Heat or wave?

A classification of semilinear damped wave equations with time-dependent coefficients [a]

In memory of Professor Kôji Kubota

Hiroyuki Takamura

Tohoku University, JAPAN

---

[a] 18 Aug. 2020. The 45th Sapporo Symposium on Partial Differential Equations.

Based on joint works with Takuto Imai (Accenture Japan Ltd), Masakazu Kato (Muroran Inst. of Tech., Japan) and Kouhei Wakasa (National Inst. of Tech., Kushiro College, Japan).

The series of works are partially supported by KAKENHI(B)(No.18H01132), (C)(No.15K04964) and Special Research Expenses in FY2017, Future Univ. Hakodate.

Prof. Kôji Kubota in 1999

Prof. Kôji Kubota in 2006

with Profs. R.Agemi, K.Mochizuki and H.Kubo

★ I.V.P. with Small Data and Critical Exponent

For scalar unknowns $u = u(x,t)$, $x \in \mathbf{R}^n$, $t \in [0,\infty)$

Classical Damped Wave : $u_{tt} - \Delta u + u_t = |u|^p$,

or Heat : $-\Delta u + u_t = |u|^p$

$\Longrightarrow$ Fujita exp. $p_F(n) := 1 + \frac{2}{n}$

Wave : $u_{tt} - \Delta u = |u|^p$ $(n \geq 2)$

$\Longrightarrow$ Strauss exp. $p_S(n) := \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}$

[Note] $p_F(n) < p_S(n)$ $(n \geq 2)$

★ I.V.P. for Semilinear Damped Wave Equations

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{(1+t)^{\beta}} u_t = |u|^p \\ \qquad \text{in} \quad \mathbf{R}^n \times (0,\infty), \\ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x) \end{cases} \qquad (1)$$

$\varepsilon > 0$: small, $\mu > 0$, $\beta \in \mathbf{R}$.

$(f,g) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$: compact supp.

$T(\varepsilon)$: maximal existence time of energy solution of (1) for $\forall (f,g) \not\equiv (0,0)$ (fixed).

★ Wirth's Classification of Linear Equations

$$u_{tt} - \Delta u + \frac{\mu}{(1+t)^\beta} u_t = 0$$

$\beta < -1$	overdamping
$-1 \le \beta < 1$	effective damping
$\beta = 1$	scaling invariant $0 < \mu < 1 \Rightarrow$ non-effective damping
$1 < \beta$	scattering damping

by Mochizuki('76), Matsumura('77) $\sim$ Wirth('07)

★ Overdamping & Effective Damping ($\beta < 1$)

$$u_{tt} - \Delta u + \frac{\mu}{(1+t)^{\beta}} u_t = |u|^p, \quad p_F(n) := 1 + \frac{2}{n}$$

by Li&Zhou('95), Li('96), Todorova&Yordanov('01), Zhang('01), Nishihara('03), Nishihara('11),

Lin&Nishihara&Zhai('12), Ikeda&Wakasugi('15) Ikeda&Ogawa('16), Lai&Zhou('19),

Fujiwara&Ikeda&Wakasugi('19), Ikeda&Inui('19), Ikeda&Wakasugi('20)

★ Scattering Damping ($\beta > 1$); No.1

$$u_{tt} - \Delta u + \frac{\mu}{(1+t)^\beta} u_t = |u|^p, \quad p_S(n) := \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}$$

Conjecture 1 When $n \geq 3$, or $n = 2$ and $p > 2$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,n)}} & \text{for } p < p_S(n), \\ \exp(C\varepsilon^{-p(p-1)}) & \text{for } p = p_S(n), \end{cases}$$

$$T(\varepsilon) = \infty \qquad \text{for} \quad p > p_S(n),$$

where $\gamma(p,n) := 2 + (n+1)p - (n-1)p^2$.

[Note] $\gamma(p,n) = 0 \iff p = p_S(n), \ p_S(2) > 2$

★ Scattering Damping ($\beta > 1$); No.2

$$u_{tt} - \Delta u + \frac{\mu}{(1+t)^\beta} u_t = |u|^p, \quad u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x)$$

Conjecture 2 When $n = 2$ and $p \leq 2 < p_S(2)$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{p-1}{3-p}} & \text{for } p < 2 \\ Ca(\varepsilon) & \text{for } p = 2 \end{cases} \quad \text{if } \int_{\mathbf{R}^2} g(x)dx \neq 0,$$

where $a = a(\varepsilon)$ satisfies $a^2\varepsilon^2 \log(1+a) = 1$. While Conjecture 1 still holds if $\int_{\mathbf{R}^2} g(x)dx = 0$.

