbol{R}^n$$

BZ reaction: spiral pattern

Experiment

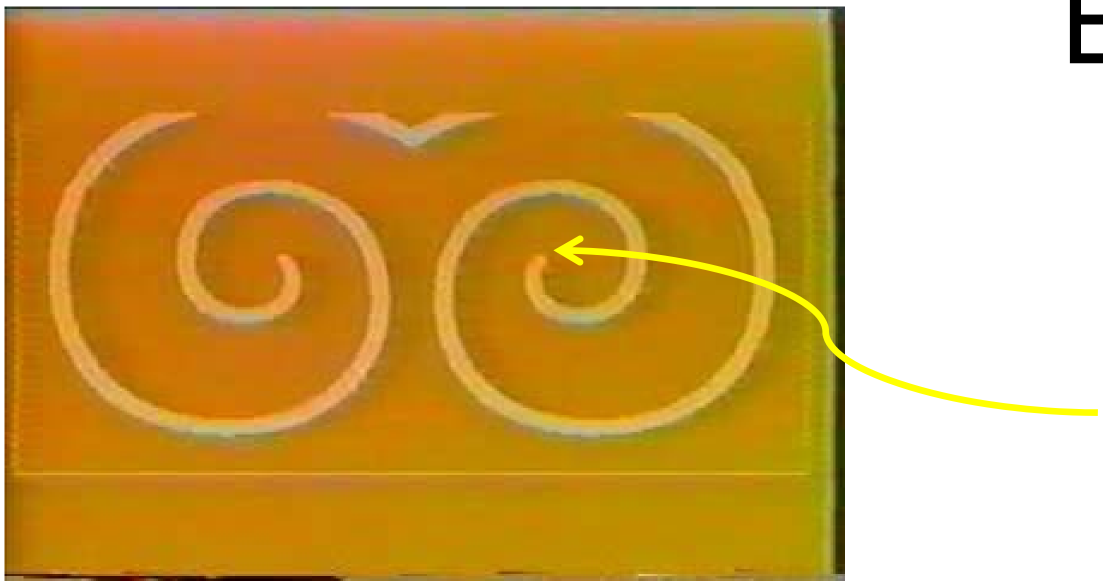
by Kitahata

Simulation of FHN

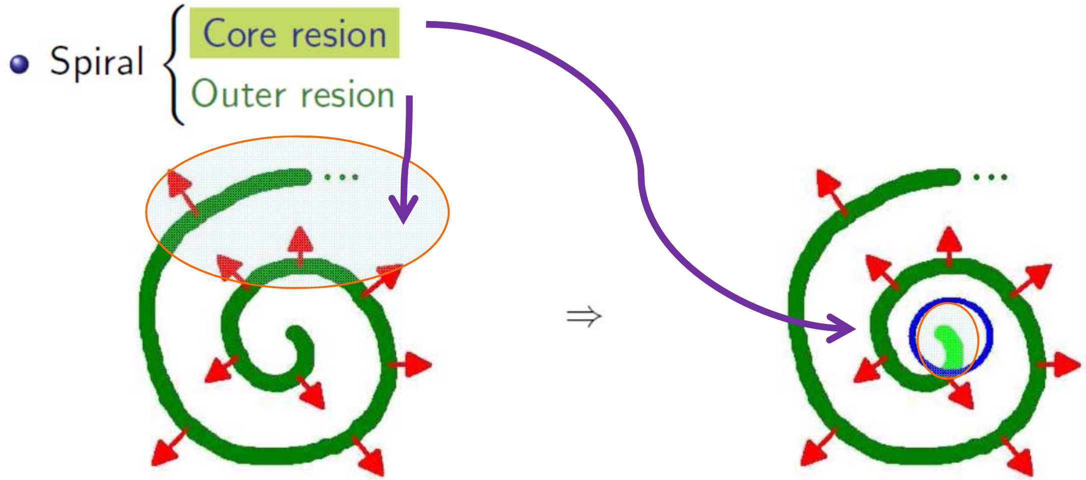
Hiroshima (Mimura, Kobayashi)

Barkley ’95, Keener ’94, Mikhailov ’94, Fife ’88,
Sandstede, Scheel ’01,
Sandstede, Scheel, wulff ’97, 99

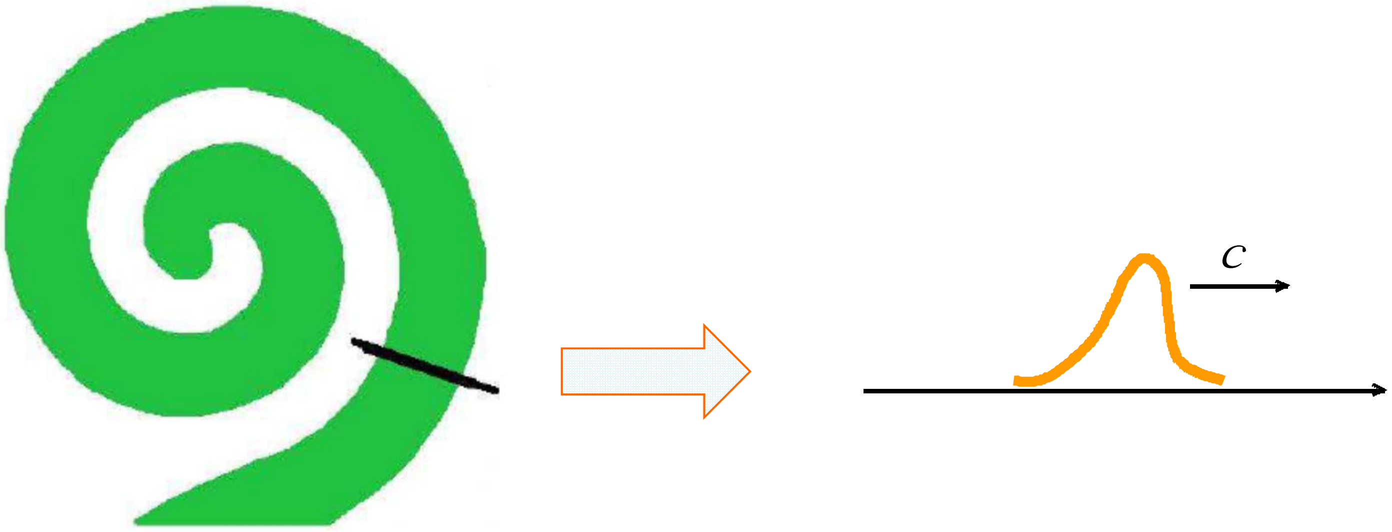
BZスパイラル



Core part (tip part)



BZスパイラルの外部領域

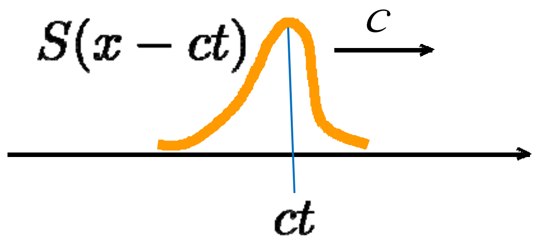


$u_t = D\Delta u + F(u),\ t > 0,\ x \in R^2$ → $U_t = DU_{xx} + F(U),\ x \in R^1$

2次元問題

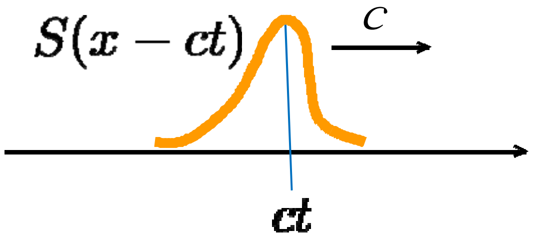
1次元問題

$U(t,x) = S(x - ct)$ 1次元進行波解

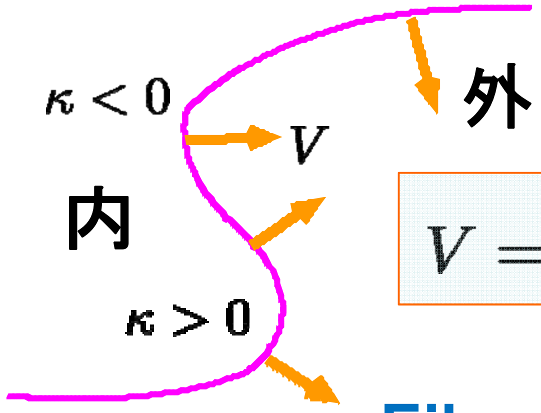


BZスパイラルの外部領域 II

$U(t,x) = S(x-ct)$ 1次元進行波解



曲線に沿って1次元進行波解が並んでいる



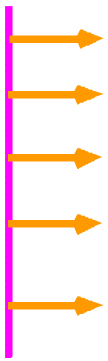
$$V = c - M\kappa$$

V: 外側法線方向への進行速度

κ: 曲線の曲率

(外側に, より曲がっているほど正で大きいとする)

Eikonal curvature equation



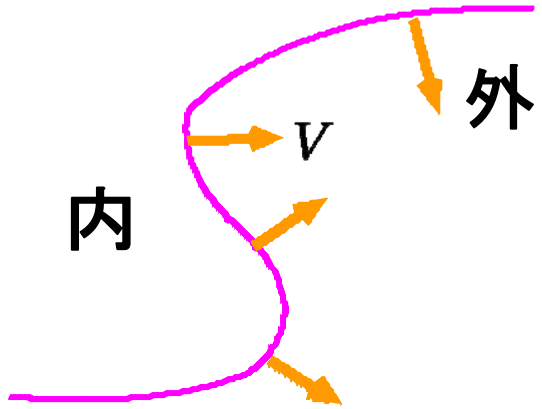
C (*κ* = 0)

$M > 0 \Rightarrow$ **planar stable**

$M < 0 \Rightarrow$ **unstable**

定数 *M* について

$V = c - M\kappa$



$u_t = D\Delta u + F(u),\ t > 0,\ x \in R^2$ 2次元問題

⇩

$U_t = DU_{xx} + F(U),\ x \in R^1$ 1次元問題

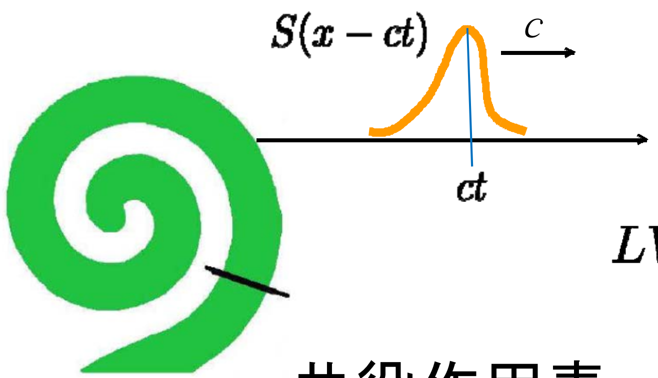
$U(t,x) = S(x-ct)$ 1次元進行波解

⇩

$-cS_z = DS_{zz} + F(U),\ z := x - ct \in R^1$

$U(t,x) \Rightarrow U(t,z)$ ⇩ 動座標系 *z = x - ct*

$U_t - cU_z = DU_{zz} + F(U),\ z \in R^1$



S(z) は定常解

$LV := DV_{zz} + cV_z + F'(S(z))V$ 線形化作用素

⇩

$LS_z = 0$

共役作用素

$L^*V := DV_{zz} - cV_z + {}^tF'(S(z))V,\ L^*\Phi^* = 0,\ \langle\ S_z, \Phi^*\ \rangle_{L^2} = 1$

