時空の問題としてみる「渦」

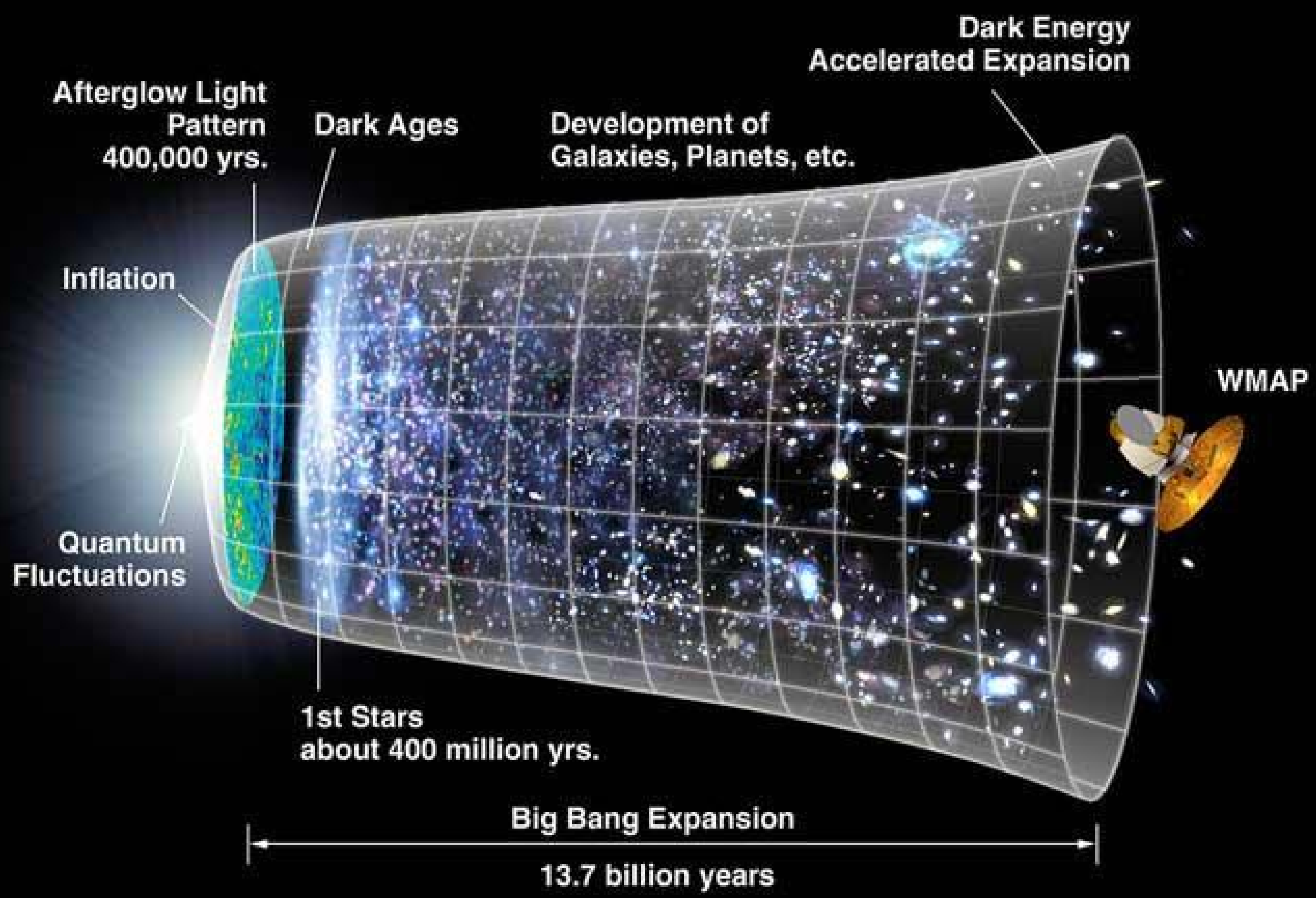
吉田善章 （東京大学・新領域）
Z. Yoshida (The University of Tokyo)

はじめに

- **渦を「時空の問題」として考える.**
- **渦＝磁場は仕事をしない（エネルギーで表現できない）力.**

- P. Cartier, *A mad day's work: from Grothendieck to Connes and Kontsevich, the evolution of concepts of space and symmetry*, Bull. Amer. Math. Soc. **38** (2001), 389.
- S.M. Mahajan & ZY, *Twisting space-time: Relativistic origin of seed magnetic field and vorticity*, PRL **105** (2010), 095005.

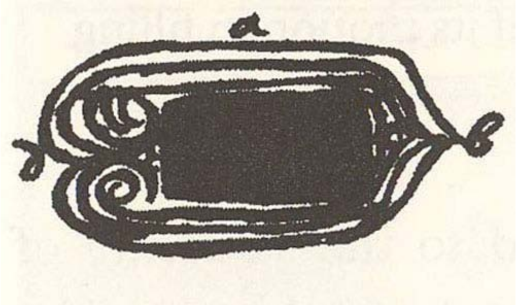
VORTEX: common “structure” in the Universe



Cited from http://www.astronomynotes.com/cosmolgy/s12.htm

〈渦〉とコスモロジー(古典)

- **デカルト**: 「宇宙に充満する物質が渦巻運動の中で変容した結果、発光体である恒星群、光を伝える媒体となる微細物質、光を遮断する不透明体である遊星が分かれて生成し、また渦動が惑星と衛星を運ぶ」
- **ニュートン**: デカルトコスモロジーの批判
- **オイラー**: 「エーテル中の幾つもの渦動」



Violent motion (Newton's drawing).

