# Numerical computations of split Bregman method for fourth order total variation flow

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# Overview

- We consider the 4-th order total variation (TV) flow, Spohn's model

$$u_t = -\Delta\left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right), \qquad u_t = -\Delta\left(\operatorname{div}\left(\beta\frac{\nabla u}{|\nabla u|} + |\nabla u|^{p-2}\nabla u\right)\right),$$

  or the OSV model for denoising in image processing:

$$\text{(OSV)} \quad u = \operatorname*{argmin}_{u\in H^{-1}(\Omega)} \left\{ \int_\Omega |Du| + \frac{\lambda}{2}\|u-f\|^2_{H^{-1}(\Omega)} \right\}.$$

- For 4-th order TV flow, a class of initial data has been studied analytically in earlier study.
  - It is proved that the solution becomes discontinuous instantaneously.
- The split Bregman method is an efficient solver for ROF model in image processing and second order TV flow.
- We provide a new numerical scheme, which is based on the split Bregman method, for fourth order problems under periodic boundary condition.

# Outline

1. Schemes
   - Preliminary
   - Discretization

2. Numerical examples
   - 1 dimensional case
   - 2 dimensional case

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# Preliminary 1. $H^{-1}_{\mathrm{av}}(\mathbb{T})$ and inverse Laplacian

- Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\mathcal{D}(\mathbb{T})$ be $C^\infty(\mathbb{T})$ endowed with suitable topology.
- (generalized) Fourier transform;

$$\widehat{f_T}(\xi) = \langle f, e^{-2\pi i \xi x} \rangle_{\mathcal{D}'(\mathbb{T}),\mathcal{D}(\mathbb{T})}$$

## Definition (Spaces of functions whose average are equal to 0)

$$\begin{aligned}
L^2_{\mathrm{av}}(\mathbb{T}) &= \left\{ f \in \mathcal{D}'(\mathbb{T}) : \textstyle\sum_{\xi \in \mathbb{Z}\setminus\{0\}} |\widehat{f_T}(\xi)|^2 < \infty \text{ and } \widehat{f_T}(0) = 0 \right\}, \\
H^1_{\mathrm{av}}(\mathbb{T}) &= \left\{ f \in \mathcal{D}'(\mathbb{T}) : \textstyle\sum_{\xi \in \mathbb{Z}\setminus\{0\}} \xi^2 |\widehat{f_T}(\xi)|^2 < \infty \text{ and } \widehat{f_T}(0) = 0 \right\}, \\
H^{-1}_{\mathrm{av}}(\mathbb{T}) &= \left\{ f \in \mathcal{D}'(\mathbb{T}) : \textstyle\sum_{\xi \in \mathbb{Z}\setminus\{0\}} \xi^{-2} |\widehat{f_T}(\xi)|^2 < \infty \text{ and } \widehat{f_T}(0) = 0 \right\}.
\end{aligned}$$

- They are Hilbert spaces, and
  - $\|f\|_{L^2_{\mathrm{av}}(\mathbb{T})} = \|f\|_{L^2(\mathbb{T})}$ for all $f \in L^2_{\mathrm{av}}(\mathbb{T})$,
  - $\|f\|_{H^1_{\mathrm{av}}(\mathbb{T})} = \|\nabla f\|_{L^2(\mathbb{T})}$ for all $f \in H^1_{\mathrm{av}}(\mathbb{T})$,
  - There exists an isometry $(-\Delta_{\mathrm{av}})^{-1} : H^{-1}_{\mathrm{av}}(\mathbb{T}) \to H^1_{\mathrm{av}}(\mathbb{T})$, that is,

$$\|f\|_{H^{-1}_{\mathrm{av}}(\mathbb{T})} = \|(-\Delta_{\mathrm{av}})^{-1} f\|_{H^1_{\mathrm{av}}(\mathbb{T})} = \|\nabla(-\Delta_{\mathrm{av}})^{-1} f\|_{L^2(\mathbb{T})} \text{ for all } f \in H^{-1}_{\mathrm{av}}(\mathbb{T}).$$