[Note] $\frac{p-1}{3-p} < \frac{2p(p-1)}{\gamma(p,2)} \Longleftrightarrow 1 < p < 2,\ \gamma(2,2) = 4,$ so that $T(\varepsilon)$ with $\int_{\mathbf{R}^2} g(x)dx \neq 0 < T(\varepsilon)$ with $= 0$.

★ Scattering Damping ($\beta > 1$); No.3

$$u_{tt} - \Delta u + \frac{\mu}{(1+t)^\beta} u_t = |u|^p, \quad u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x)$$

Conjecture 3 When $n = 1$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{p-1}{2}} & \text{if } \int_{\mathbf{R}} g(x)dx \neq 0, \\ C\varepsilon^{-p\frac{p-1}{p+1}} & \text{if } \int_{\mathbf{R}} g(x)dx = 0. \end{cases}$$

[Note] $\frac{p-1}{2} < p\frac{p-1}{p+1} \Longleftrightarrow p > 1$,

so that $T(\varepsilon)$ with $\int_{\mathbf{R}} g(x)dx \neq 0 < T(\varepsilon)$ with $= 0$.

Conjecture 1-3 are partially verified by Lai&T('19),

Wakasa&Yordaov('19), Liu&Wang('20).

★ Main Question

In view of the critical exponent and the lifespan estimate of

$$u_{tt} - \Delta u + \frac{\mu}{(1+t)^\beta} u_t = |u|^p,$$

- Effective damping without threshold $(-1 < \beta < 1)$
  $\sim -\Delta u + \dfrac{\mu}{(1+t)^\beta} u_t = |u|^p.$
- Scattering damping $(\beta > 1)$
  $\sim u_{tt} - \Delta u = |u|^p.$

So, how about the scale-invariant damping $(\beta = 1)$?

★ Scale-Invariant Damping; No.1

$$u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = |u|^p, \quad u(x,0) = \varepsilon f(x), \; u_t(x,0) = \varepsilon g(x)$$

Conjecture 4 The critical exponent is

$$\begin{cases} p_F(n) & \text{for } \mu > \mu_0(n) : \text{heat-like}, \\ p_F(n) = p_S(n+\mu) & \text{for } \mu = \mu_0(n) : \text{intermediate}, \\ p_S(n+\mu) & \text{for } \mu < \mu_0(n) : \text{wave-like}, \end{cases}$$

where $\mu_0(n) := \dfrac{n^2+n+2}{n+2}$.

[Note] $p_F(n) < p_S(n)$ and $p_S(n+\mu) \searrow (\mu \nearrow)$.

★ Scale-Invariant Damping; No.2

Conjecture 4 is partially verified by Waksasugi('14) with

- partial heat-like blow-up for
  $\mu \geq 1$ & $1 < p < p_F(n)$.
- partial wave-like blow-up (super-Fujita exp.) for
  $0 < \mu < 1$ & $1 < p < p_F(n + \mu - 1)$.

and by D'Abbicco('15) with partial heat-like existence for

$$p > p_F(n) \;\&\; \begin{cases} n = 1 \;\&\; \mu \geq 5/3 > \mu_0(1) = 4/3, \\ n = 2 \;\&\; \mu \geq 3 > \mu_0(2) = 2, \\ n \geq 3 \;\&\; \mu \geq n + 2 \;{\scriptstyle (p \leq p_F(n-2))}. \end{cases}$$

★ Scale-Invariant Damping; No.3

$$u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = |u|^p, \ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x).$$

Liouville trans.

Set $v(x,t) := (1+t)^{\frac{\mu}{2}} u(x,t)$. Then we have

$$\begin{cases} v_{tt} - \Delta v + \dfrac{\mu(2-\mu)}{4(1+t)^2} v = \dfrac{|v|^p}{(1+t)^{\frac{\mu}{2}(p-1)}} \\ \qquad \text{in} \quad \mathbf{R}^n \times (0,\infty), \\ v(x,0) = \varepsilon f(x), \ v_t(x,0) = \varepsilon \{\mu f(x)/2 + g(x)\}. \end{cases}$$

$\Longrightarrow$ Conjecture 4 is also partially verified in $\mu = 2$ by

D'Abbicco&Lucente&Reissig('15), D'Abbicco&Lucente('15).