L. Euler『惑星と彗星の運動理論』(1744).

〈渦〉の物理･数理　（近代～現代）

- **「渦度」による定式化**

  “curl”　→　一般化：　微分形式とコホモロジー

- **「一般渦度」による一般化**

  $$\mathrm{curl}\,\boldsymbol{P} = m\boldsymbol{\omega} + q\boldsymbol{B} + (\mathrm{d}S_1 \wedge \mathrm{d}S_2)/S_3 + \cdots$$

- **「ヘリシティー」による数量化**

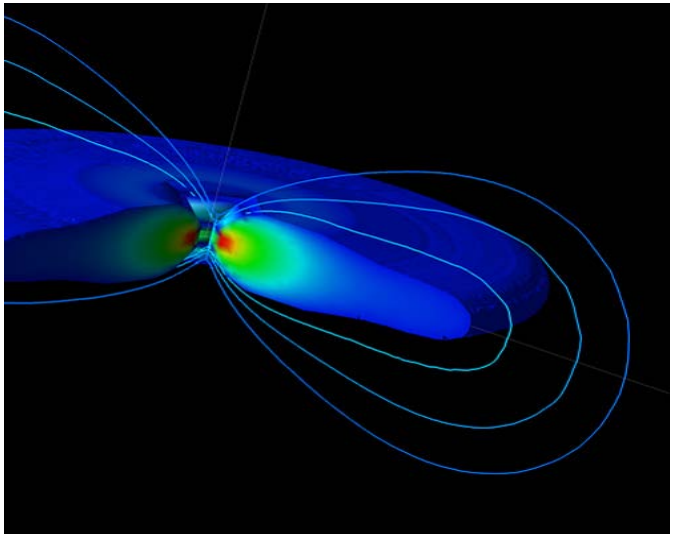
  →トポロジー制約を与えるラグランジアン

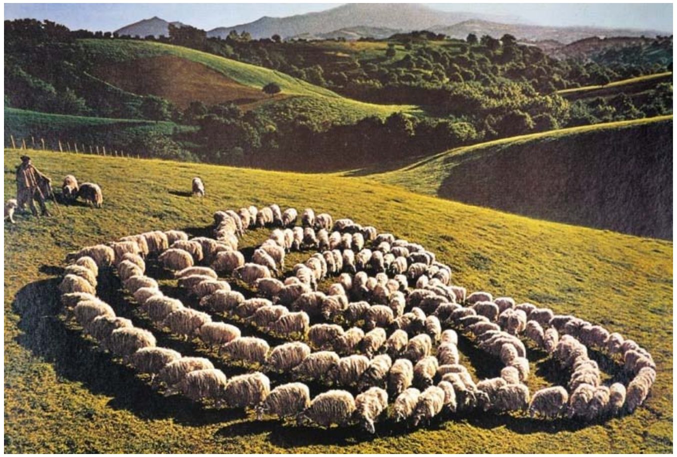
- **「カシミール元」による幾何学化**

  → 葉層構造

時空の歪みと渦　生成するもの・生成されるもの

Photo by K. Okano





kurasse.jp/member/little-kinoko778/note/93577

Plasma physics as a problem of *space-time*

- Plasma = $L_{\text{matter}} + L_{\text{field}}$

- 〈渦〉を表現することの難しさ

  Hamilton-Jacobi eq: $\boldsymbol{P} = \nabla S$

  ♦ R. Jackiw, *Lectures on fluid dynamics--a particle theorist's view of super-symmetric, non-Abelian, noncommutative fluid mechanics and d-branes* (Springer, 2002).

  ♦ Z. Yoshida & S.M. Mahajan, Plasma Phys. Contr. Fusion **54** (2012) 014003.

- 〈渦〉＝ non-canonicality of symplectic geometry

Space-time: *a priori* of theory ?

- *Kopernikanische Wendung*

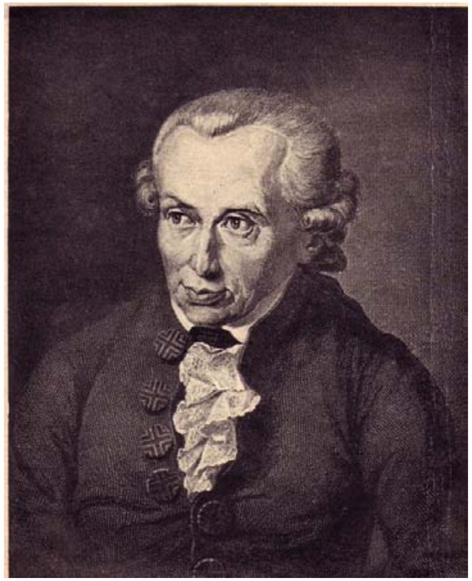
  a priori of recognition
- space-time & matter

  geometry & energy

$$\frac{d}{dt}F=\left[H,F\right]$$

$[\ ,\ ]$: geometry, H: energy

Mach: Space-time ←→ matter

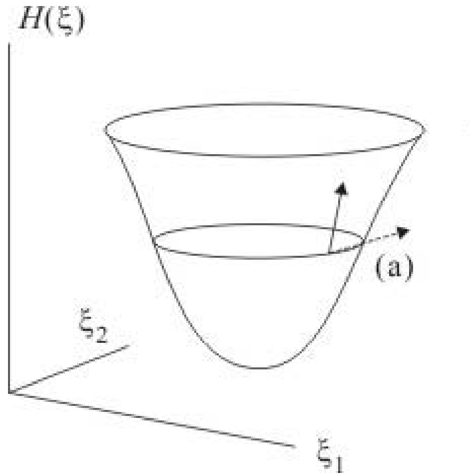


Immanuel Kant
(1724-1804)

General syntax of Hamiltonian systems

- Hamiltonian mechanics is dictated by
  J (symplectic operator) and H (Hamiltonian)

$$\boxed{\frac{d}{dt}u = J\partial_u H(u)}$$



Poisson bracket : $[G,F] = \langle \partial_u G(u), J\partial_u F(u) \rangle$

$$\frac{d}{dt}F(u) = [H,F]$$

$$[[F,G],H]+[[G,H],F]+[[H,F],G]=0$$

Examples of Hamiltonian systems

classical mechanics:

$$\frac{d}{dt}\boldsymbol{z} = \frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix} = J\partial_{\boldsymbol{z}} H$$

quantum mechanics:

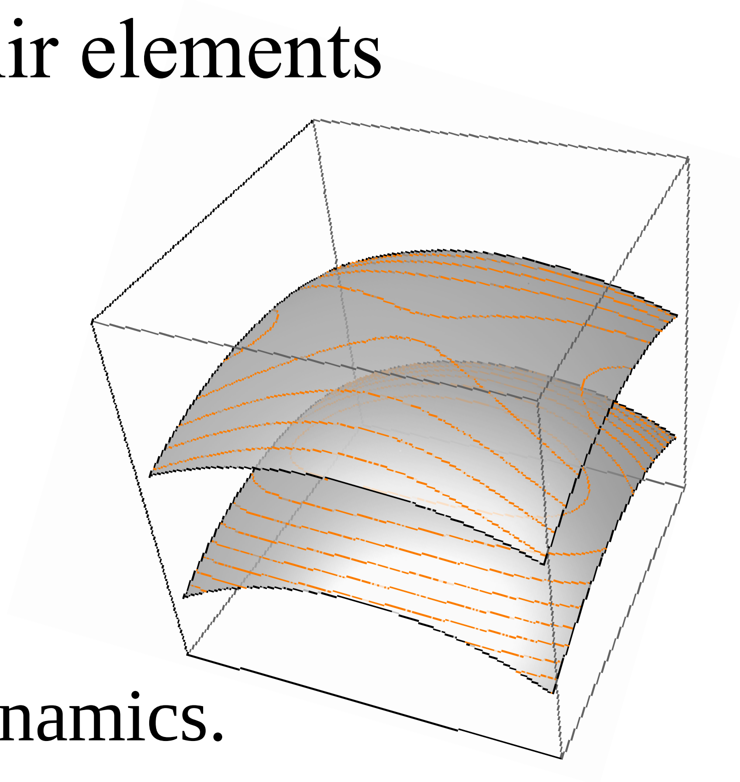
$$\partial_t \psi = -i\partial_\psi \left\langle \mathscr{H}\psi, \psi \right\rangle / 2 = J\partial_\psi H$$