# Preliminary 2. Total Variation and fourth order TV flow

## Definition (Total variation and Bounded variation space)

$$\int_{\mathbb{T}} |Df| = \operatorname{esssup} \left\{ \sum_{j=1}^{M} |f(x_j) - f(x_{j-1})| : 0 = x_0 < x_1 < \cdots < x_M = 1 \right\},$$

$$BV(\mathbb{T}) = \left\{ f \in \mathcal{D}'(\mathbb{T}) : \int_{\mathbb{T}} |Df| < \infty \right\},$$

$$E(f) = \begin{cases} \int_{\mathbb{T}} |Df| & \text{if } f \in H^{-1}_{\text{av}}(\mathbb{T}) \cap BV(\mathbb{T}), \\ 0 & \text{otherwise.} \end{cases}$$

where the supremum is taken over all partition of the interval.

- $E : H^{-1}_{\text{av}}(\mathbb{T}) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is proper, l.s.c. and convex functional.
- Fourth order TV flow can be described as

$$u_t = -\Delta \left( \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right) \right) \in -\partial_{H^{-1}_{\text{av}}(\mathbb{T})} E(u),$$

where $\partial_{H^{-1}_{\text{av}}(\mathbb{T})} E(u)$ is a subdifferential.

# Earlier studies

1. Y. Kashima (2012):
   Characterization of subdifferential $H^{-1}_{\mathrm{av}}(\mathbb{T}^d)$
2. M.-H. Giga and Y. Giga (2010):
   Exact profile under periodic B.C.
3. Y. Giga and R.V. Kohn (2011):
   Extinction time estimate under periodic B.C.
4. Y. Giga, M. Muszkieta and P. Rybka (2019):
   A duality based numerical scheme which applies forward-backward splitting
5. R. V. Kohn and H. M. Versieux (2010):
   Numerical computation for Spohn's model, based on mixed FEM and regularization for singularity

# Outline

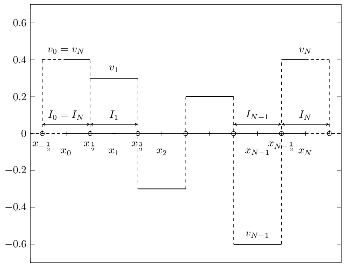
# Spaces of piecewise constant functions

Let $N \in \mathbb{N}$, $h = 1/N$, $x_n = nh$, $x_{n+1/2} = (n+1/2)h$,

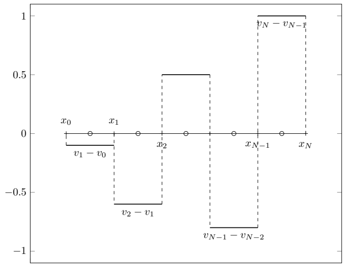
$$I_n = [x_{n-1/2}, x_{n+1/2}), \qquad I_{n+1/2} = [x_n, x_{n+1}).$$

## Definition (Spaces of piecewise constant functions)

$$\begin{aligned} V_h &= \{v_h : \mathbb{T} \to \mathbb{R} : v_h|_{I_n} \in \mathbb{P}_0(I_n) \text{ for all } n = 0, \dots, N\} \\ V_{h0} &= \left\{v_h = \textstyle\sum_{n=1}^N v_n \mathbf{1}_{I_n} \in V_h : \sum_{n=1}^N v_n = 0\right\}, \\ \widehat{V}_h &= \{d_h : [0,1) \to \mathbb{R} : d_h|_{I_{n+1/2}} \in \mathbb{P}_0(I_{n+1/2}) \text{ for all } n = 0, \dots, N\}. \end{aligned}$$




(a) $V_{h0}$




(b) $\widehat{V}_h$

# Discretization

## Time discretization

The backward Euler method for $u_t \in -\partial_{H^{-1}_{\mathrm{av}}(\mathbb{T})} E(u)$ gives:
For given $u^k \in H^{-1}_{\mathrm{av}}(\mathbb{T})$, find $u^{k+1} \in H^{-1}_{\mathrm{av}}(\mathbb{T})$ s.t.