$M = \langle\ DS_z, \Phi^*\ \rangle_{L^2}$

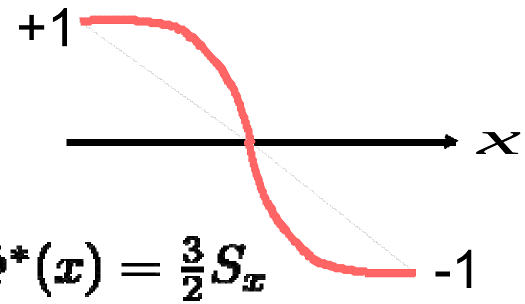
GL のフロント: *M = 1*, Gray-Scottの定常パルス: *M < 0*,
FitzHugh-Nagumo の進行パルス: *M > 0* など

線形作用素と共役作用素の例 I

$$\boldsymbol{u}_t = D\Delta \boldsymbol{u} + \boldsymbol{F}(\boldsymbol{u}),\ t>0,\ x \in \boldsymbol{R}^2$$

$$L^*V := DV_{zz} - cV_z + {}^t F'(S(z))V,\ L^*\Phi^* = 0,\ \langle\, S_z, \Phi^*\,\rangle_{L^2} = 1$$

<u>Ginzburg-Landau (Allen-Cahn)方程式</u>

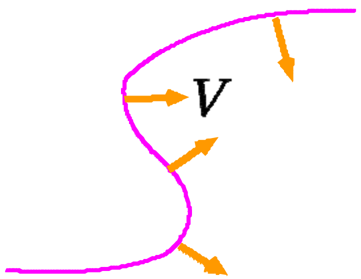


$$u_t = \Delta u + \tfrac{1}{2}u(1-u^2)$$

$\Downarrow$ $c = 0$

$$S(x) = \tanh(x/2), Lv = L^*v = v_{xx} + \tfrac{1}{2}(1-3S^2(x))v,\quad \Phi^*(x) = \tfrac{3}{2}S_x$$

$$M = \langle\, DS_z, \Phi^*\,\rangle_{L^2} = 1$$



$$V = c - M\kappa \Rightarrow V = -\kappa$$

DeMottoni, Schatzman 95, X.F.Chen 92,
Barles, Soner, Souganidis 93 …

線形作用素と共役作用素の例 II

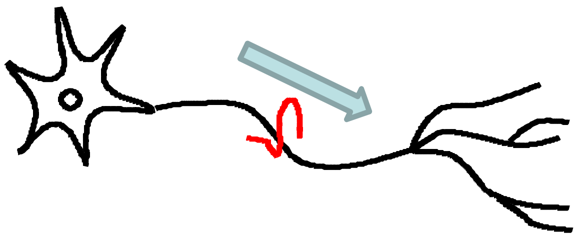
FitzHugh-Nagumo 方程式 (FHN)

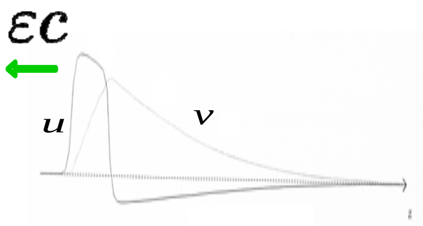
$$\begin{cases} u_t = u_{xx} + f(u) - v, \\ v_t = \varepsilon(u - \gamma v), \end{cases} \quad t > 0,\ x \in \mathbf{R}^1$$

$$f(u) = u(1-u)(u-a)$$

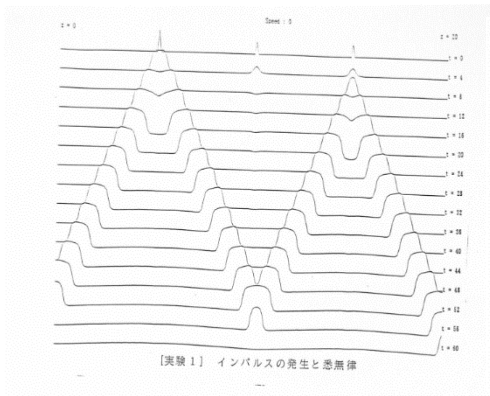
$$0 < \varepsilon << 1,\ \gamma > 0$$

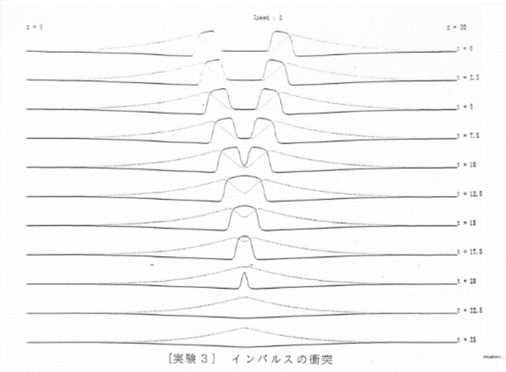
Existence, Hastings ‘76
Stability, Jones ‘84
Yanagida ‘85

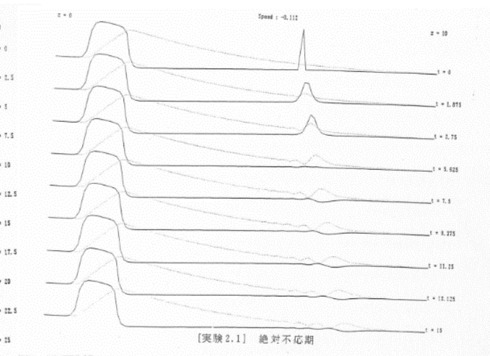




Traveling solution





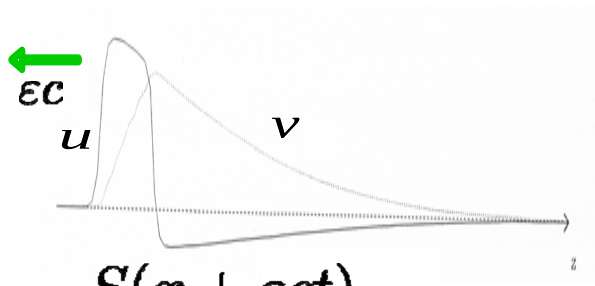


FHNの共役作用素

$$\boldsymbol{u}_t = D\Delta \boldsymbol{u} + \boldsymbol{F}(\boldsymbol{u}),\ t>0,\ x\in \boldsymbol{R}^2$$

$$L^*V := DV_{zz} - cV_z + {}^tF'(S(z))V,\ L^*\Phi^* = 0,\ \langle\, S_z, \Phi^* \,\rangle_{L^2} = 1$$

$$\textbf{(FHN)}\begin{cases} u_t &= \varepsilon^2 u_{xx} + f(u) - v, \\ \tau v_t &= \varepsilon(u-\gamma v), \end{cases}$$



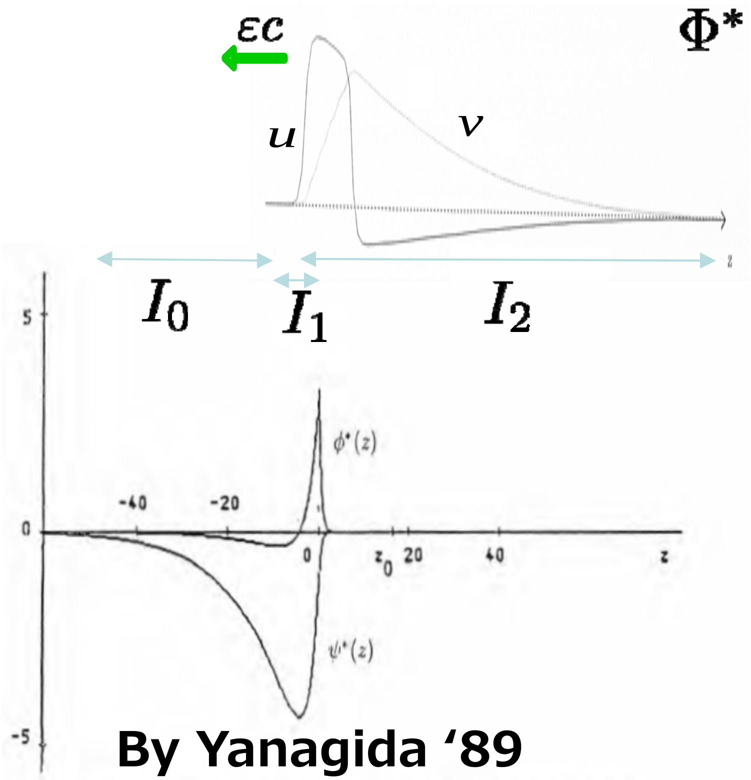
$S(x+\varepsilon ct)$

$$LV := \begin{pmatrix} \varepsilon^2 u_{zz} - \varepsilon c u_z + f'(\Phi)u - v \\ -\varepsilon c v_z + \frac{\varepsilon}{\tau}(u-\gamma v) \end{pmatrix},$$

$$L^*V := \begin{pmatrix} \varepsilon^2 u_{zz} + \varepsilon c u_z + f'(\Phi)u + \frac{\varepsilon}{\tau} v \\ \varepsilon c v_z - u - \frac{\varepsilon}{\tau}\gamma v \end{pmatrix},$$

$$V := (u,v),\ S := (\Phi, \Psi),$$

Adjoint eigenfunction



By Yanagida '89

$$\Phi^* = (\phi^*, \psi^*) \quad c = c_0 + O(\varepsilon),\ (\Theta^0(\xi) = \frac{e^{\xi/\sqrt{2}}}{1+e^{\xi/\sqrt{2}}})$$

$$M_1 := \int_{-\infty}^{\infty} e^{-c_0\xi}\Theta^0(\xi)d\xi,\ M_2 := \int_{-\infty}^{\infty} e^{-c_0\xi}(\Theta^0_\xi(\xi))^2 d\xi$$

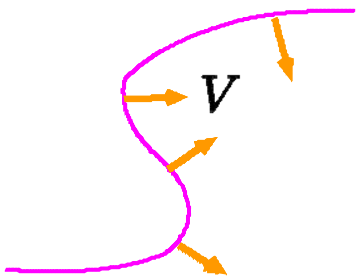
$$I_0)\quad \begin{cases} \phi^*(z) &= \dfrac{\varepsilon M_1}{\tau f'(0) M_2} exp\left\{\dfrac{1}{\tau c_0}\left(\gamma - \dfrac{1}{f'(0)}\right) z\right\} + O(\varepsilon^2), \\ \psi^*(z) &= -\dfrac{M_1}{M_2} exp\left\{\dfrac{1}{\tau c_0}\left(\gamma - \dfrac{1}{f'(0)}\right) z\right\} + O(\varepsilon), \end{cases}$$

$$I_1)\quad \begin{cases} \phi^*(\xi) &= \dfrac{1}{M_2} e^{-c_0\xi}\Theta^0_\xi(\xi) + O(\varepsilon), \\ \psi^*(\xi) &= \dfrac{1}{c_0 M_2}\displaystyle\int_{-\infty}^{\xi} e^{-c_0\xi}\Theta^0_\xi(\xi)d\xi - 1 + O(\varepsilon), \end{cases}$$

$$I_2)\quad \begin{cases} \phi^*(z) &= O(\varepsilon^2), \\ \psi^*(z) &= O(\varepsilon). \end{cases}$$