These examples are “canonical” because *J* are regular operators.

non-canonical Hamiltonian mechanics

- Ker (J) = Coker$(J) \rightarrow$ "topological constraint" *
- Foliation of Ker $(J) \rightarrow$ Casimir elements

$$\exists C \ s.t. \left[G,C\right]=0 \ (\forall G)$$
$$i.e. \quad \partial_u C \in \mathrm{Ker}(J)$$



- $H \rightarrow H+C$ does not change the dynamics.

* Ker$(J\partial_z)$=Hom$_D$(Coker$(J\partial_z)$, F)

The Euler eq. and vorticity eq.

- Euler equation of ideal fluid:

$$\partial_t u + (u \cdot \nabla) u = -\nabla p, \quad \nabla \cdot u = 0,$$
$$n \cdot u = 0$$

- Vorticity formulation:

$$\partial_t \omega = \nabla \times (u \times \omega) \quad (\omega := \nabla \times u)$$
$$= \{\omega, \phi\} \qquad (\mathrm{2D} : u = \mathrm{curl}\,\phi, \omega = -\Delta \phi)$$

- *Vorticity eq. in $H^{-1}(\Omega)$ is equivalent to Euler eq. in $L^2{}_\sigma(\Omega)$.*

Hamiltonian formalism of the Euler eq.

- Hamiltonian formalism

$$\partial_t \omega = J(\omega)\partial_\omega H(\omega) \quad [\text{in } H^{-1}(\Omega)]$$

- Hamiltonian

$$H(\omega) = \frac{1}{2}\int_\Omega (K\omega)\cdot\omega\, dx \quad [K := -\Delta^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)]$$

- Poisson operator

$$J(\omega) = \{\omega, \cdot\} : \; H_0^1(\Omega) \to H^{-1}(\Omega)$$

$$[\psi, \varphi] = \langle J(\omega)\psi, \varphi\rangle := \boxed{\langle \omega, \{\psi, \varphi\}\rangle} \quad [\omega \in C(\Omega)]$$

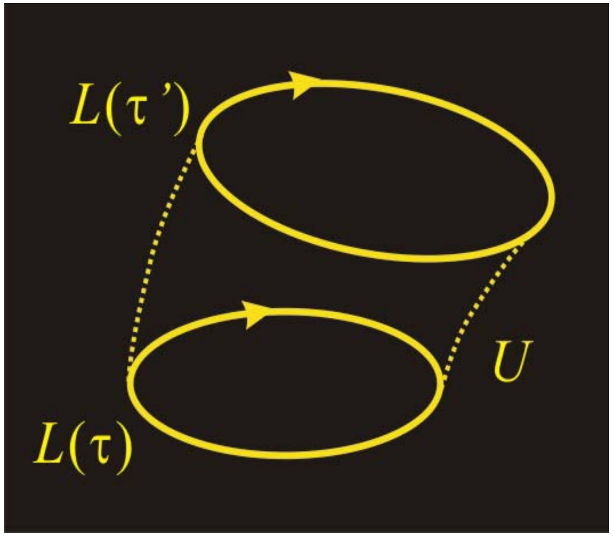
- *Known to be classically solvable for a Hölder continuous initial value: T. Kato (1967)*

中間的まとめ (I)

- 渦（ω）はメトリックである.
- 渦が規定するsymplectic幾何は非正準である.

  Casimir element = $F(\omega)$
- 渦は渦の中で運動する. あるいは, 渦が運動する空間は渦によって構造が与えられている.
- MHDなども同様の構造をもつ.

Circulation theorem



$$\frac{d}{dt}\left(\oint_{L(t)} \boldsymbol{P}\cdot d\boldsymbol{x}\right) = \oint_{L(t)} \left[\partial_t \boldsymbol{P} + (\nabla\times\boldsymbol{P})\times\boldsymbol{V}\right]\cdot d\boldsymbol{x} = 0$$

if canonical momentum $\boldsymbol{P} = m\boldsymbol{V} + (q/c)\boldsymbol{A}$ obeys

$$\partial_t \boldsymbol{P} + (\nabla\times\boldsymbol{P})\times\boldsymbol{V} = -\nabla\varphi \quad (\varphi = H + h).$$

Relativistic circulation theorem

$$\frac{d}{ds}\left(\oint_{L(s)} P^{\mu}\cdot dx_{\mu}\right)=\oint_{L(s)}\left(\partial^{\mu}P^{\nu}-\partial^{\nu}P^{\mu}\right)U_{\nu}dx_{\mu}$$

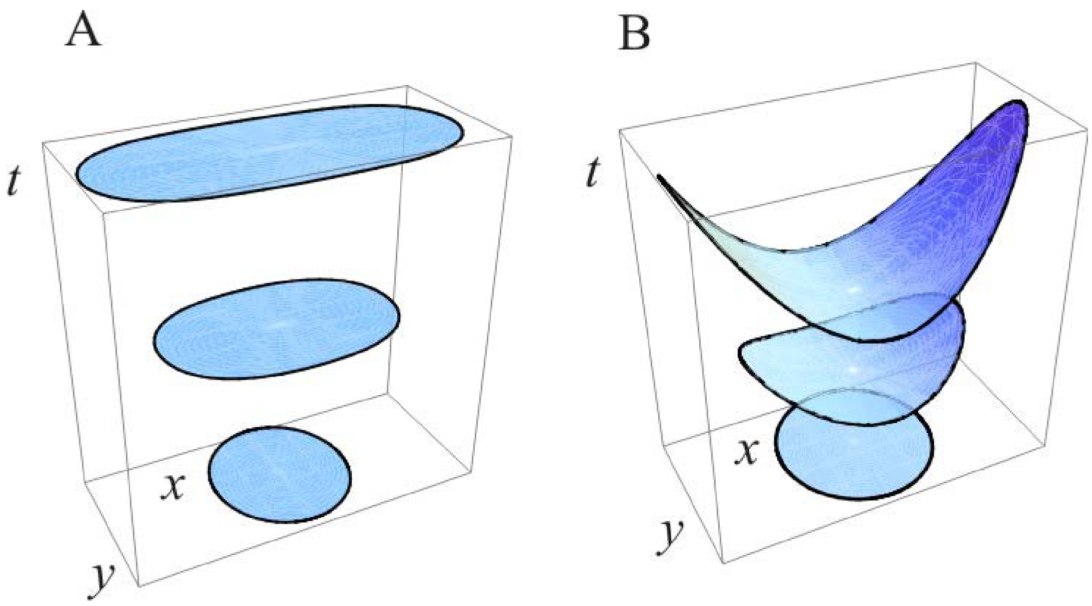
s : proper time, $U^{\mu}=(\gamma,\gamma V^{j}/c)$

