Existence of traveling wave solutions to a nonlocal scalar equation with sign-changing kernel

The 45th Sapporo Symposium on Partial Differential Equations (2020.8.17-19)

Speaker : Hiroshi Ishii (Hokkaido University, Japan)

Joint work with
Shin-Ichiro Ei (Hokkaido University, Japan)
Jong-Shenq Guo (Tamkang University [淡江大学], Taiwan)
Chin-Chin Wu (National Chung Hsing University [国立中興大学], Taiwan)

Introduction

We consider the following nonlocal scalar equation

$$u_t = \underline{du_{xx}} + \underline{K * u - \alpha u} + \underline{f(u)}, \ t > 0, \ x \in \mathbb{R},$$

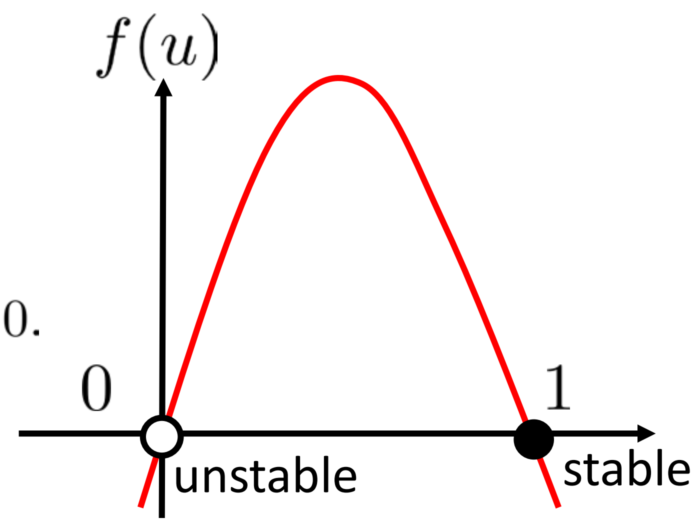
Diffusion   Nonlocal effects   Reaction

where $u(t,x) \in \mathbb{R}$, $d \geq 0$, $K * u(t,x) = \int_{\mathbb{R}} K(x-y)u(t,y)dy$, $\alpha := \int_{\mathbb{R}} K(y)dy$

Condition

(K1) $C(\mathbb{R}) \cap L^1(\mathbb{R}) \ni K \not\equiv 0, \ K(x) = K(-x), \ \forall x \in \mathbb{R}, \quad \int_{\mathbb{R}} K(y)dy = \alpha \geq 0$

(F1) $\begin{cases} f \in \mathrm{Lip}_{loc}(\mathbb{R}), \ f(0) = f(1) = 0, \\ f < 0 \text{ in } (1,\infty), \ f > 0 \text{ in } (0,1), \\ f \text{ is differentiable at } u = 0, \text{ and } f'(0) > 0. \end{cases}$



Backgrounds

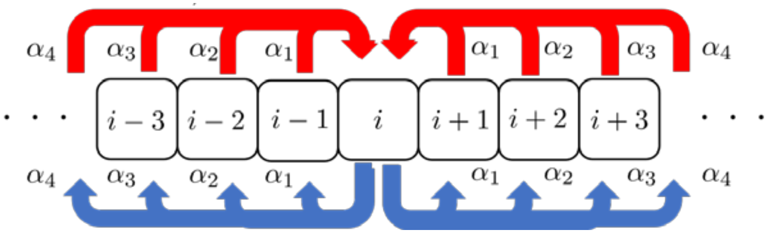
$$u_t = \underbrace{du_{xx}}_{\text{Diffusion}} + \underbrace{K*u - \alpha u}_{\text{Nonlocal effects}} + \underbrace{f(u)}_{\text{Reaction}},\ t>0,\ x\in\mathbb{R},$$

where $u(t,x)\in\mathbb{R},\ d\geq 0,\ K*u(t,x) = \int_{\mathbb{R}} K(x-y)u(t,y)dy,\ \alpha := \int_{\mathbb{R}} K(y)dy$