★ Lifespan Estimates for Scale-Invariant Damping

$$u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = |u|^p, \; u(x,0) = \varepsilon f(x), \; u_t(x,0) = \varepsilon g(x).$$

- By Lai&T&Wakasa('17) partially and Tu&Lin(arXiv) finally,

$$T(\varepsilon) \leq C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,n+\mu)}} \quad \text{for } p < p_S(n+\mu).$$

- By Ikeda&Sobajima('19),

$$T(\varepsilon) \leq \exp\left(C\varepsilon^{-p(p-1)}\right) \quad \text{for } p = p_S(n+\mu).$$

[Rem] In view of $\mu = 0$, the estimates above may be optimal when $\mu < \mu_0$ (wave-like) except for $n = 1$, or $n = 2$ and $p \leq 2$.

★ Scale-Invariant Damping in the Special Case

$$u_{tt} - \Delta u + \frac{2}{1+t} u_t = |u|^p, \ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x).$$

- By Wakasa('16), in case of $n = 1$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{p-1}{3-p}} & \text{for } p < p_F(1) = 3, \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{for } p = p_F(1) = 3. \end{cases}$$

- By Kato&Sakuraba('19), in case of $n = 3$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,5)}} & \text{for } p < p_S(5), \\ \exp\left(C\varepsilon^{-p(p-1)}\right) & \text{for } p = p_S(5). \end{cases}$$

[Note] $\mu_0(1) = 4/3 < 2$ (heat-like) and $\mu_0(3) > 2$ (wave-like).

★ Expectation (?) on Scale-Invariant Damping

$$u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = |u|^p, \; u(x,0) = \varepsilon f(x), \; u_t(x,0) = \varepsilon g(x).$$

So, one may expect that

- For $\mu > \mu_0(n)$: heat-like, i.e. $p_S(n+\mu) < p_F(n)$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{p-1}{2-n(p-1)}} & \text{for } p < p_F(n), \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{for } p = p_F(n). \end{cases}$$

- For $\mu < \mu_0(n)$: wave-like, i.e., $p_S(n+\mu) > p_F(n)$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,n+\mu)}} & \text{for } p < p_S(n+\mu), \\ \exp\left(C\varepsilon^{-p(p-1)}\right) & \text{for } p = p_S(n+\mu). \end{cases}$$

★ Lifespan Estimates in the Special Case; No.1

$$u_{tt} - \Delta u + \frac{2}{1+t} u_t = |u|^p, \ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x).$$

The expectation is not always true as

Thm 1 [Kato&T.&Wakasa('19)] For $n = 1$ and $\mu = 2$, then

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,3)}} & \text{for } 1 < p < 2, \\ Cb(\varepsilon) & \text{for } p = 2, \\ C\varepsilon^{-\frac{p(p-1)}{3-p}} & \text{for } 2 < p < 3, \\ \exp\left(C\varepsilon^{-p(p-1)}\right) & \text{for } p = p_F(1) = 3, \end{cases}$$

where $b\varepsilon^2 \log(1+b) = 1$ provided $\int_{\mathbb{R}} \{f(x) + g(x)\} dx = 0$.

★ Lifespan Estimates in the Special Case; No.2

$$u_{tt} - \Delta u + \frac{2}{1+t} u_t = |u|^p,\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x).$$

As a conclusion, Wakasa's result on heat-like in $n = 1$;

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{p-1}{3-p}} & \text{for } p < p_F(1) = 3, \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{for } p = p_F(1) = 3, \end{cases}$$

is valid only for $\int_{\mathbf{R}} \{f(x) + g(x)\}dx \neq 0$.

So, we may say that

$$u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = |u|^p \sim v_{tt} - \Delta v = \frac{|v|^p}{(1+t)^{\frac{\mu}{2}(p-1)}}.$$

★ Lifespan Estimates in 2D Case; No.1

$$u_{tt} - \Delta u + \frac{2}{1+t} u_t = |u|^p, \ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x).$$

[Note] $\mu_0(2) = 2$: intermediate.