relativistic equation of motion

$$\left(\partial^{\mu}P^{\nu}-\partial^{\nu}P^{\mu}\right)U_{\nu}=T\partial^{\mu}S$$

Relativistic circulation theorem applies on space-time:
relativistic distortion of space-time
→non-exact thermodynamic force on the reference frame
→Cosmological origin of vortex/magnetic field

Relativistic distortion of space-time



$$\boxed{\frac{d}{dt}x_{\mu} = V_{\mu}}$$

$$\boxed{\frac{d}{ds}x_{\mu} = U_{\mu}}$$

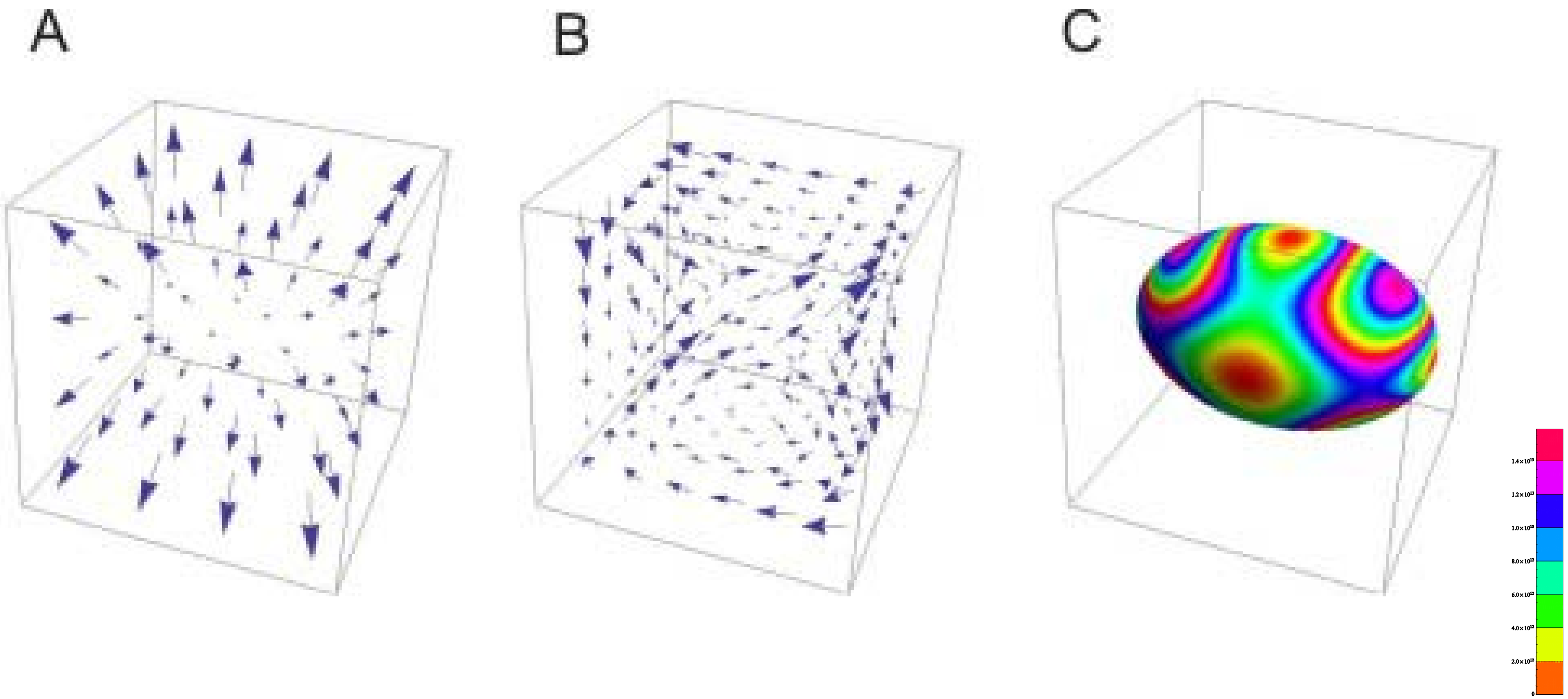
Relativity destroys the “synchronity” of a cycle.

→ “twists” space-time

→ generate circulation

$$\boxed{\oint_{L(t)} \gamma^{-1} T dS}$$

scalar → vector → axial vector (vortex)



膨張 → 回転運動 → 渦＝磁場

中間的まとめ (II)

- Unification: vorticity + magnetic field
- Distortion → Creation
- Non-exactness (entropy) → Vorticity

Distortion by *scale hierarchy*

- 構造形成とエントロピーの関係　?
- 等分配の “measure” ?
- 「マクロ階層」の相空間 ?