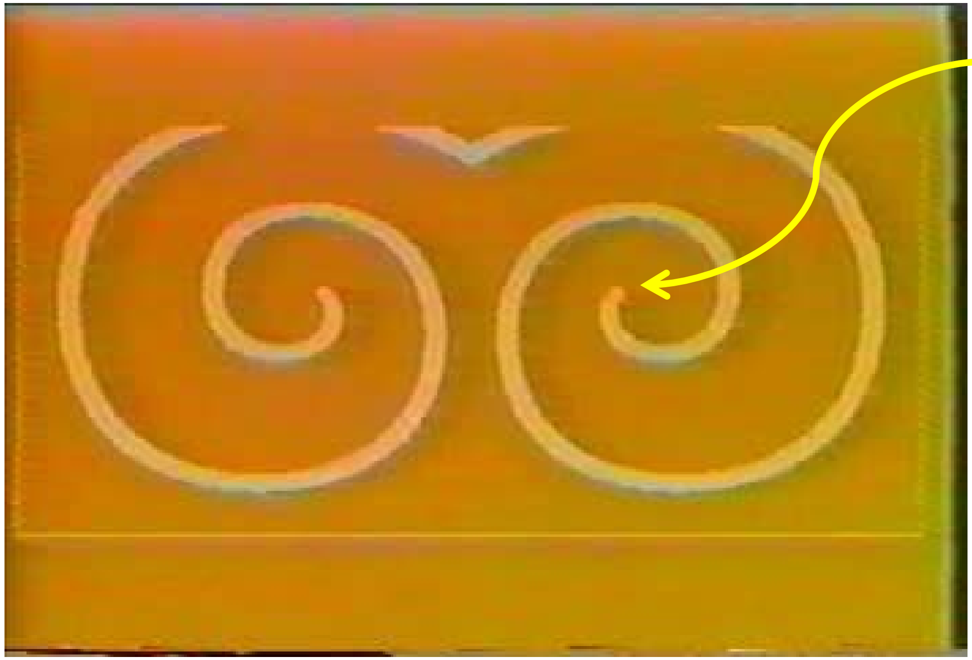
$$M = \langle\, DS_z, \Phi^* \,\rangle_{L^2} > 0$$

Ei '95, Ei, H. Ikeda, K. Ikeda, Yanagida '08



$$V = c - M\kappa \Rightarrow V = \varepsilon c - M\kappa$$

Spiral Pattern

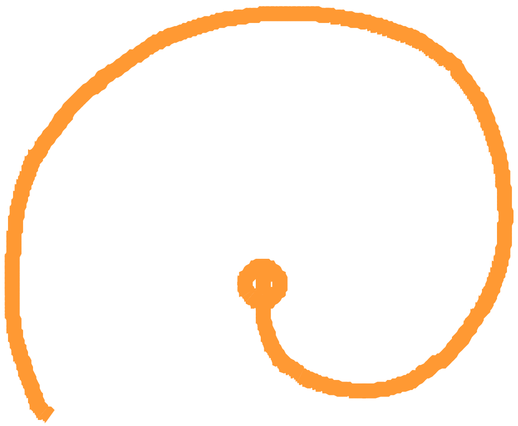


Motion of tip part

Heuristic conditions were added.

$$V = c - M\kappa$$

Eikonal curvature equation

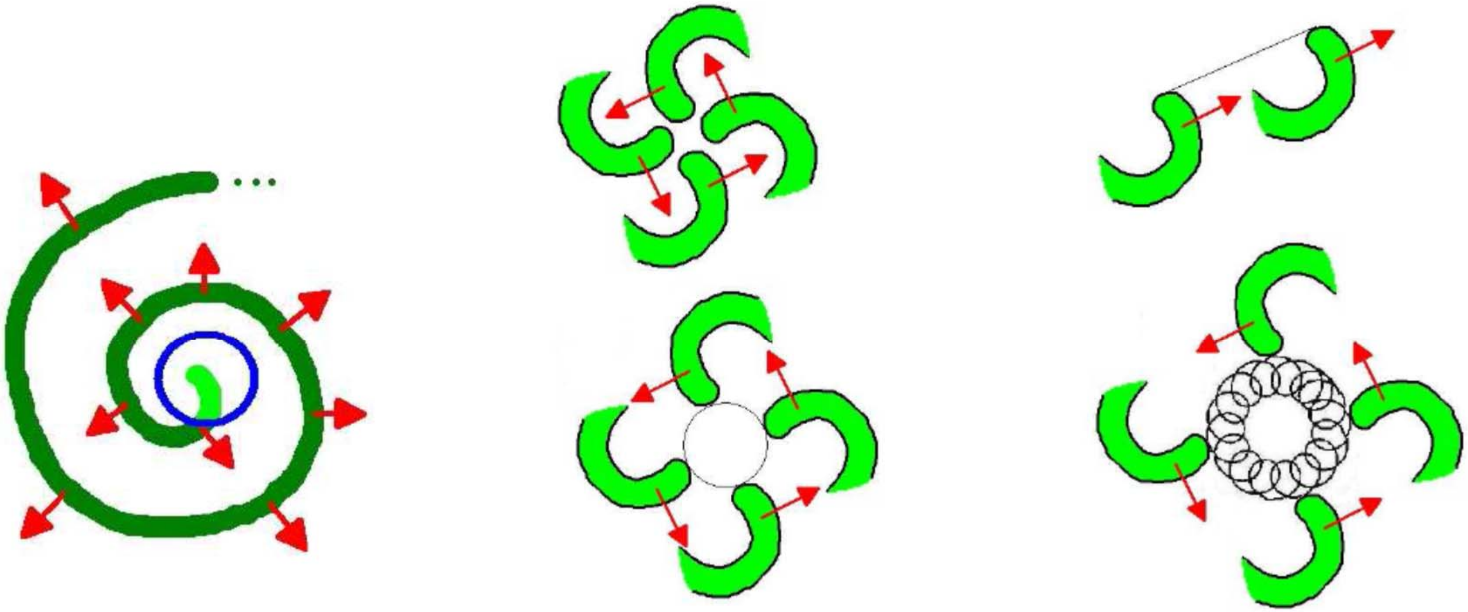


Boundary conditions on tip is necessary

Core part

- Spiral { Core resion / Outer resion

- Typical behavior of Core : rotating, meandering, · · ·



Known Result (Keener)

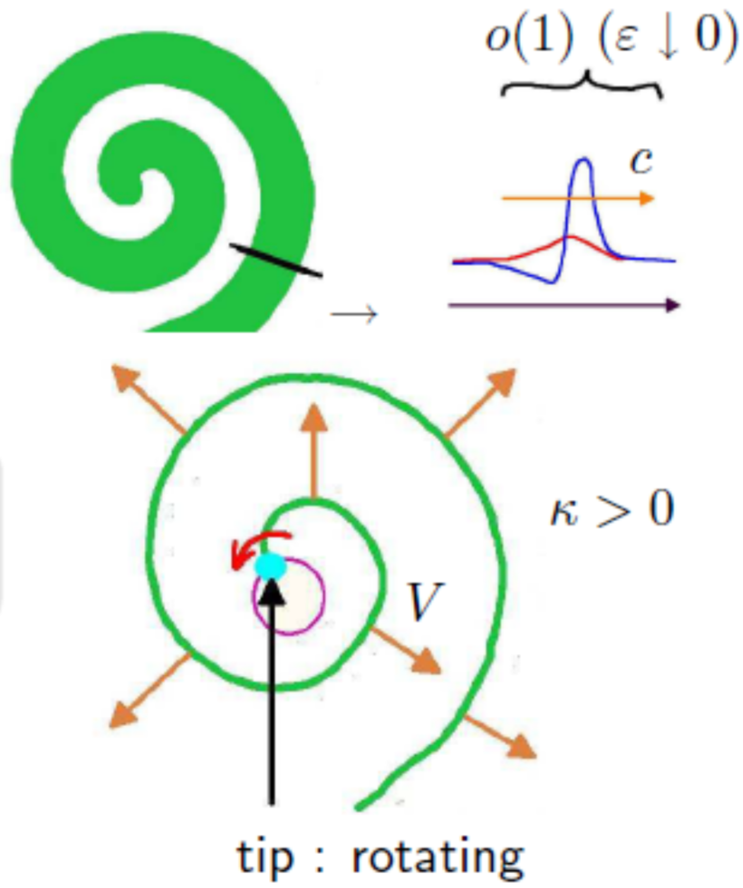
$$\begin{cases} \varepsilon u_t = \varepsilon^2 \Delta u + f(u,v) \\ v_t = g(u,v) \end{cases}$$

- $\varepsilon \downarrow 0$

Eikonal equation (Keener 1986)

curve : $V = c - \varepsilon\kappa$

- V : normal velocity
- c : velocity of 1-D pulse
- κ : curvature



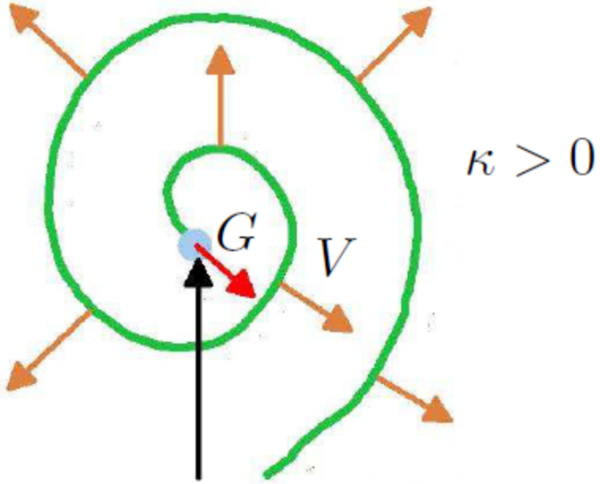
Known Result (Mikhailov & Zykov)

Mikhailov & Zykov, 1991

curve : $V = V_0 - \gamma_1 \kappa,$

tip : $G = \gamma_2(\kappa_c - \kappa_0).\quad (*)$

- V : normal velocity
- κ : curvature
- G : tangential velocity of tip
- $V_0,\ \gamma_1,\ \gamma_2$: const.
- κ_c : critical curvature
- $\kappa_0 = \lim_{\sigma \to 0} \kappa,\ \ \sigma$: arc length



tip : $\sigma = 0$

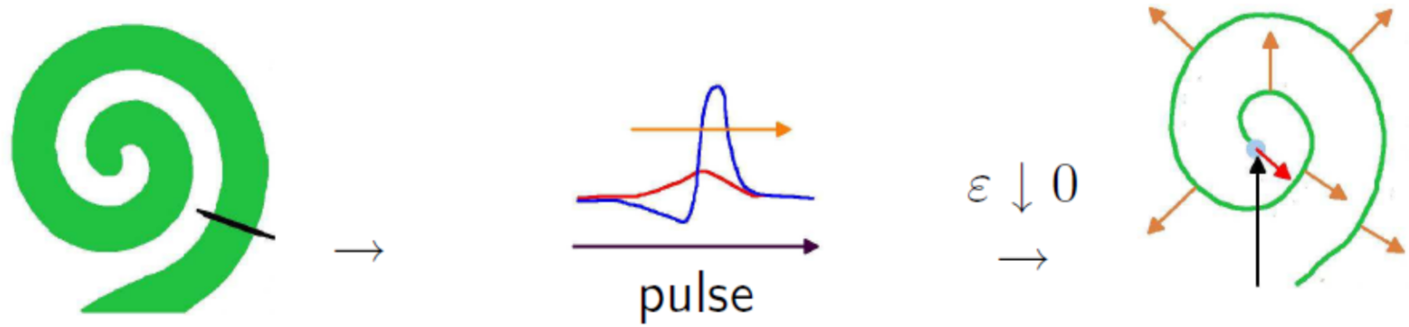
- Note : $(*)$ is given by a physical background, not derived from RD system.