- Nonlocal-diffusion [V. Huston et al. (2003)F. A. Vaillo et al. (Springer, 2006)]

$$u_t = K*u - \alpha u,\ t>0,\ x\in\mathbb{R}$$

$$\sum_j \alpha_{|i-j|}u_j - \left(\sum_j \alpha_j\right)u_i$$

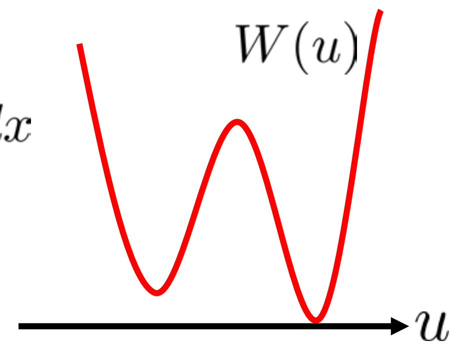


- Phase transition [P. W. Bates et al. (1997, 2005, …)]

$$E(u) = \frac{1}{4}\int_{\mathbb{R}^2} K(x-y)(u(x)-u(y))^2 dxdy + \int_{\mathbb{R}} W(u(x))dx$$

L^2 - gradient flow

$$u_t = K*u - \alpha u - W'(u),\ t>0,\ x\in\mathbb{R}$$



- Neural science [J. D. Murray (1993)]
- Pattern formation problem [S. Kondo (2017), S.-I. Ei, H. I., S. Kondo, T. Miura, Y. Tanaka (preprint)]
- dispersal motion of cell or organism [V. Huston (2003)]

Backgrounds

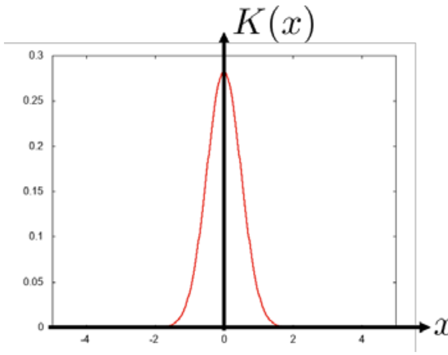
$$u_t = du_{xx} + K * u - \alpha u + f(u), \ t > 0, \ x \in \mathbb{R},$$

Diffusion　　Nonlocal effects　　Reaction

$K(x)$: non-negative function

There are many results of localized patterns.

- J. Coville and C. Dupaigne (2007)
- H. Yagishita (2009)
- Y. –J. Sun, W.-T. Li and Z.-C. Wang (2011)



$K(x)$: sign-changing function

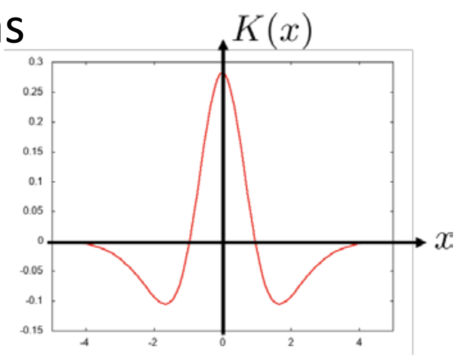
There are few results. In numerical simulation, this equation has

→**spatial periodic solutions** (induced by Turing instability)

[J. Siebert, E. Schöll (2015), S. Kondo(2017)]

→**non-monotonic traveling wave solutions**

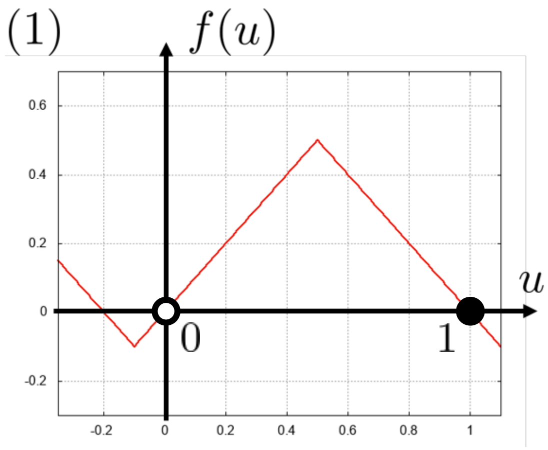
[J. Siebert, E. Schöll (2015)]