Thm 2 [Imai&Kato&T.&Wakasa('20)] For $n = 2$ and $\mu = 2$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{p-1}{4-2p}} & \text{for } 1 < p < 2, \\ \exp\left(C\varepsilon^{-\frac{1}{2}} \right) & \text{for } p = p_S(4) = p_F(2) = 2 \end{cases}$$

provided $\int_{\mathbf{R}^2} \{f(x) + g(x)\} dx \neq 0$.

★ Lifespan Estimates in 2D Case; No.2

$$u_{tt} - \Delta u + \frac{2}{1+t} u_t = |u|^p, \ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x).$$

[Note] $\mu_0(2) = 2$: intermediate.

Thm 3 [Imai&Kato&T.&Wakasa('20)] For $n = 2$ and $\mu = 2$,

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,4)}} & \text{for } 1 < p < 2, \\ \exp\left(C\varepsilon^{-\frac{2}{3}}\right) & \text{for } p = p_S(4) = p_F(2) = 2 \end{cases}$$

provided $\int_{\mathbf{R}^2} \{f(x) + g(x)\} dx = 0$.

★ Mechanism in 1D Case with $\mu = 2$; No.1

Proof of Thm 1 is proceeded with transformed problem;

$$\begin{cases} v_{tt} - \Delta v = \dfrac{|v|^p}{(1+t)^{p-1}} \quad \text{in } \mathbf{R} \times (0, \infty), \\ v(x,0) = \varepsilon f(x),\ v_t(x,0) = \varepsilon\{f(x) + g(x)\} \end{cases}$$

$$\Updownarrow$$

$$v = \varepsilon v^0 + L\left(\frac{|v|^p}{(1+t)^{p-1}}\right),$$

where

v^0	:	sol. of free wave eq. with data $(f, f+g)$,
$L(F)$	:	Duhamel term of $F = F(x,t)$.

★ Mechanism in 1D Case with $\mu = 2$; No.2

With the well-known formula;

$$v^0(x,t) = \frac{f(x+t)+f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t}\{f(y)+g(y)\}dy$$

and assumption: supp $(f,g) \subset \{x \in \mathrm{R} : |x| \leq R\},\ R > 0,$
it is possible to establish key point-wise a priori, or blow-up estimate in a main domain

$$D := \{(x,t) \in \mathrm{R} \times (0,\infty) : x+t \geq R \;\&\; x-t \leq -R\}.$$

The key fact ($\leftarrow$ Zhou's observation for N.W. in '92) is that

$$v(x,t) = \frac{\varepsilon}{2}\int_{\mathrm{R}}\{f(x)+g(x)\}dx + L\left(\frac{|v|^p}{(1+t)^{p-1}}\right) \text{ in } D$$

$$\Longrightarrow v \sim O(\varepsilon), \text{ or } O(\varepsilon^p).$$

★ Mechanism in 2D Case with $\mu = 2$; No.1

The existence part is obtained in weighted L^∞ space by

Prop.1 [Hörmander('97),Lindblad('90),Imai&Kato&T.&Wakasa('19)]

For supp $(f, g) \subset \{x \in \mathbf{R}^2 : |x| \leq R\},\ R > 0$,

$$\sum_{|\alpha|\leq 1} \left|\nabla_x^\alpha v^0(x,t)\right| \leq \frac{C_R \left|\int_{\mathbf{R}^2} \{f(x)+g(x)\}dx\right|}{\sqrt{t+|x|+2R}\sqrt{t-|x|+2R}} + \frac{C_{f,g,R}}{\sqrt{t+|x|+2R}(t-|x|+2R)^{\frac{3}{2}}}$$

holds in $\mathbf{R}^2 \times (0,\infty)$.

★ Mechanism in 2D Case with $\mu = 2$; No.2

The blow-up part for $\displaystyle\int_{\mathbf{R}^2}\{f(x)+g(x)\}dx \neq 0$ is obtained by

Prop.2 [Imai&Kato&T.&Wakasa('20)]

With supp $(f, g) \subset \{x \in \mathbf{R}^2 : |x| \leq R\},\ R > 0$,

$$\left| v^0(x,t) - \frac{\displaystyle\int_{\mathbf{R}^2}\{f(x)+g(x)\}dx}{2\pi\sqrt{t+|x|}\sqrt{t-|x|}} \right| \leq \frac{C_{f,g,R}}{\sqrt{t+|x|}(t-|x|)^{\frac{3}{2}}}$$

holds for $t-|x| \geq 2R,\ t \geq 4R$.