Purpose

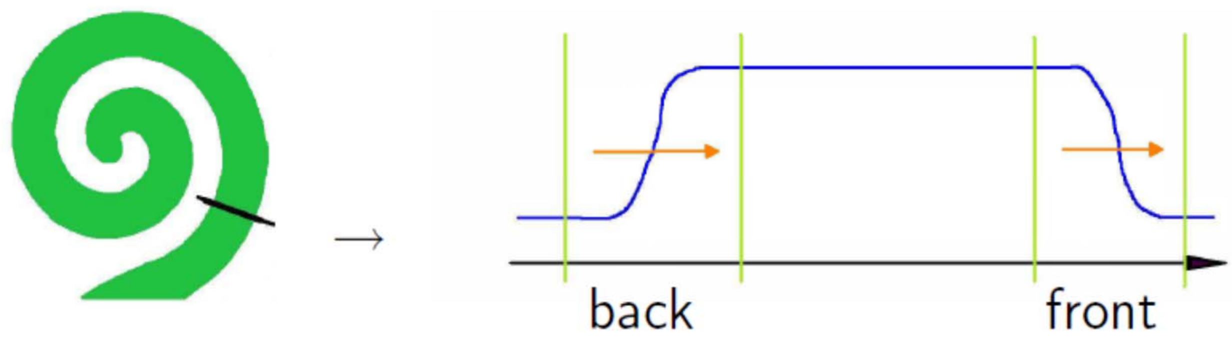
Derivation of explicit dynamics of spiral wave from RD, including the tip parts

Another approach -front & back- (1)

monostable (pulse)



bistable (front & back)

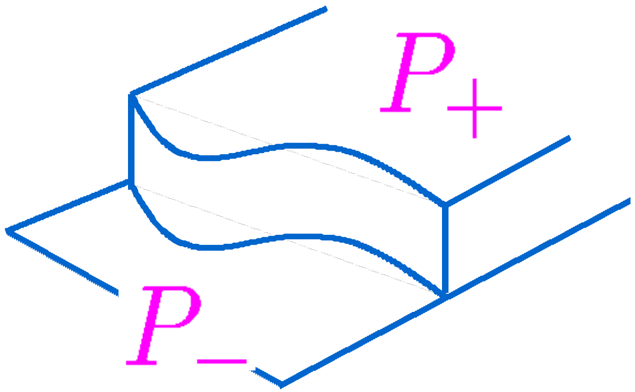


Bistable Nonlinearity

$$\boldsymbol{u}_t = D\Delta \boldsymbol{u} + \boldsymbol{F}(\boldsymbol{u}), \ t > 0, \ x \in \boldsymbol{R}^2$$

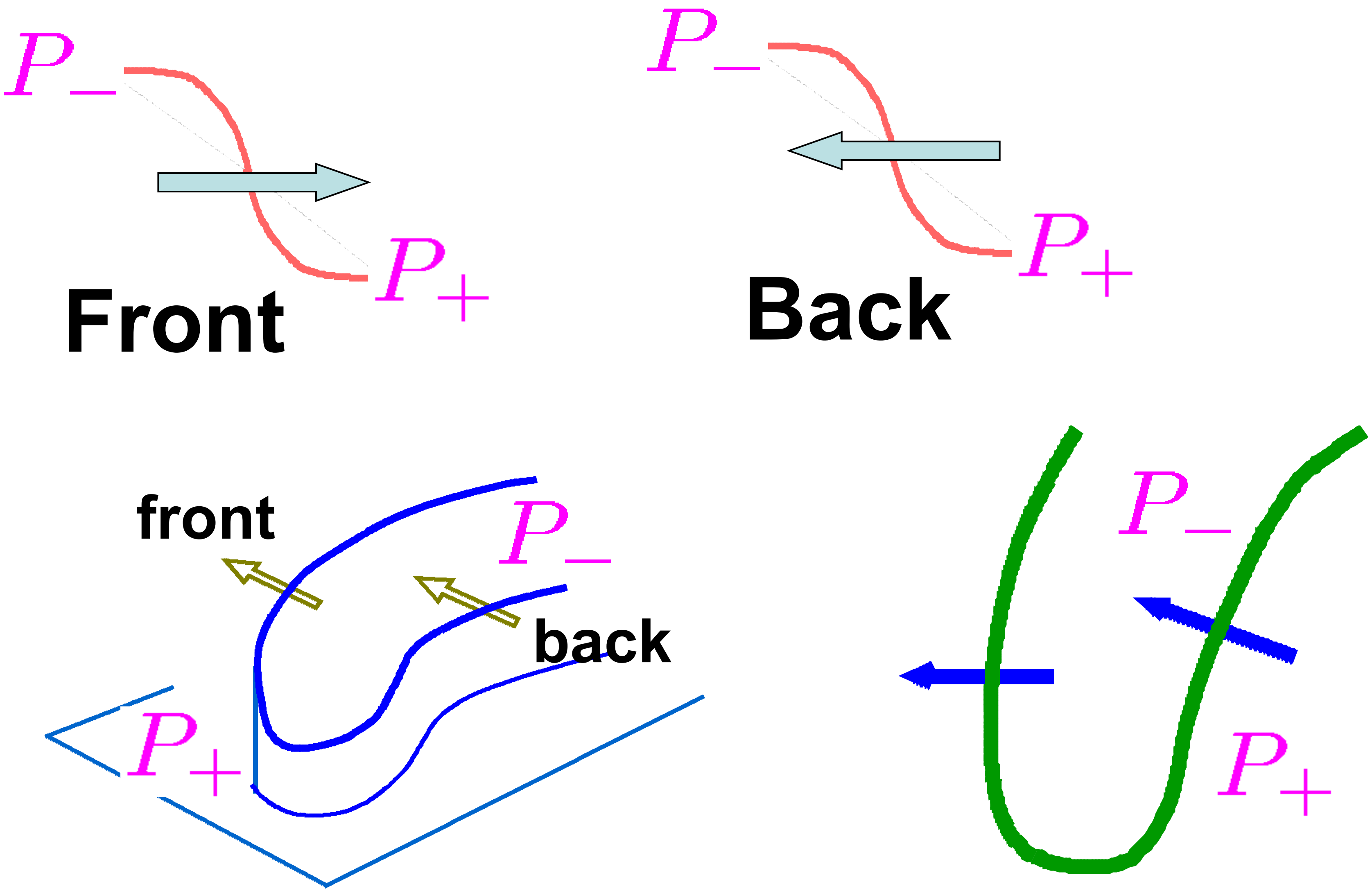
Assume: $\boldsymbol{F}$ has two stable equilbria, say $P_{\pm}$

Consider interfacial dynamics connecting P_- and P_+



Front and Back traveling solutions

$$u_t = Du_{xx} + F(u),\ t > 0,\ x \in R$$

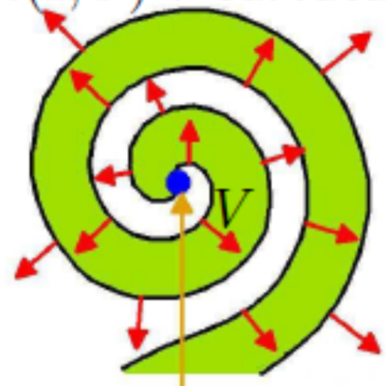


Known Result(Hagberg & Meron)

- 2-comp. Bistable RD system on 2D

$$\begin{cases} \varepsilon\tau u_t = \varepsilon^2 \Delta u + f(u,v) \\ v_t = d_2 \Delta v + g(u,v) \end{cases}$$

- $\tau = \tau_c - \eta \ (|\eta| \ll 1)$
- $\boxed{\varepsilon = \tau_c + \varepsilon_1 \ (|\varepsilon_1| \ll 1)}$



Order parameter equations (Hagberg & Meron, 1998)

curve : $\begin{cases} V = r - \gamma_1 \kappa, \\ r_t = \gamma_2 r - \gamma_3 r^3 + \gamma_4 \kappa + r_{\sigma\sigma} - r_\sigma \int_0^\sigma \kappa V \, d\sigma \end{cases}$

- V : Outward normal velocity
- $r = r(t,\sigma)$: "Order parameter", γ_i : const.

1D problem

$$\boldsymbol{u}_t = D\boldsymbol{u}_{xx} + \boldsymbol{F}(\boldsymbol{u}) =: \mathcal{L}_1(\boldsymbol{u}),\ t > 0,\ x \in \boldsymbol{R}$$

$\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{u}; k)$, depending on a parameter k

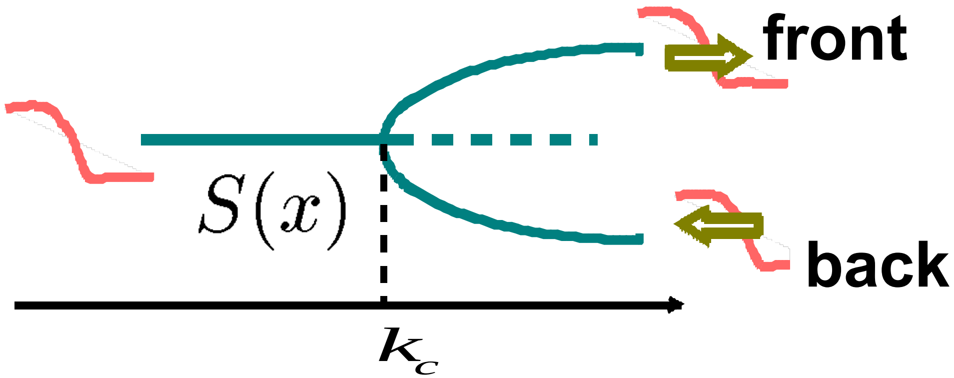
Assumption:

$\Rightarrow \mathcal{L}_1(\boldsymbol{u}) = \mathcal{L}_1(\boldsymbol{u}; k)$

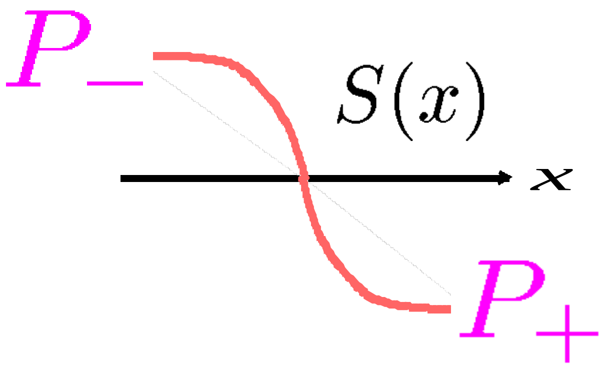
$S(x)$; stationary front solution s.t. $\mathcal{L}_1(S(x); k) = 0$

$S(x) \to P_\pm$ as $x \to \pm\infty$, (For simplicity) $S(x)$, odd

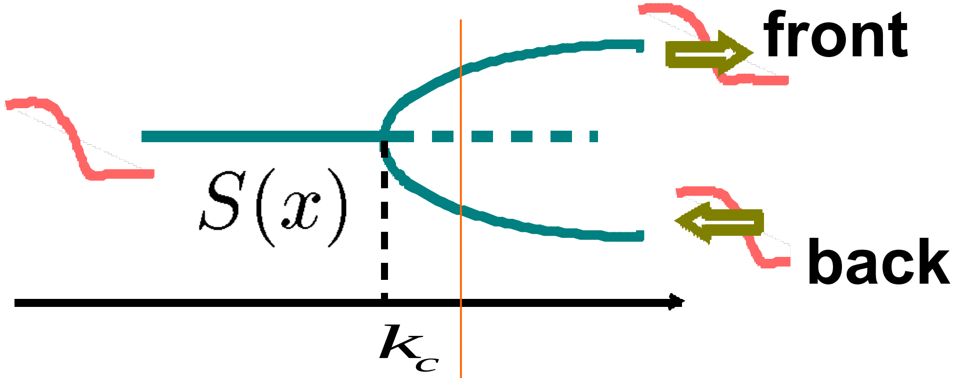
$k = k_c$, s.t. pitchfork type bifurcation of traveling fronts occurs.