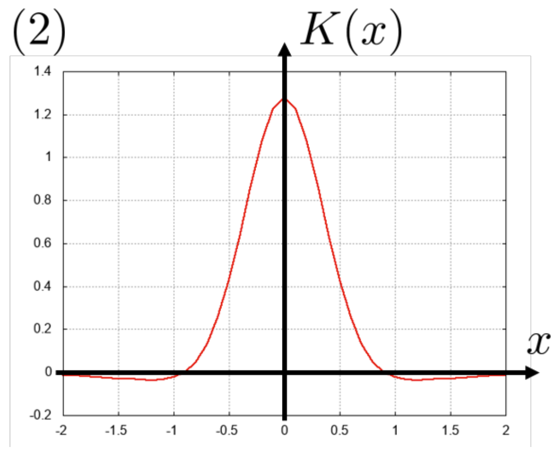


Numerical simulation

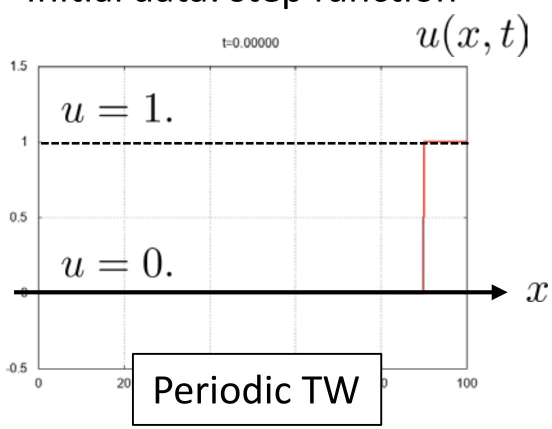
$$u_t = du_{xx} + K * u - \alpha u + f(u), \ t > 0, \ x \in \mathbb{R},$$

When the kernel has negative parts, this equation does not have comparison principle.

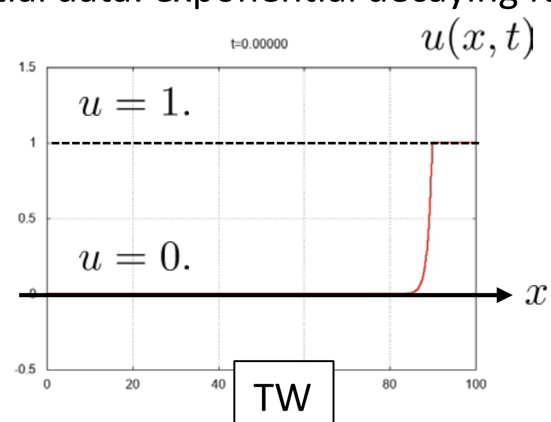




Initial data: step function



Initial data: exponential decaying function



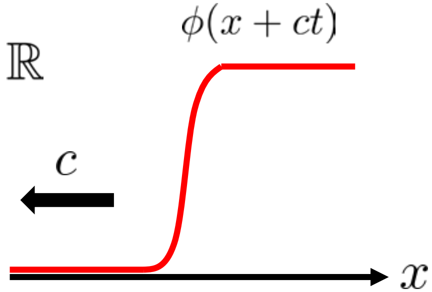
Traveling wave solution

$$u_t = du_{xx} + K * u - \alpha u + f(u),\ t > 0,\ x \in \mathbb{R},$$

Today, we consider that the existence of traveling wave solutions connecting two states.

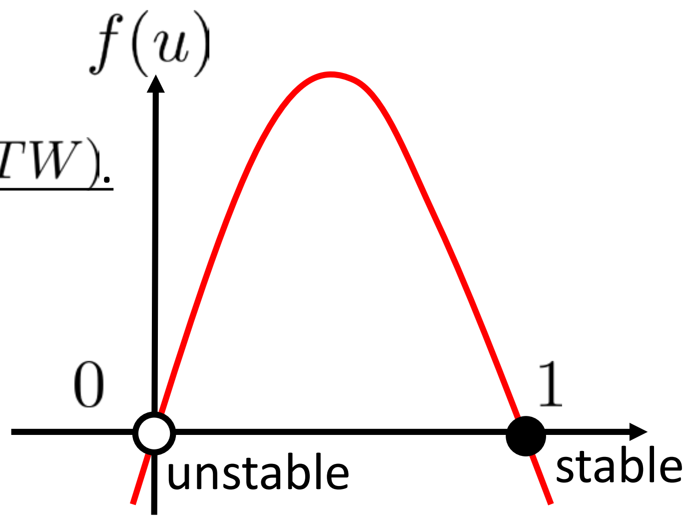
$u(t,x) = \phi(x+ct) = \phi(\xi)$ (wave speed c , wave profile ϕ)