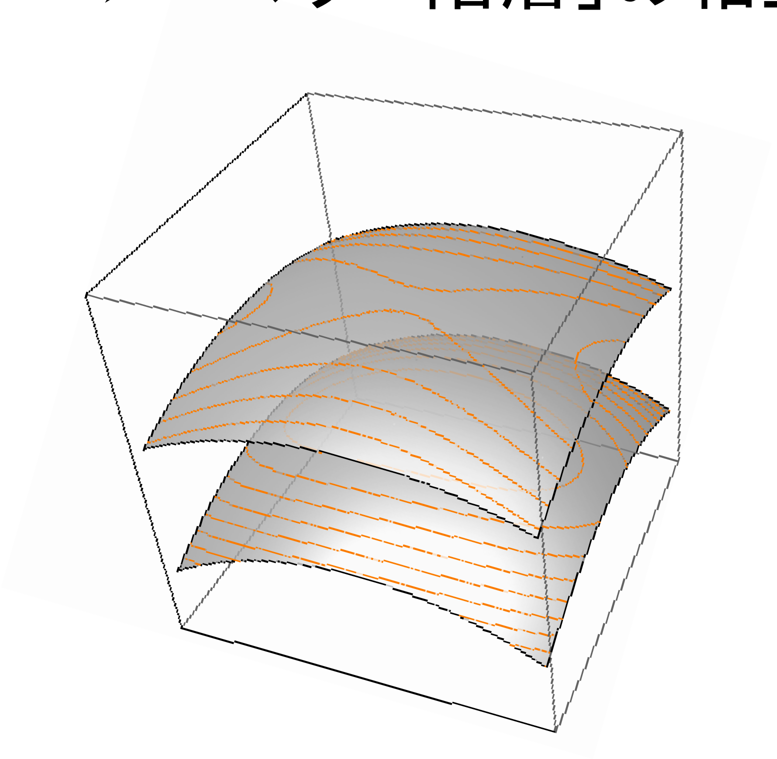


Image of constrained (foliated) phase space

General Hamiltonian system

- Hamiltonian mechanics is dictated by
  J (symplectic operator) and H (Hamiltonian)

$$\frac{d}{dt}u = J\partial_u H(u)$$

$$\text{Poisson bracket}: \left[G, F\right] = \left\langle J\partial_u G(u), \partial_u F(u) \right\rangle$$

$$\frac{d}{dt}F(u) = \left[H, F\right]$$

$$\text{canonical form}: \ J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Non-canonical Hamiltonian mechanics

- Non-canonicality : Ker(J)

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & -I & 0 \end{pmatrix}$$

Effective phase space

- Ker(J) = Coker(J) → "topological defect"

Scale hierarchy of magnetized particles

$$H = \frac{m}{2}\left(V_c^2 + V_\parallel^2 + V_\perp^2\right) + q\phi$$

$$\begin{aligned} H_c &= \mu\omega_c + \frac{m}{2}\left(V_\parallel^2 + V_\perp^2\right) + q\phi \\ &= \mu\omega_c + \frac{\left(P_\theta - q\psi\right)^2}{2mr^2} + \frac{p_\parallel^2}{2m} + q\phi \end{aligned}$$

$$\boldsymbol{z} = \left(\vartheta_c, \mu; \zeta, p_\parallel; \theta, P_\theta\right)$$

“Quantization”

$$\mathbf{z} = (\vartheta_c, p_c; \zeta, p_\parallel; \theta, P_\theta) \rightarrow (\cancel{\vartheta_c}, \mu; \zeta, p_\parallel; \theta, P_\theta)$$

Coarse-graining → non-canonicalizetion

$$J = \begin{pmatrix} J_c & 0 & 0 \\ 0 & J_c & 0 \\ 0 & 0 & J_c \end{pmatrix} \rightarrow J_{nc} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & J_c & 0 \\ 0 & 0 & J_c \end{pmatrix}$$

Boltzmann distribution on Casimir leaf

$$\boxed{\delta\left(S-\alpha N-\beta E-\gamma M\right)=0}$$

$$S=-\int f\log f\,d^6z$$

$$N=\int f\,d^6z$$

$$E=\int H_c\,f\,d^6z$$

$$M=\int \mu\,f\,d^6z$$

$$\boxed{f=c\exp\left(-\beta H_c-\gamma\mu\right)}$$

Chemical potential

Quasi-particle number

Embedding into the lab-frame space

$$\rho(x) = \int f\, d^3v = \int f\,(2\pi\omega_c / m)\, d\mu\, dv_{\|}\, dv_\theta$$

$$= c\int \exp\left(-\beta H_c - \gamma\mu\right)\frac{2\pi\omega_c d\mu}{m} dv_{\|}\, dv_\theta$$

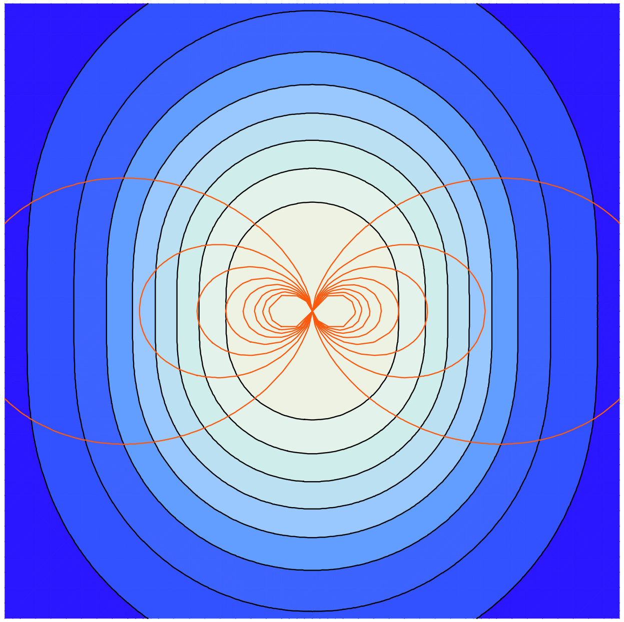
$$= \frac{\omega_c(x)}{\omega_c(x)+\gamma}$$

Density clump in lab-frame space

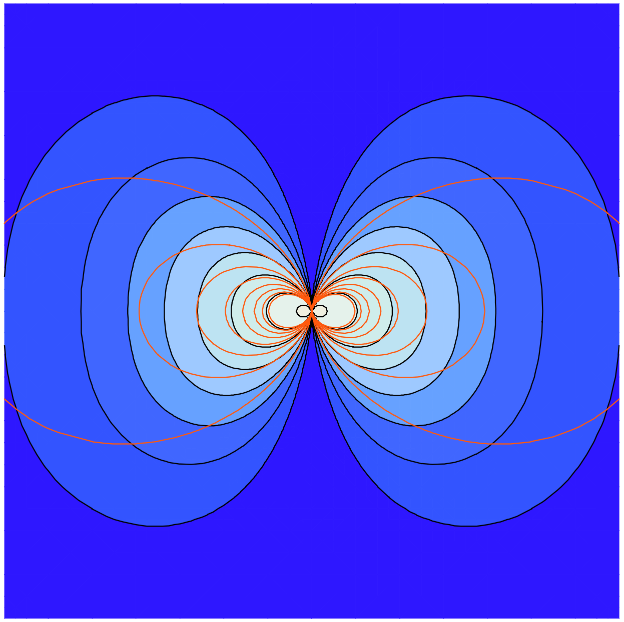
Adiabatic invariant μ

→ *quantization* of quasi-particles

→ thermodynamic distribution on a Casimir leaf



(A)