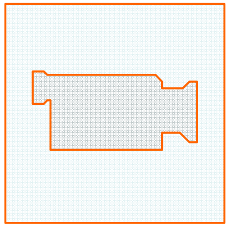


$L := \mathcal{L}_1'(S(x); k_c)$

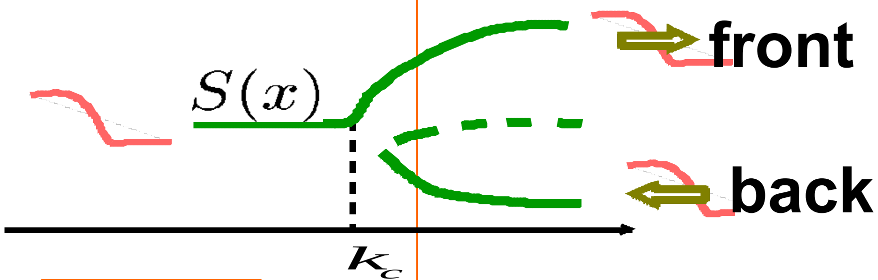


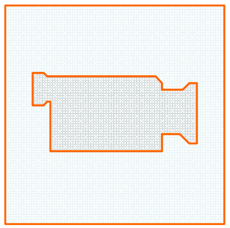
Numerical simulations

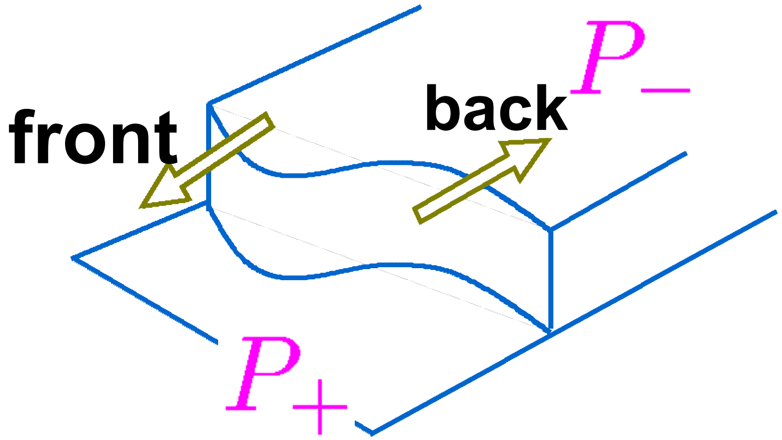




imperfection







Center Manifold near $k = k_c$

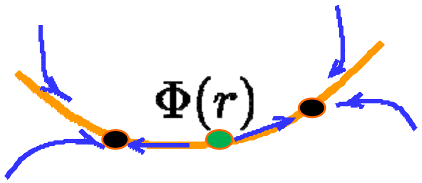
$$\boldsymbol{u}_t = D\boldsymbol{u}_{xx} + \boldsymbol{F}(\boldsymbol{u};k) =: \mathcal{L}_1(\boldsymbol{u};k),\ k = k_c + \eta$$

$^{\exists}\Phi(r;\eta)$; Center Manifold and $^{\exists}H_1(r;\eta) \in \boldsymbol{R}$ s.t.

$$\Phi(r;\eta) = \Phi(r;\eta)(x) \in \boldsymbol{R}^n,\ -H_0\Phi_x + H_1\Phi_r = \mathcal{L}_1(\Phi;k_c+\eta),$$

$$\boldsymbol{u}(t,x) = \Phi(r(t);\eta)(x - l(t)),$$

$$\begin{cases} \dot{l} &= H_0(r;\eta), \\ \dot{r} &= H_1(r;\eta) \end{cases}$$

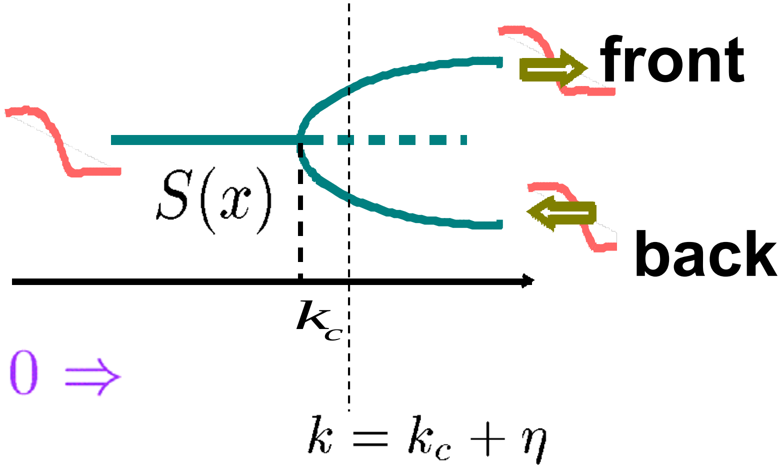


$^{\exists}\psi(x),\ ^{\exists}M_j > 0,$

$$\Phi(r;\eta) = S(x) + r\psi(x) + \cdots,$$

$$H_0(r;\eta) = r + \cdots,$$

$$H_1(r;\eta) = -(M_1 r^2 - M_2\eta)r + \cdots$$



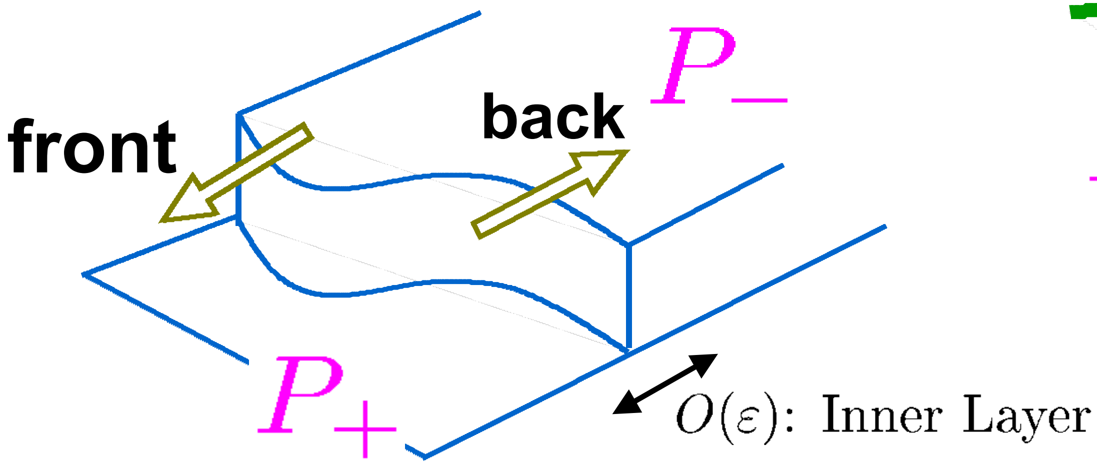
$L := \mathcal{L}_1'(S(x);k_c)$ $\quad M_1,\ M_2 > 0 \Rightarrow$

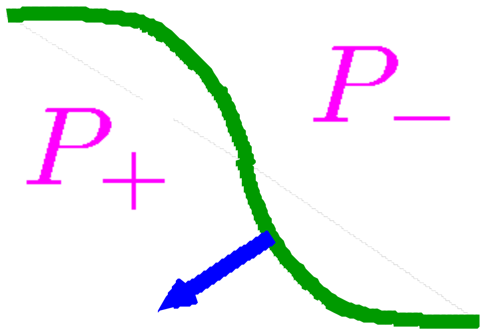
Linearized operator at $k = k_c$

2D problem

$$\boldsymbol{u}_t = \varepsilon^2 D \Delta \boldsymbol{u} + \boldsymbol{F}(\boldsymbol{u}; k),\ t > 0,\ x \in \boldsymbol{R}^2$$

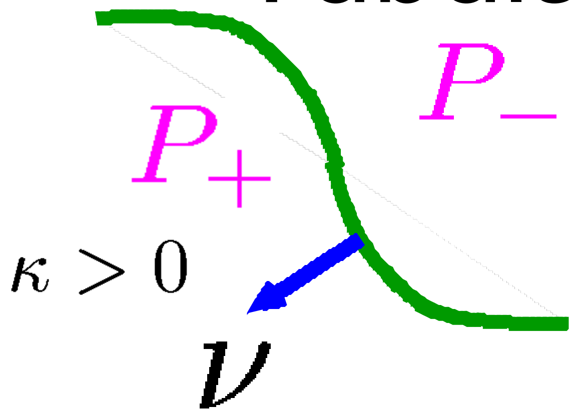
$$k = k_c + \eta \ (0 < \varepsilon,\ \eta << 1)$$





Dynamics of interfaces

Tubular Nbd of Interface



$\Gamma = \{\Gamma(t, \sigma)\}$ $\kappa = \kappa(t, \sigma)$ **: Curvature**

$\nu = \nu(t, \sigma)$: Outward normal unit vector

$$\boldsymbol{x} = \Gamma(t, \sigma) + \lambda \nu(t, \sigma),\ \sigma = \Sigma(t, \boldsymbol{x}),\ \lambda = \Lambda(t, \boldsymbol{x})$$

$$\boldsymbol{x} = (x, y) \iff (\lambda, \sigma) \underset{\lambda = \varepsilon\mu}{\iff} (\mu, \sigma)$$

$$\boldsymbol{u}(t, \boldsymbol{x}) = \Phi(r(t, \sigma); \eta)(\mu) + \boldsymbol{v}(t, \sigma, \mu)$$

Derivation of Interface equations I

$$\boldsymbol{u}(t,\boldsymbol{x}) = \Phi(r(t,\Sigma(t,\boldsymbol{x}));\eta)(\Lambda(t,\boldsymbol{x})/\varepsilon) + \boldsymbol{v}$$

Substitute ⇓

$$\boldsymbol{u}_t = \varepsilon^2 D\Delta\boldsymbol{u} + \boldsymbol{F}(\boldsymbol{u};k_c+\eta) =: \mathcal{L}_2(\boldsymbol{u};k_c+\eta)$$