$$u^{k+1} = \operatorname*{argmin}_{u \in H^{-1}_{\mathrm{av}}(\mathbb{T})} \left\{ \int_{\mathbb{T}} |d| + \frac{\tau^{-1}}{2} \|u - u^k\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})} : d = Du \right\}.$$

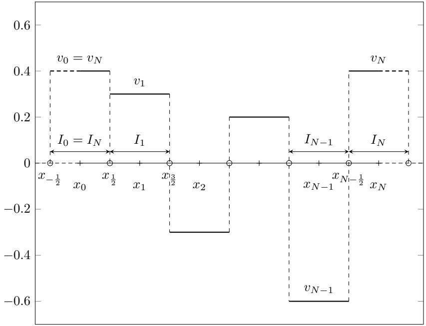
## Spatial discretization

For given $u_h^k \in V_{h0}$, find $u_h^{k+1} \in V_{h0}$ and $d_h^{k+1} \in \widehat{V}_h$ s.t.

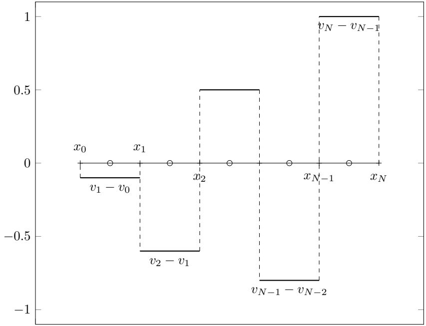
$$(u_h^{k+1}, d_h^{k+1}) = \operatorname*{argmin}_{u_h, d_h} \left\{ \int_{\mathbb{T}} |d_h| + \frac{\tau^{-1}}{2} \|u_h - u_h^k\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})} + \frac{\mu}{2} \|d_h - D_h u_h^k\|^2_{L^2(I)} \right\},$$

where $I = (0,1)$, $\mu > 0$, $D_h : \sum_{n=1}^N v_n \mathbf{1}_{I_n} \mapsto \sum_{n=1}^N (v_n - v_{n-1}) \mathbf{1}_{I_{n+1/2}}$.

# Discretized differential operator




(c) $v_h \in V_{h0}$




(d) $D_h v_h \in \widehat{V}_h$

- Note that $D_h v_h \notin L^2(\mathbb{T})$.

# Matrix form

We propose two schemes for $\|u_h - u_h^k\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})}$.

## Scheme 1: Using discrete gradient and discrete Laplacian

- $\nabla_{\mathrm{av},h} \in \mathbb{R}^{N\times(N-1)}$ : discrete gradient on $\mathbb{T}$ with average zero condition.
- $-\Delta_{\mathrm{av},h} \in \mathbb{R}^{(N-1)\times(N-1)}$ : discrete Laplacian.
- $\|u_h - u_h^k\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})} = \|\nabla(-\Delta_{\mathrm{av}})^{-1}(u_h - u_h^k)\|^2_{L^2(\mathbb{T})} \approx h\|\nabla_{\mathrm{av},h}(-\Delta_{\mathrm{av},h})^{-1}(\boldsymbol{u} - \boldsymbol{u}^k)\|_2^2$.

## Scheme 2. Using second degree B-spline

- Let $B_n(x)$ be the second degree periodic B-spline basis functions.
- $\forall u_h, u_h^k \in V_{h0}, \exists w_h = \sum_{n=1}^N w_n B_n \in H^1_{\mathrm{av}}(\mathbb{T})$ s.t. $w_h = (-\Delta_{\mathrm{av}})^{-1}(u_h - u_h^k)$.
- $\|u_h - u_h^k\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})} = \|\nabla w_h\|^2_{L^2(\mathbb{T})} = h\|\sqrt{hM}\nabla_{\mathrm{av},h}(-\Delta_{\mathrm{av},h})^{-1}(\boldsymbol{u} - \boldsymbol{u}^k)\|_2^2$, where $M \in \mathbb{R}^{N\times N}$ is the mass matrix for tent functions and $\sqrt{hM} \in \mathbb{R}^{N\times N}$ is a matrix such that $\sqrt{hM}^{\mathrm{T}}\sqrt{hM} = hM$.