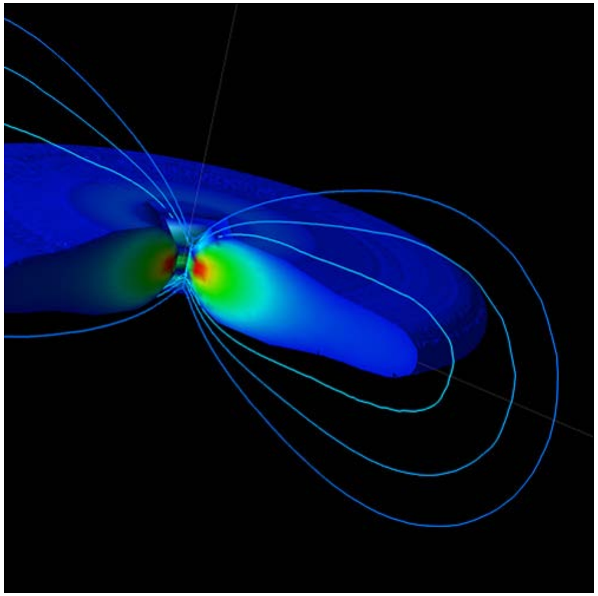
(B)

中間的まとめ (III)

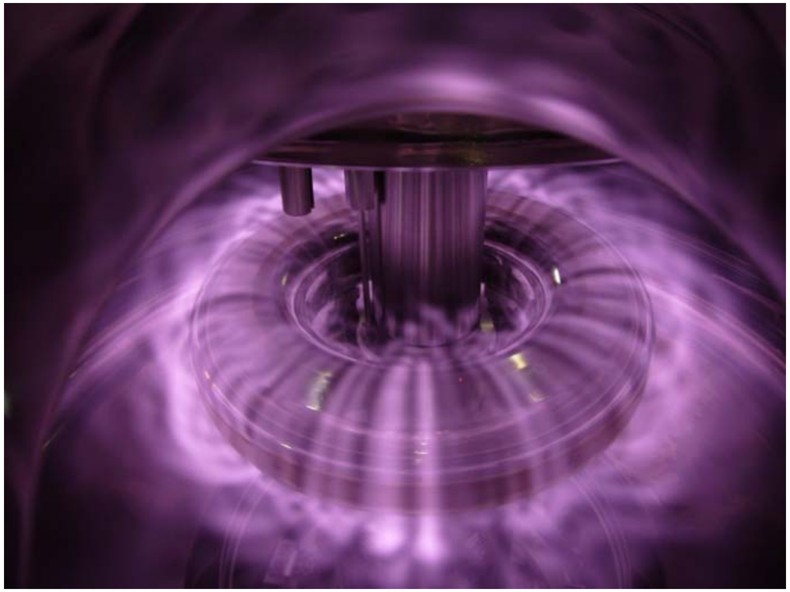
- スケール階層 = 相空間の葉層構造
- 断熱不変量 = *Casimir element* → 葉層
- 葉層上の歪んだメトリック → 自己組織化