⇓ $\boldsymbol{u} = \boldsymbol{u}(t,\sigma,\mu)$

$$\begin{aligned}\boldsymbol{u}_t + (\Lambda_t/\varepsilon)\boldsymbol{u}_\mu + \Sigma_t\boldsymbol{u}_\sigma \\ &= D\left\{\boldsymbol{u}_{\mu\mu} \underbrace{- \frac{\varepsilon\kappa}{1-\varepsilon\kappa\mu}\boldsymbol{u}_\mu + \frac{\varepsilon^2}{1-\varepsilon\kappa\mu}\left(\frac{1}{1-\varepsilon\kappa\mu}\boldsymbol{u}_\sigma\right)_\sigma}_{\varepsilon K(\varepsilon)\boldsymbol{u}}\right\} + \boldsymbol{F}(\boldsymbol{u};k_c+\eta) \\ &= \mathcal{L}_1(\boldsymbol{u};k_c+\eta) + \varepsilon K(\varepsilon)\boldsymbol{u},\end{aligned}$$

$$\mathcal{L}_1(\boldsymbol{u};k) := D\boldsymbol{u}_{xx} + \boldsymbol{F}(\boldsymbol{u};k),\ \boldsymbol{u} = \boldsymbol{u}(t,\sigma,\mu) = \Phi + \boldsymbol{v}$$

Derivation of Interface equations II

$$\boldsymbol{u}_t = \varepsilon^2 D\Delta \boldsymbol{u} + \boldsymbol{F}(\boldsymbol{u}; k_c + \eta)$$

$\boldsymbol{u} = \boldsymbol{u}(t, \sigma, \mu)$

$$\boldsymbol{u}_t + (\Lambda_t/\varepsilon)\boldsymbol{u}_\mu + \Sigma_t \boldsymbol{u}_\sigma = \mathcal{L}_1(\boldsymbol{u}; k_c + \eta) + \varepsilon K(\varepsilon)\boldsymbol{u}$$

$\boldsymbol{u} = \Phi(r(t,\sigma); \eta)(\mu) + \boldsymbol{v}(t, \sigma, \mu)$

$$\begin{aligned}\boldsymbol{v}_t + (\Lambda_t/\varepsilon)\boldsymbol{v}_\mu + \Sigma_t \boldsymbol{v}_\sigma + (\Lambda_t/\varepsilon)\Phi_\mu + D_t r \Phi_r \\ = \mathcal{L}_1(\Phi; k) + \varepsilon K \Phi + \mathcal{L}_1'(\Phi; k)\boldsymbol{v} + \varepsilon K \boldsymbol{v} + O(\boldsymbol{v}^2),\end{aligned}$$

$(k = k_c + \eta,\ K = K(\varepsilon),\ D_t r := r_t + \Sigma_t r_\sigma)$

$-H_0 \Phi_\mu + H_1 \Phi_r = \mathcal{L}_1(\Phi; k_c + \eta)$ holds.

$$\Lambda_t/\varepsilon = -(H_0 + H_0^*),\ D_t r = H_1 + H_1^*$$

$$\begin{aligned}\boldsymbol{v}_t - (H_0 + H_0^*)\boldsymbol{v}_\mu + \Sigma_t \boldsymbol{v}_\sigma - H_0^* \Phi_\mu + H_1^* \Phi_r \\ = \varepsilon K \Phi + \mathcal{L}_1'(\Phi; k)\boldsymbol{v} + \varepsilon K \boldsymbol{v} + O(\boldsymbol{v}^2),\end{aligned}$$

Derivation of Interface equations III

$$\boldsymbol{v}_t - (H_0 + H_0^*)\boldsymbol{v}_\mu + \Sigma_t \boldsymbol{v}_\sigma - H_0^*\Phi_\mu + H_1^*\Phi_r = \varepsilon K\Phi + \mathcal{L}_1'(\Phi;k)\boldsymbol{v} + \varepsilon K\boldsymbol{v} + O(\boldsymbol{v}^2),$$

$$\boldsymbol{v} = \boldsymbol{v}_0 + \varepsilon \boldsymbol{v}_1 + \cdots,$$

Standard perturbation method

$$H_0^* = \varepsilon\gamma_1\kappa + O(\varepsilon^2),\ H_1^* = \varepsilon\gamma_2\kappa + \varepsilon^2\gamma_3 r_{\sigma\sigma} + O(\varepsilon^2) \quad ({}^\exists\gamma_1,\ \gamma_2,\ \gamma_3)$$

Thus,

$$\begin{cases} \Lambda_t/\varepsilon &= -(H_0(r) + \varepsilon\gamma_1\kappa) + \cdots = -(r + \varepsilon\gamma_1\kappa) + \cdots, \\ D_t r &= H_1(r) + \varepsilon\gamma_2\kappa + \varepsilon^2\gamma_3 r_{\sigma\sigma} + \cdots \\ &= (M_2\eta - M_1 r^2)r + \varepsilon\gamma_2\kappa + \varepsilon^2\gamma_3 r_{\sigma\sigma} + \cdots \end{cases}$$

$$r = \sqrt{\eta}R,$$

$$\left(\begin{array}{c} D_t R = R_t + \Sigma_t R_\sigma, \\ Q(R) := (M_2 - M_1 R^2)R \end{array} \right)$$

$$\begin{cases} \Lambda_t &= -(\varepsilon\sqrt{\eta}R + \varepsilon^2\gamma_1\kappa) + \cdots, \\ D_t R &= \eta Q(R) + (\varepsilon/\sqrt{\eta})\gamma_2\kappa + \varepsilon^2\gamma_3 R_{\sigma\sigma} + \cdots \end{cases}$$

Derivation of Interface equations IV

$$\begin{cases} \Lambda_t &= -(\varepsilon\sqrt{\eta}R + \varepsilon^2\gamma_1\kappa) + \cdots, \\ D_tR &= \eta Q(R) + (\varepsilon/\sqrt{\eta})\gamma_2\kappa + \varepsilon^2\gamma_3 R_{\sigma\sigma} + \cdots \end{cases}$$

$$\varepsilon = \eta^{3/2}$$

$$\left(\begin{array}{c} D_tR = R_t + \Sigma_t R_\sigma, \\ Q(R) := (M_2 - M_1R^2)R \end{array} \right)$$

$$\begin{cases} \Lambda_t &= -\eta^2(R + \eta\gamma_1\kappa) + \cdots, \\ D_tR &= \eta(Q(R) + \gamma_2\kappa + \eta^2\gamma_3 R_{\sigma\sigma}) + \cdots \end{cases}$$

$$\tau := \eta t$$

$$\begin{cases} \Lambda_\tau &= -\eta(R + \eta\gamma_1\kappa) + \cdots, \\ D_\tau R &= Q(R) + \gamma_2\kappa + \eta^2\gamma_3 R_{\sigma\sigma} + \cdots \end{cases}$$

$$(D_\tau R := R_\tau + \Sigma_\tau R_\sigma)$$

Or $T := \eta^2 t$

$$\begin{cases} \Lambda_T &= -(R + \eta\gamma_1\kappa) + \cdots, \\ \eta D_T R &= Q(R) + \gamma_2\kappa + \eta^2\gamma_3 R_{\sigma\sigma} + \cdots \end{cases}$$

Dynamics of Interfaces

$$\begin{cases} \Lambda_T = -(R + \eta\gamma_1\kappa) + \cdots, \\ \eta D_T R = Q(R) + \gamma_2\kappa + \eta^2\gamma_3 R_{\sigma\sigma} + \cdots \end{cases}$$

$$\Gamma = \Gamma(T) = \{\Gamma(T,\sigma)\}$$

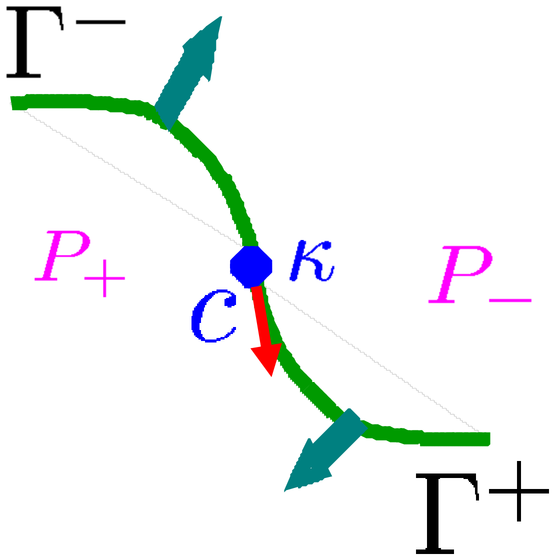
$$\eta \downarrow 0$$

Note: $\Lambda_T = -V$ (normal velocity)

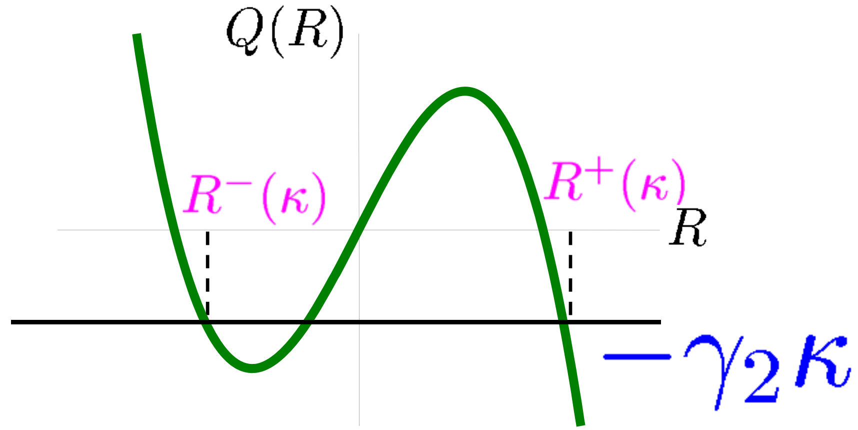
$$\begin{cases} \Lambda_T = -R, \\ 0 = Q(R) + \gamma_2\kappa \end{cases}$$

$$\Gamma = \Gamma^+ \cup \Gamma^-$$

$$\begin{cases} \Gamma^+;\ V^+ = R^+(\kappa), \\ \Gamma^-;\ V^- = R^-(\kappa), \end{cases}$$



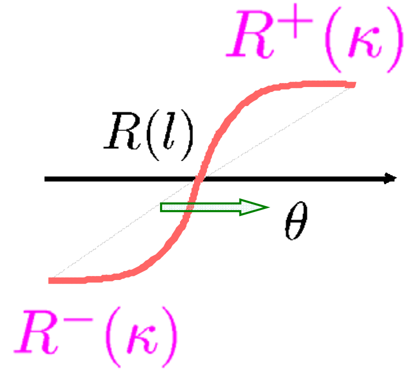
$$c = \Gamma(T, \sigma^*(T))$$



Inner Layer

$$\begin{cases} \Lambda_T = -(R + \eta\gamma_1\kappa) + \cdots, \\ \eta D_T R = Q(R) + \gamma_2\kappa + \eta^2\gamma_3 R_{\sigma\sigma} + \cdots \end{cases}$$