$$(TW) \begin{cases} c\phi'(\xi) = d\phi''(\xi) + K * \phi(\xi) - \alpha\phi(\xi) + f(\phi(\xi)),\ \xi \in \mathbb{R} \\ \phi(-\infty) = 0,\ \phi(+\infty) = 1. \end{cases}$$



<u>The existence of the traveling wave solutions when $K \geq 0$ (p.73).</u>

<u>When K has negative parts, there is no results of the existence of traveling wave solutions satisfying (TW).</u>
(to the best of my knowledge).



Motivation

We show the existence of traveling wave solutions satisfying

$$(TW)\begin{cases} c\phi'(\xi) = d\phi''(\xi) + K * \phi(\xi) - \alpha\phi(\xi) + f(\phi(\xi)), \ \xi \in \mathbb{R} \\ \phi(-\infty) = 0, \ \phi(+\infty) = 1. \end{cases}$$

where K has negative parts.

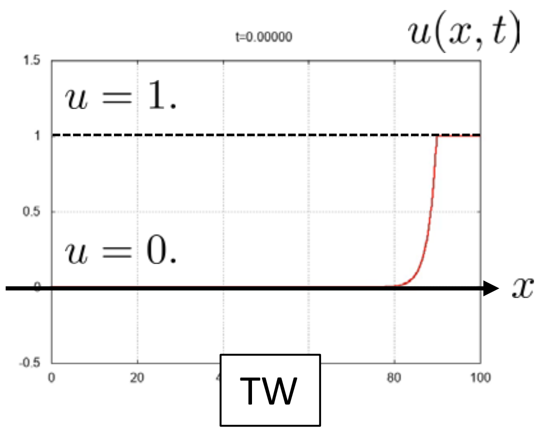
We always assume that

(K1) $C(\mathbb{R}) \cap L^1(\mathbb{R}) \ni K \not\equiv 0, \ K(x) = K(-x), \ \forall x \in \mathbb{R}, \quad \int_{\mathbb{R}} K(y)dy = \alpha \geq 0$

(K2) $\forall \lambda > 0, \ \int_{\mathbb{R}} |K(x)|e^{-\lambda x}dx < \infty$

(F1) $\begin{cases} f \in \mathrm{Lip}(\mathbb{R}), \ f(0) = f(1) = 0, \\ f < 0 \text{ in } (1,\infty), \ f > 0 \text{ in } (0,1), \\ f \text{ is differentiable at } u = 0, \text{ and } f'(0) > 0. \end{cases}$

(F2) $f(u) \leq f'(0)u$ for $u \in [0, u^+]$,
for some constant $u^+ > 1$.



Main results

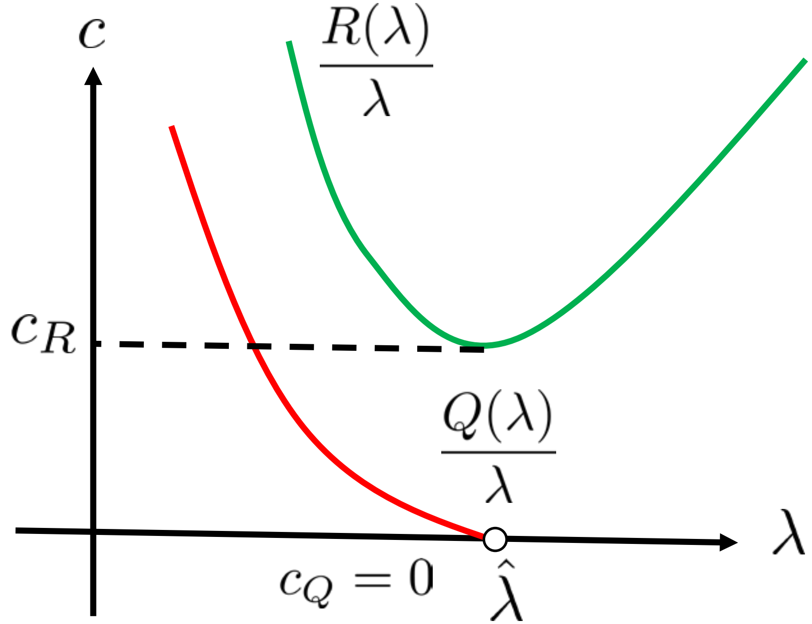
$$(TW)\begin{cases} c\phi'(\xi) = d\phi''(\xi) + K * \phi(\xi) - \alpha\phi(\xi) + f(\phi(\xi)),\ \xi \in \mathbb{R} \\ \phi(-\infty) = 0,\ \phi(+\infty) = 1. \end{cases}$$