# Split Bregman framework for fourth order TV flow

## Scheme

For given $\boldsymbol{u}^k \in \mathbb{R}^{N-1}$, find $\boldsymbol{u}^{k+1} \in \mathbb{R}^{N-1}$ and $\boldsymbol{d}^{k+1} \in \mathbb{R}^N$ such that

$$(\boldsymbol{u}^{k+1}, \boldsymbol{d}^{k+1}) = \operatorname*{argmin}_{\boldsymbol{u},\boldsymbol{d}} \left\{ \|\boldsymbol{d}\|_1 + \frac{\tau^{-1}h}{2} \|K(\boldsymbol{u} - \boldsymbol{u}^k)\|_2^2 + \frac{\mu h}{2} \|\boldsymbol{d} - h\nabla_{\mathrm{av},h}\boldsymbol{u}^k\|_2^2 \right\},$$

where $K = \nabla_{\mathrm{av},h}(-\Delta_{\mathrm{av},h})^{-1}$ or $\sqrt{hM}\nabla_{\mathrm{av},h}(-\Delta_{\mathrm{av},h})^{-1}$, $\tau = O(h^3)$, $\mu = O(h^{-1})$.

This is very similar to the matrix form of ROF model, therefore we apply the split Bregman framework.

## Split Bregman framework for fourth order TV flow

$$\begin{aligned}
\boldsymbol{u}^{k,j+1} &= \left(\tau^{-1}hK^{\mathrm{T}}K + \mu h^3 \nabla_{\mathrm{av},h}^{\mathrm{T}} \nabla_{\mathrm{av},h}\right)^{-1} \left(\tau^{-1}hK^{\mathrm{T}}K\boldsymbol{u}^k + \mu h^2 \nabla_{\mathrm{av},h}^{\mathrm{T}} (\boldsymbol{d}^{k,j} - \boldsymbol{\alpha}^{k,j})\right), \\
d_n^{k,j+1} &= \operatorname{shrink}\left(\left(h\nabla_{\mathrm{av},h}\boldsymbol{u}^{k,j+1} + \boldsymbol{\alpha}^{k,j+1}\right)_n, 1/(\mu h)\right) \text{ for all } n = 1, \dots, N, \\
\boldsymbol{\alpha}^{k,j+1} &= \boldsymbol{\alpha}^{k.j} - \boldsymbol{d}^{k,j+1} + h\nabla_{\mathrm{av},h}\boldsymbol{u}^{k,j+1},
\end{aligned}$$

where $\operatorname{shrink}(\rho, a) = \dfrac{\rho}{|\rho|} \max\{|\rho| - a, 0\}$.

This gives $\boldsymbol{u}^{k+1} = \lim_{j\to\infty} \boldsymbol{u}^{k,j}$.

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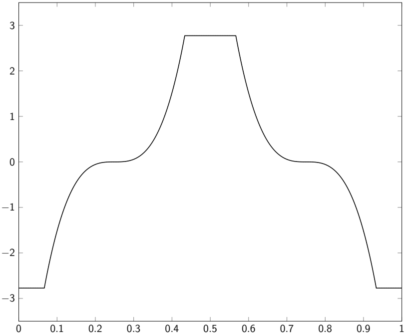
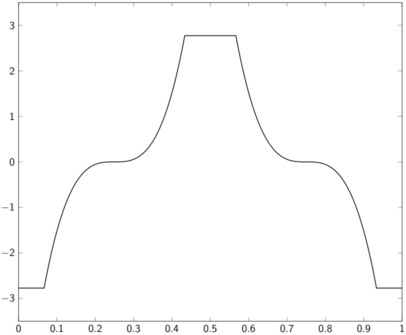
# Fourth order TV flow




(e) First scheme




(f) Second scheme

- It is proved that the exact solution becomes discontinuous instantaneously.
- The symmetry of initial profile is preserved during the evolution.