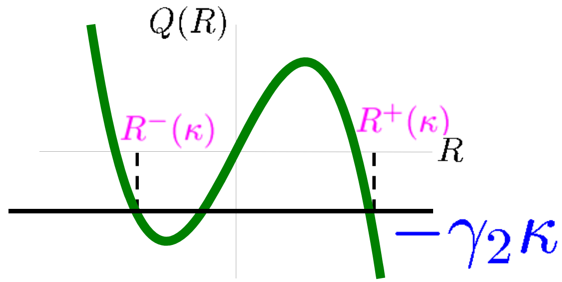
$$R = R\left(T, \frac{\sigma - \sigma^*(T)}{\eta}\right) \qquad l := \frac{\sigma - \sigma^*(T)}{\eta}$$



$$\begin{cases} \Lambda_T = -(R + \eta\gamma_1\kappa) + \cdots, \\ \eta R_T + (-\sigma_T^* + \Sigma_T)R_l = Q(R) + \gamma_2\kappa + \gamma_3 R_{ll} + \cdots \end{cases}$$

$\eta \downarrow 0$

$$\begin{cases} -V = \Lambda_T = -R, \\ (-\sigma_T^* + \Sigma_T)R_l = Q(R) + \gamma_2\kappa + \gamma_3 R_{ll} \end{cases}$$

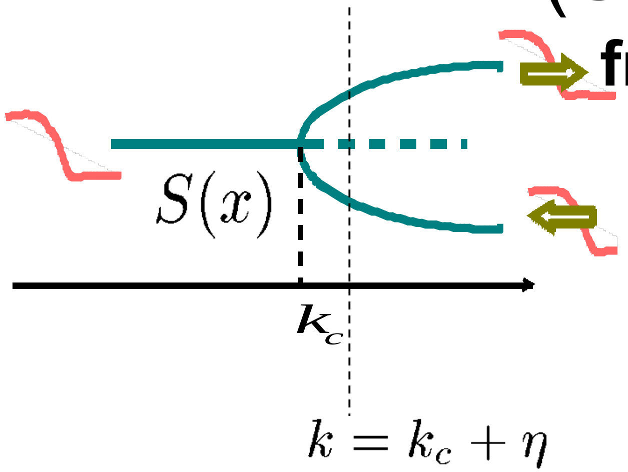


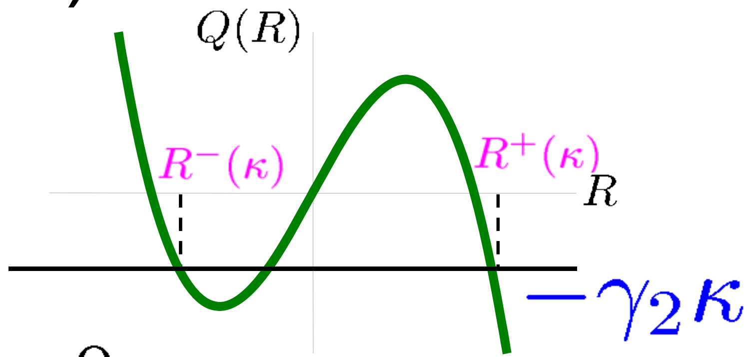
$\sigma_T^* - \Sigma_T = \theta$ s.t.

$$\begin{cases} -\theta R_l = Q(R) + \gamma_2\kappa + \gamma_3 R_{ll} \\ R(\pm\infty) = R^\pm(\kappa) \end{cases}$$

Dynamics of interfaces II

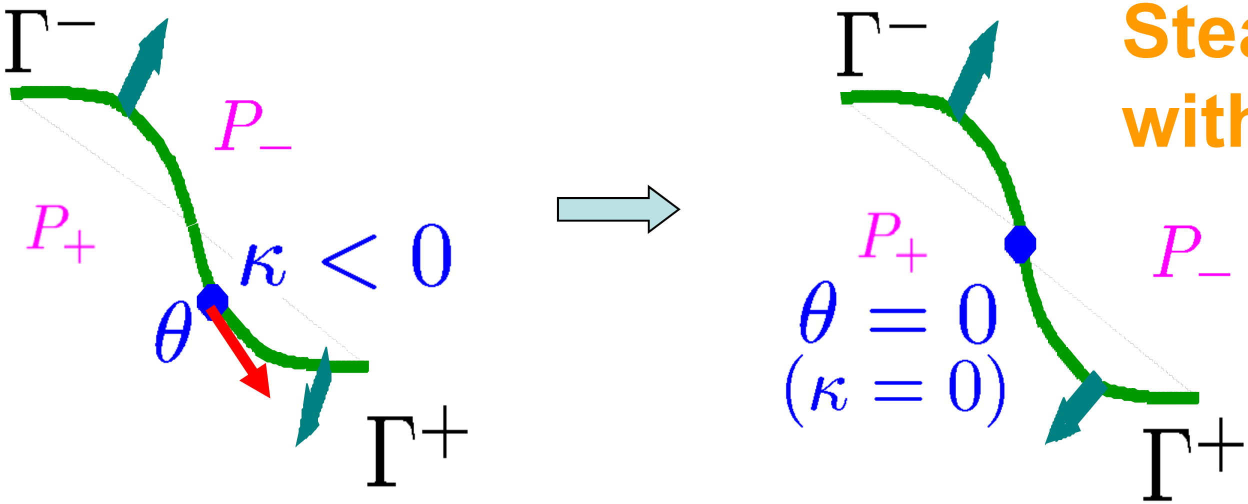
(Symmetric case)





Assume $\gamma_2 > 0$

$$\kappa > 0 \Rightarrow \theta < 0, \; \kappa < 0 \Rightarrow \theta > 0$$

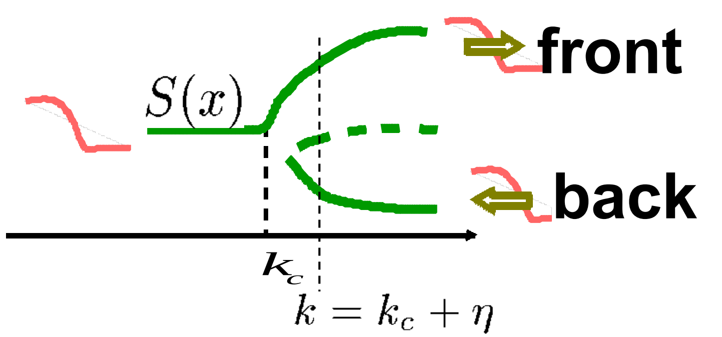


Steadily rotating with fixed center

Dynamics of interfaces III

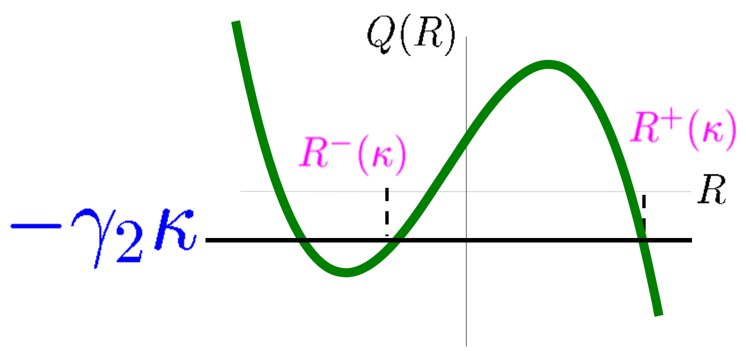
(unsymmetrical case)

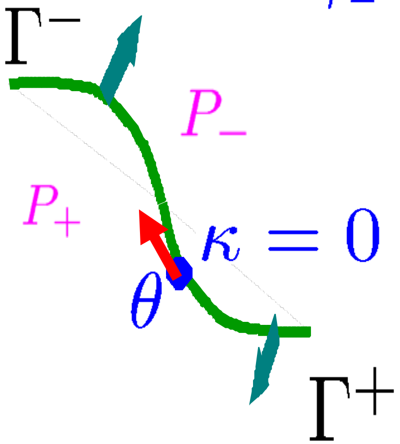
imperfection

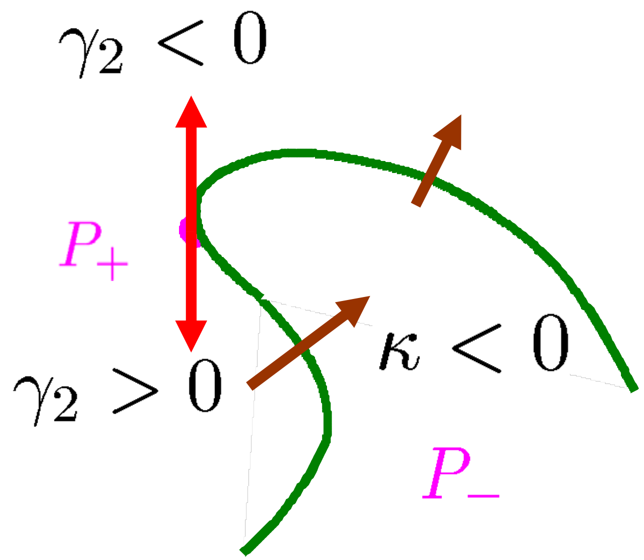


$$Q(R) = (M_2 - M_1 R^2)R + \delta\widehat{Q}(R)$$

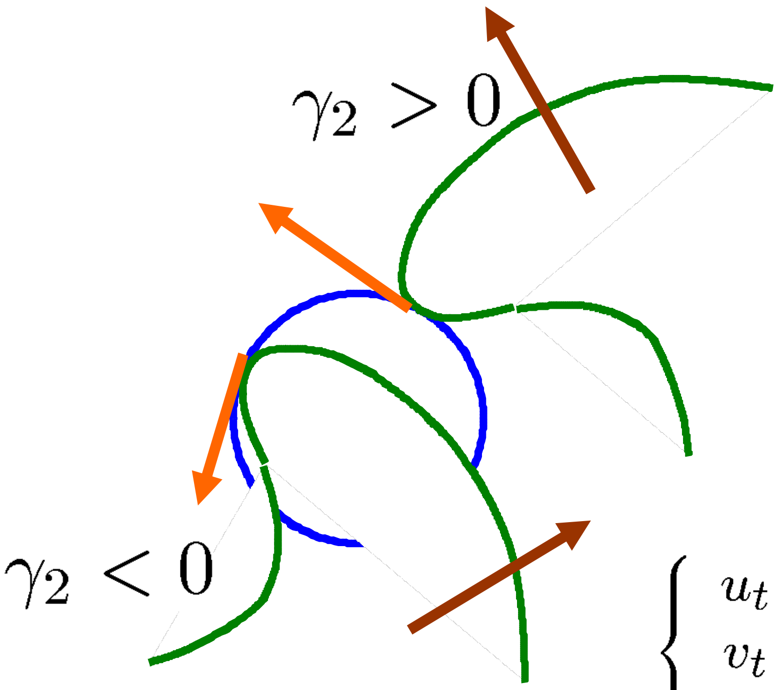
$$\begin{cases} -\theta R_l = Q(R) + \gamma_2 \kappa + \gamma_3 R_{ll} \\ R(\pm\infty) = R^{\pm}(\kappa) \end{cases}$$







Dynamics of interfaces IV



$$\begin{cases} \varepsilon\tau u_t &= \varepsilon^2\Delta u + f(u,v), \\ v_t &= d\Delta v + g(u,v), \end{cases} \quad t>0,\ x\in \boldsymbol{R}^2.$$