A magnetosphere on the Earth

RT-1 project

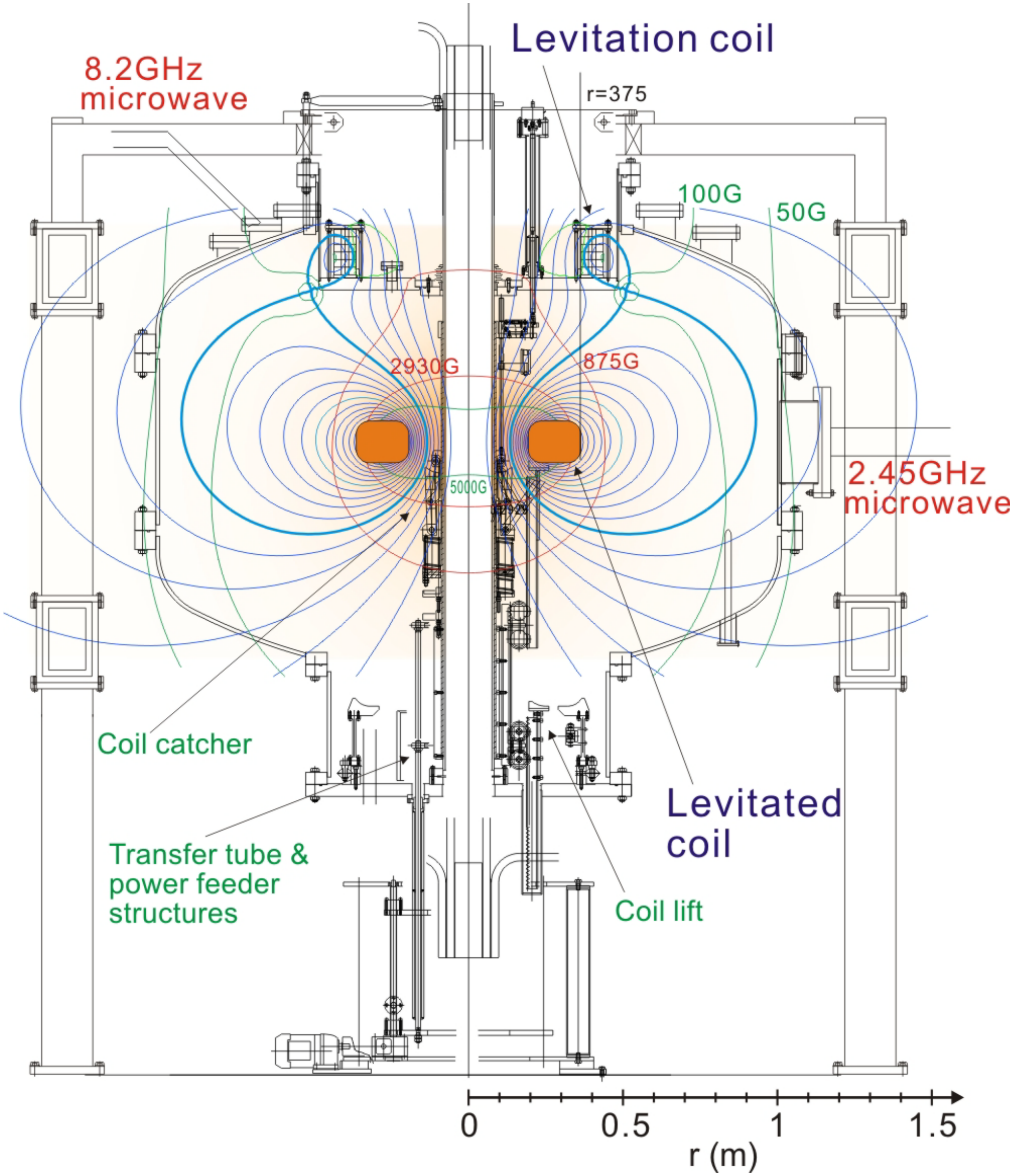


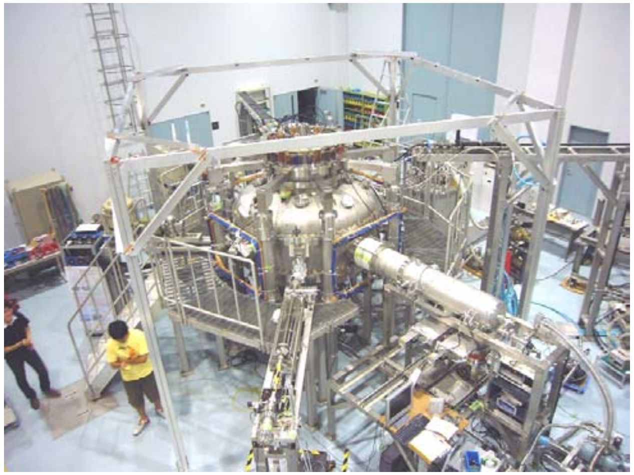
Jovian magnetosphere



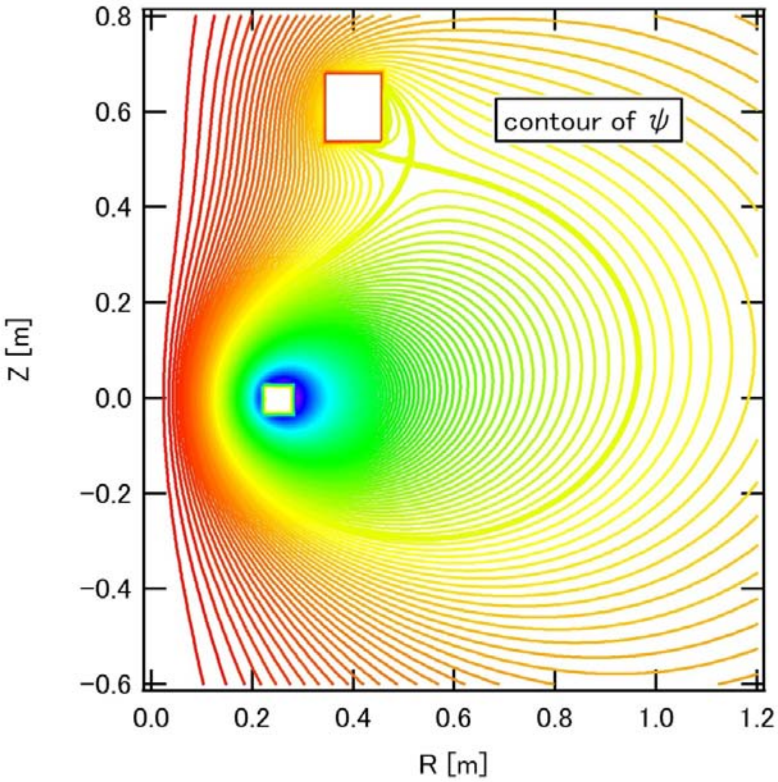
RT-1 magnetospheric plasma

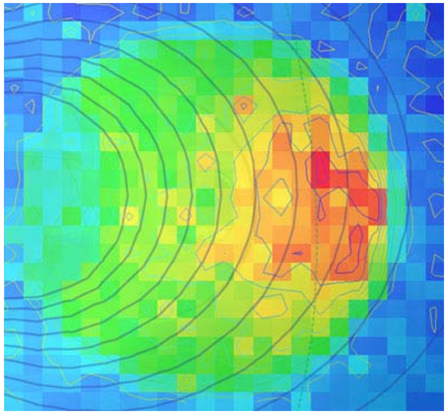
Levitating HTC superconducting magnet system