★ Mechanism in 2D Case with $\mu = 2$; No.3

The blow-up part for $\displaystyle\int_{\mathbf{R}^2}\{f(x)+g(x)\}dx = 0$ is obtained by

Prop.3 [Imai&Kato&T.&Wakasa('20)]

With supp $(f,g) \subset \{x \in \mathbf{R}^2 : |x| \le R\},\ R>0$ and $f(x)+g(x) \equiv 0$,

$$\left| v^0(x,t) + \frac{t\displaystyle\int_{\mathbf{R}^2} f(x)dx}{2\pi(t+|x|)^{\frac{3}{2}}(t-|x|)^{\frac{3}{2}}} \right| \le \frac{C_{f,g,R}}{\sqrt{t+|x|}(t-|x|)^{\frac{5}{2}}}$$

holds for $t-|x| \ge 2R,\ t \ge 4R$.

★ Critical Blow-up for $\int_{\mathbf{R}^2}(f+g)dx > 0$; No.1

The proof proceeds with iteration argument $\&$ "slicing method".

We may assume that $v(x,t) = v(r,t), r = |x|$. By linear estimate $\&$ positivity of the kernel, Prop.2 gives us

$$v(r,t) \geq \frac{E\varepsilon}{\sqrt{t+r}\sqrt{t-r}} \quad \text{for } t-r \geq K,$$

where E and $K(>1)$ are independent of ε. If one plugs this estimate into Duhamel term, one obtains that

$$v(r,t) \geq \frac{E^2\varepsilon^2}{\sqrt{t+r}\sqrt{t-r}} I(r,t) \quad \text{for } t-r \geq K,$$

where the integral term $I(r,t)$ is defined and estimated by...

★ Critical Blow-up for $\int_{\mathbf{R}^2}(f+g)dx > 0$; No.2

$$
\begin{aligned}
I(r,t) \quad &:= \frac{1}{2}\int_K^{t-r}\frac{1}{\alpha}d\alpha\int_K^{\alpha}\frac{(\alpha-\beta)/2}{1+(\alpha+\beta)/2}\cdot\frac{1}{\beta}d\beta \\
&\geq \frac{1}{8}\int_K^{t-r}\frac{1}{\alpha^2}d\alpha\int_K^{\alpha}(\alpha-\beta)\frac{d}{d\beta}\log\frac{\beta}{K}d\beta \\
&= \frac{1}{8}\int_K^{t-r}\frac{1}{\alpha^2}d\alpha\int_K^{\alpha}\log\frac{\beta}{K}d\beta \\
&\geq \frac{1}{8}\int_{K/(1+1/2)}^{t-r}\frac{1}{\alpha^2}d\alpha\int_{\alpha/(1+1/2)}^{\alpha}\log\frac{\beta}{K}d\beta \\
&\geq \frac{1/2}{8}\int_{K/(1+1/2)}^{t-r}\frac{1}{\alpha}\log\frac{\alpha}{K(1+1/2)}d\alpha \\
&\geq \frac{1/2}{16}\left(\log\frac{t-r}{K(1+1/2)}\right)^2
\end{aligned}
$$

for $t-r \geq (1+1/2)K$. $\longleftarrow$ “slicing” of the blow-up set.

★ Critical Blow-up for $\int_{\mathbf{R}^2}(f+g)dx > 0$; No.3

Proceeding this way $j(\in \mathbf{N})$-times, we reach to

$$v(r,t) \geq \frac{\exp\left\{2^j J(r,t)\right\}}{\sqrt{t+r}\sqrt{t-r}} \log^{-2}\frac{t-r}{2K} \quad \text{for } t-r \geq 2K,$$

where

$$J(r,t) := \log\left(C\varepsilon \log^2 \frac{t-r}{2K}\right).$$

This reads to the desired result: $T(\varepsilon) \leq \exp\left(C\varepsilon^{-\frac{1}{2}}\right)$.

[Rem] The functional (test function?) method is not useful.

★ Conclusion

Semilinear damped wave equation

$\sim$ Semilinear wave equation

at least

in the scale-invariant and scattering cases

according to

the lifespan estimates.