Bistable systems,
(E, Ikeda, kawana)

$\gamma_2 < 0$ $\gamma_2 > 0$

$$\begin{cases} u_t &= \varepsilon^2 d_1\Delta u + f_1(u,v,w), \\ v_t &= \varepsilon^2 d_2\Delta v + f_2(u,v,w), \\ w_t &= \varepsilon^2 d_2\Delta w + f_3(u,v,w) \end{cases} \quad t>0,\ x\in \boldsymbol{R}^2.$$

3 species competition diffusion systems
(C.C.Chen, L.C.Hung, M.Mimura)

Remark:

Planar front: $V = R^+(\kappa)$ or $(V = R^-(\kappa))$

$\gamma_2 > 0$; stable, $\gamma_2 < 0$, unstable

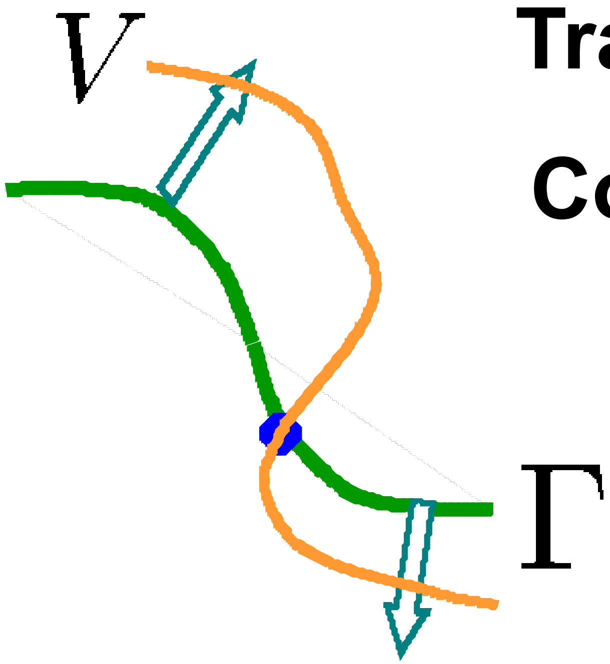
Summary

- Front and back traveling solutions are considered for RD systems with bistable nonlinearity.
- Interfacial dynamics is formally derived. It includes the tip motion explicitly.
- Several statinary rotating spiral solutions are mentioned.
- Other dynamics of spirals such as meandering are still in progress even in formal level.

Problems

$$\begin{cases} \Lambda_T = -(R + \eta\gamma_1\kappa) + \cdots, \\ \eta D_T R = Q(R) + \gamma_2\kappa + \eta^2\gamma_3 R_{\sigma\sigma} + \cdots \end{cases}$$

$\Lambda_T = -V$: Normal velocity



Transition layer of V appears

Consider stretched equation

No reduced equation

Scaling has problems?

RG method is effective

interfacial equation describing spiral waves

$$\begin{cases} V &= \epsilon r + \epsilon^2 \gamma_1 \kappa \\ D_t r &= (\eta M_2 - M_1 r^2) r + \gamma_5 + \epsilon \gamma_2 \kappa + \epsilon^2 \gamma_4 r_{\sigma\sigma} \end{cases}$$

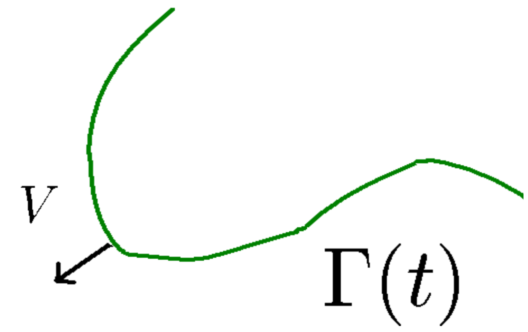
V:outward normal velocity, κ:curvature measured outward

σ:arc length parameter, $r = r(t, \sigma) \in \boldsymbol{R}$

$0 < \gamma_i, M_j, \eta$

$0 < \epsilon \ll 1$

$\Gamma(t) = \{\Gamma(t, \sigma)\}$:curved line



Rem.r is regard as normal velocity of Γ

How to simulate this interface equation ?

Level surface method

$$\Gamma(t) = \{(x,y) | u(t,x,y) = 0\}$$

$$V = \frac{u_t}{|\nabla u|}, \kappa = div(\frac{\nabla u}{|\nabla u|})$$

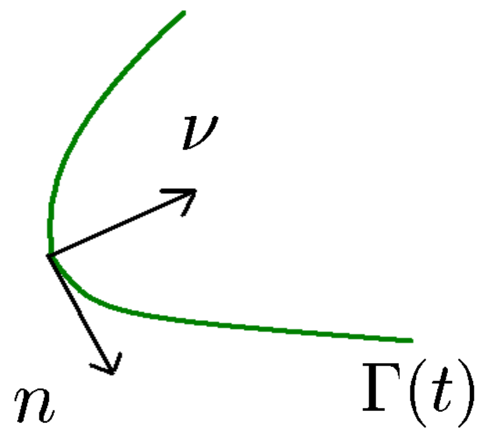
we extend $r = r(t,x,y)$

$$\begin{cases} \frac{u_t}{|\nabla u|} & = \epsilon r + \epsilon^2 \gamma_1 div(\frac{\nabla u}{|\nabla u|}) \\ r_t & = (\eta M_2 - M_1 r^2) r + \gamma_5 + \epsilon \gamma_2 div(\frac{\nabla u}{|\nabla u|}) + \epsilon^2 \gamma_4 \underline{r_{\sigma\sigma}} \end{cases}$$

Difference of $r_{\sigma\sigma}$

ν:unit normal vector

n:unit tangent vector



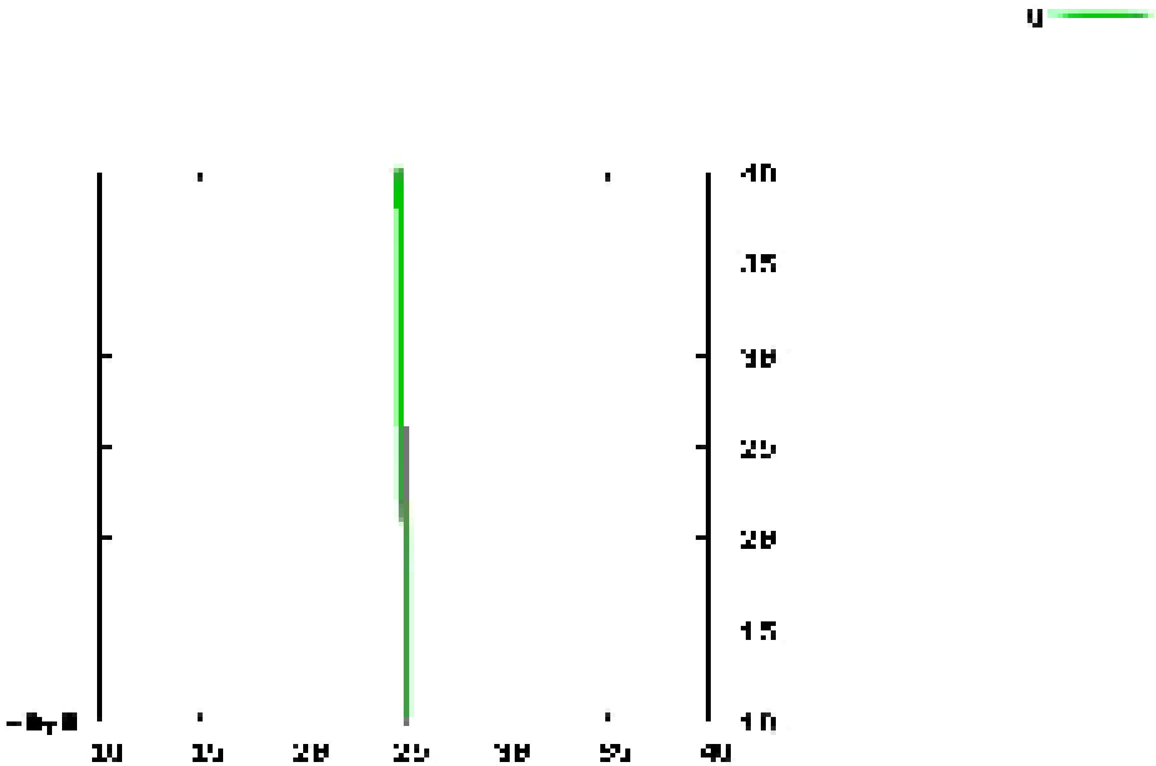
$$\nu = \frac{\nabla u}{|\nabla u|}$$

$$n = \frac{1}{|\nabla u|}(-u_y, u_x)$$

$$r_\sigma = <\nabla r, n>$$

$$r_{\sigma\sigma} = <\nabla <\nabla r, n>, n>$$

$< \, , \, >$:inner product

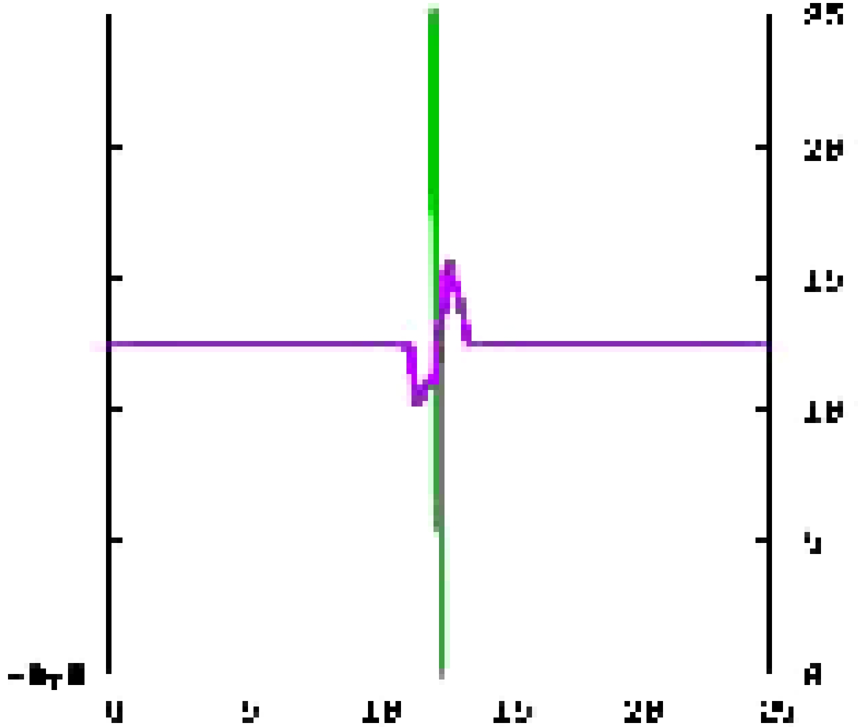


0 level line of u

$$\epsilon = 0.35, \eta = 1, M_2 = 1, M_1 = 1, \gamma_1 = 1, \gamma_2 = 1, \gamma_4 = 1$$

$\gamma_5 = 0$ ⇨ **Symmetry case**

Simulation of Interfacial equation



0 level line of u

0 level line of r

rotate with respect to one point

Summray

- We consider interfacial equation describing spiral waves
- We simulate interfacial equation by using level surface method
- Spiral motions are reappeared by numerical simulations of reduced interfacial equations

Thank you for your attention