# Spohn's fourth order model

## Spohn's fourth order model

$$u_t = -\Delta\left(\operatorname{div}\left(\beta\frac{\nabla u}{|\nabla u|} + |\nabla u|^{p-2}\nabla u\right)\right), \quad \text{where } \beta > 0 \text{ and } p > 1.$$

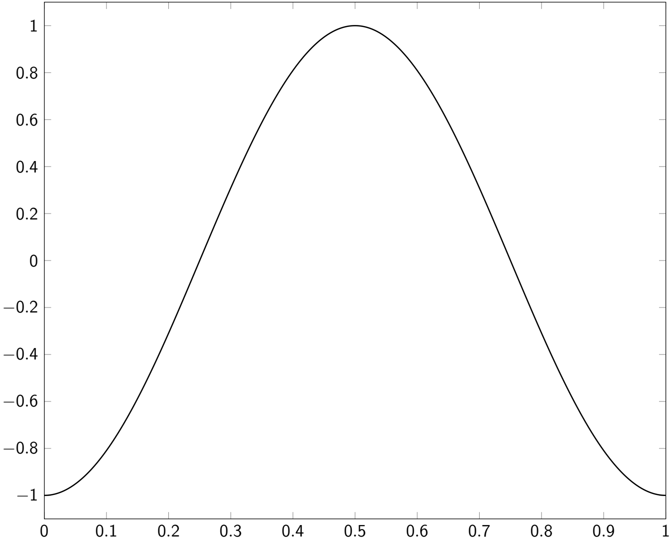
We propose a new shrinkage operator for the case $p = 3$.

## Split Bregman framework for Spohn's model

$$\begin{aligned}
\boldsymbol{u}^{k,j+1} &= \left(\tau^{-1}hK^{\mathrm{T}}K + \mu h^3 \nabla_{\mathrm{av},h}^{\mathrm{T}}\nabla_{\mathrm{av},h}\right)^{-1}\left(\tau^{-1}hK^{\mathrm{T}}K\boldsymbol{u}^k + \mu h^2\nabla_{\mathrm{av},h}^{\mathrm{T}}(\boldsymbol{d}^{k,j} - \boldsymbol{\alpha}^{k,j})\right),\\
d_n^{k,j+1} &= \mathrm{shrink}_{\mathrm{Spohn}}\left(\left(h\nabla_{\mathrm{av},h}\boldsymbol{u}^{k,j+1} + \boldsymbol{\alpha}^{k,j}\right)_n, 1/(\mu h)\right) \text{ for all } n = 1, \ldots, N,\\
\boldsymbol{\alpha}^{k,j+1} &= \boldsymbol{\alpha}^k - \boldsymbol{d}^{k,j+1} + h\nabla_{\mathrm{av},h}\boldsymbol{u}^{k,j+1},
\end{aligned}$$

where $\mathrm{shrink}_{\mathrm{Spohn}}(\rho, a) = \dfrac{\rho}{2a|\rho|}\left(-1 + \sqrt{1 + 4a\max\{|\rho| - a\beta, 0\}}\right)$.

# Numerical result of one dimensional Spohn's model



# Outline

# Fourth order problems

## 2 dimensional TV flow

(anisotropic)

$$u_t = -\Delta\left(\operatorname{div}\left(\frac{\nabla_x u}{|\nabla_x u|}, \frac{\nabla_y u}{|\nabla_y u|}\right)\right).$$

(isotropic)

$$u_t = -\Delta\left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right).$$

- $\nabla_{\mathrm{av},xh}$, $\nabla_{\mathrm{av},yh}$: discrete derivetive on $\mathbb{T}^2$ with average zero condition.
- $-\Delta_{\mathrm{av},h}$: discrete Laplacian.