High-beta plasma confinement

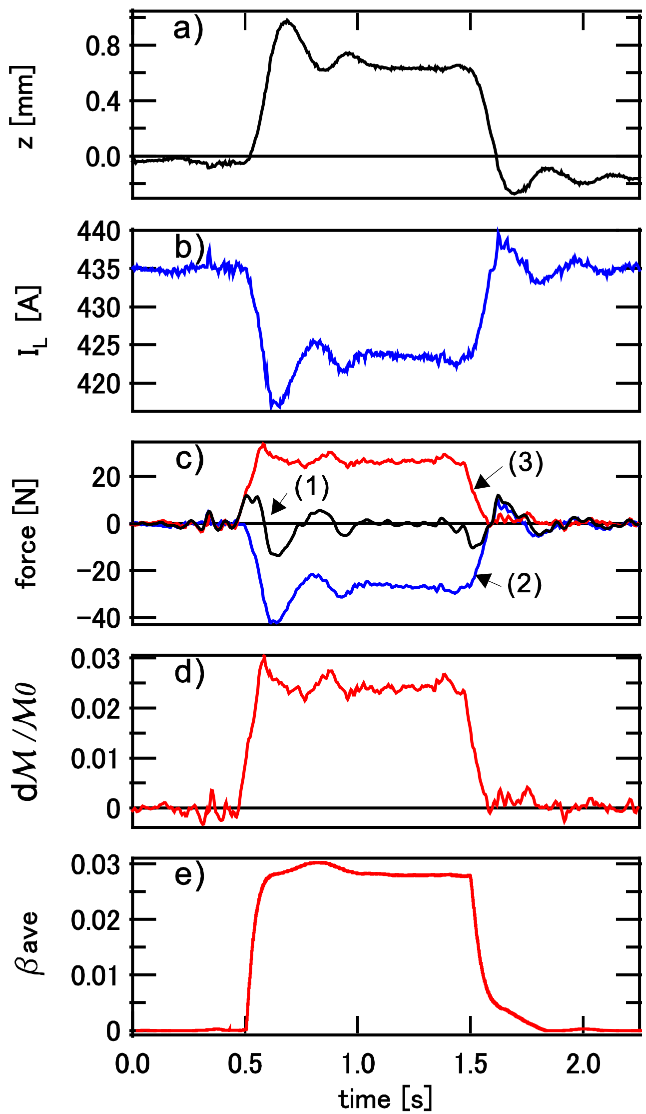


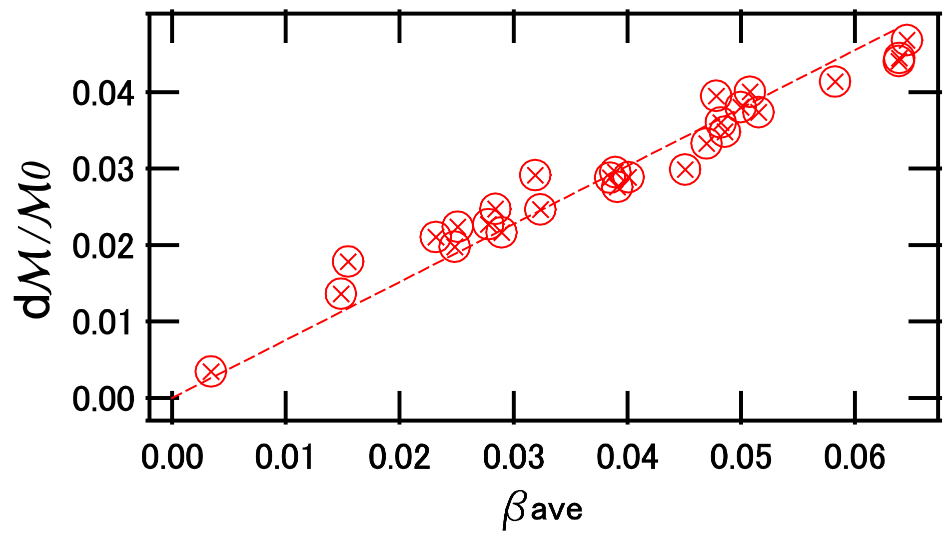


$$T_e \approx 10 \sim 30\text{keV}, \quad n_e = 10^{16} - 10^{17}\ \text{m}^{-3}$$

$$\beta \approx \beta_e \approx 0.7, \quad \tau_E \approx 0.5\,\text{sec}$$

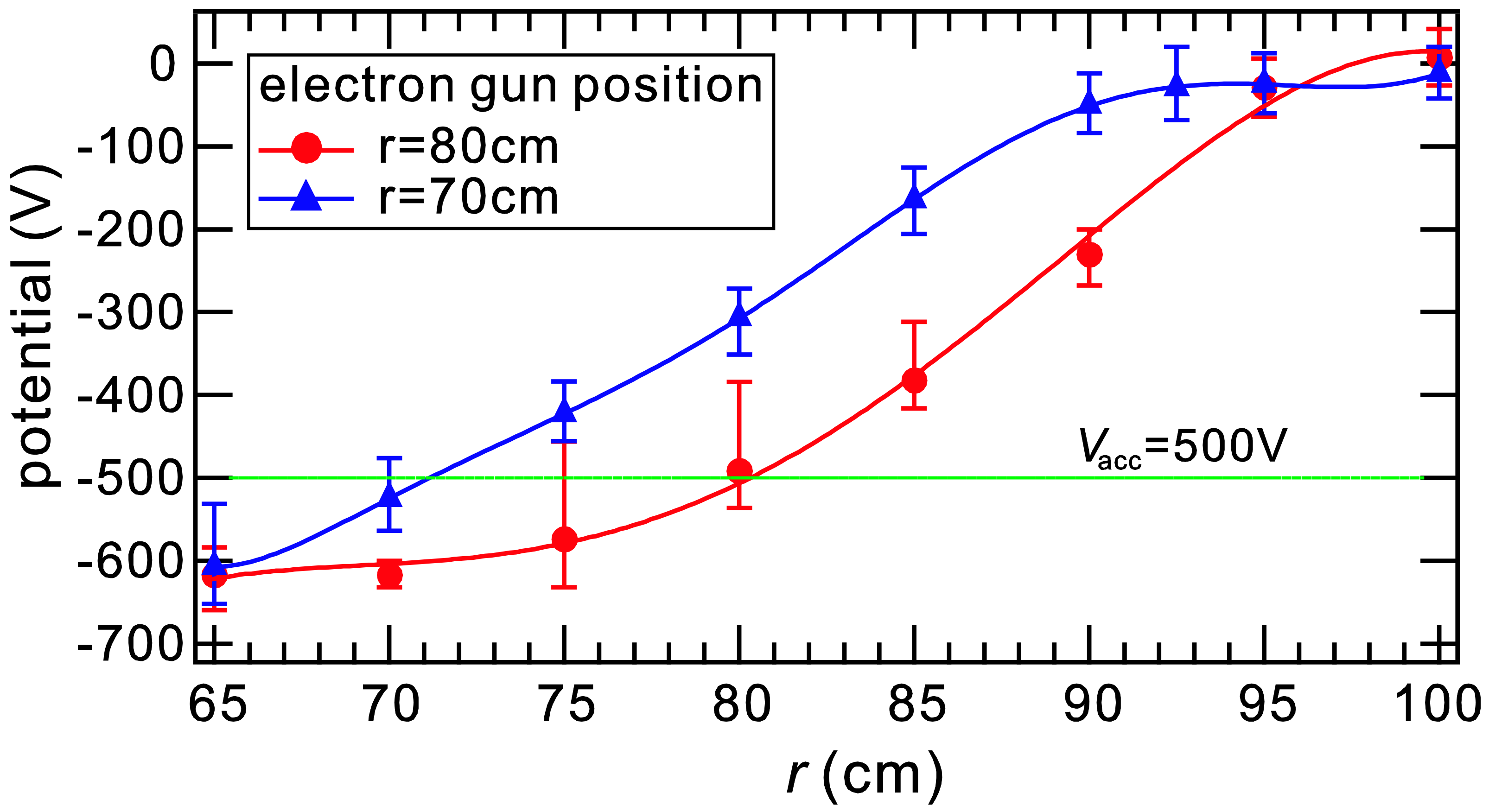
Creation of magnetic moment by heating





$$\delta Q_{\mathrm{ECH}} = \overline{B}\,\mathrm{d}M = \overline{B}\sum_j \mathrm{d}\mu_j$$

Inward (up-hill) diffusion



実験のまとめ

- *self-organized confinement* の実証
- 加熱 ＝ 磁気モーメント入射 $\times B$

  $\boxed{\text{渦場のメトリック}}$
- 内向き拡散　→ メトリックの歪みの実証

Summary

- VORTEX = distorted space-time
- effective phase space → foliation (Casimir leaf)

Newton-Kant "space" → relativity, foliation
and beyond