## Scheme

$$\underset{\boldsymbol{u},\boldsymbol{d}_x,\boldsymbol{d}_y}{\text{minimize}} \left\{ \begin{array}{l} \Phi(\boldsymbol{d}_x, \boldsymbol{d}_y) + \dfrac{\tau^{-1} h_x h_y}{2}\left(\|K_x(\boldsymbol{u}-\boldsymbol{u}^k)\|_2^2 + \|K_y(\boldsymbol{u}-\boldsymbol{u}^k)\|_2^2\right) \\ \quad + \dfrac{\mu h_x h_y}{2}\left(\|\boldsymbol{d}_x - h_x \nabla_{\mathrm{av},xh}\boldsymbol{u}^k\|_2^2 + \|\boldsymbol{d}_y - h_y \nabla_{\mathrm{av},yh}\boldsymbol{u}^k\|_2^2\right) \end{array} \right\}$$

where

$$\Phi(\boldsymbol{d}_x, \boldsymbol{d}_y) = \begin{cases} \|\boldsymbol{d}_x\|_1 + \|\boldsymbol{d}_y\|_1 & \text{for anistropic TV flow,} \\ \left\| \left(\sqrt{d_{xn}^2 + d_{yn}^2}\right)_n \right\|_1 & \text{for isotropic TV flow.} \end{cases}$$

# Remark

- $K_x$, $K_y$ can be defined as the similar way to one dimensional case.
- If $h_x = h_y = h$, we let $\tau = O(h^4)$, $\mu = O(h^{-2})$.
- In the split Bregman framework, the shrinkage operator for anistropic TV flow is the same as one dimensional case.
- Shrinkage formula for isotropic TV flow;

$$d_{xn}^{k+1} = \frac{s_{xn}^k}{|s_{xn}^k|} \max\left\{|s_{xn}^k| - \frac{|s_{xn}^k|}{\mu h_x h_y s_n^k}, 0\right\}, \quad d_{yn}^{k+1} = \dots$$

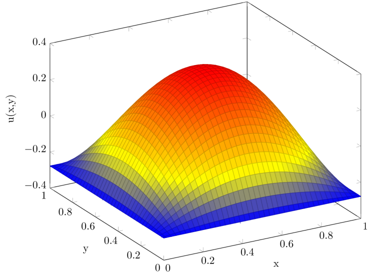
  where

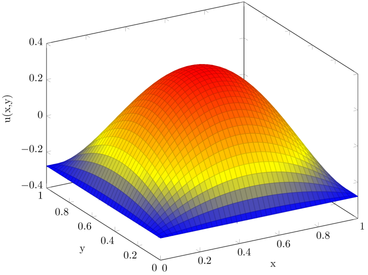
$$s_n^k = \sqrt{(s_{xn}^k)^2 + (s_{yn}^k)^2}, \quad s_{xn}^k = \left(h_x \nabla_{\mathrm{av},xh} \boldsymbol{u}^{k+1} + \boldsymbol{\alpha}_x^{k+1}\right)_n, \quad s_{yn}^k = \dots$$

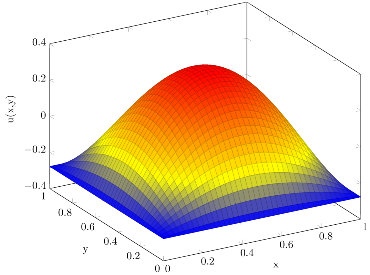
- We propose the shrinkage formula for 2 dimensional Spohn's model;

$$d_{xn}^{k+1} = \frac{muh_x h_y |s_{xn}^k|}{2s_n^k} \cdot \frac{s_{xn}^k}{|s_{xn}^k|}\left(-1 + \sqrt{1 + \frac{4s_n^k}{\mu h_x h_y |s_{xn}^k|} \max\left\{|s_{xn}^k| - \frac{\beta |s_{xn}^k|}{\mu h_x h_y s_n^k}, 0\right\}}\right).$$

# Numerical example for 2 dimensional case




- Upper left: Anisotropic TV flow.
- Upper right: Isotropic TV flow.
- Lower left: Spohn's model.

# Summary

- We propose a new numerical scheme for fourth order problems.
- We also propose a shrinkage operator for Spohn's fourth order model.
- Numerical examples show that our scheme works very well.

Thank you for your attention !