Define

$K^+(x) := \max\{K(x), 0\}$, (positive part)

$K^-(x) := \max\{-K(x), 0\}$ (negative part)

- $K(x) = K^+(x) - K^-(x)$
- $|K(x)| = K^+(x) + K^-(x)$



We define two quantities

$$c_Q := \inf_{\lambda \in (0,\hat{\lambda})} \frac{Q(\lambda)}{\lambda}, \quad Q(\lambda) := d\lambda^2 + \int_{\mathbb{R}} K(y)e^{-\lambda y}dy - \alpha + f'(0),$$

$$c_R := \inf_{\lambda \in (0,\infty)} \frac{R(\lambda)}{\lambda}, \quad R(\lambda) := d\lambda^2 + \int_{\mathbb{R}} |K(y)|e^{-\lambda y}dy - \alpha + f'(0),$$

where $\hat{\lambda}$ is defined to be the first positive zero of $Q(\lambda)$, if it exists, otherwise, set $\hat{\lambda} = \infty$.

Main results

<u>Assumption 3.2</u>

$(1)\exists \eta \in (0,1)$ s.t. $f(u) = f'(0)u \quad \text{for } u \in [0,\eta].$

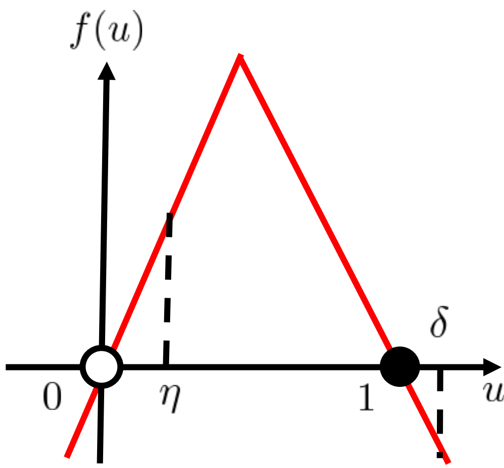
linear

$(2) f$ satisfies $f'(0) > \alpha.$

$(3) K$ satisfies

$$\int_{\mathbb{R}} K^-(y)dy \le \min\left\{\frac{-f(\delta)}{\delta}, \frac{(f'(0)-\alpha)\eta}{\delta}\right\}$$

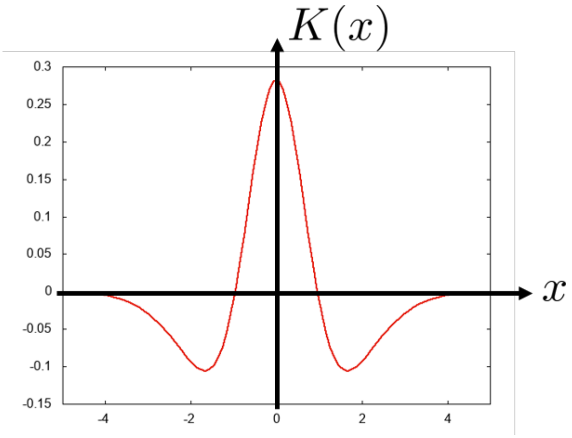
for some constant $\delta \in (1, u^+)$.



<u>Example</u>

$$f(u) = \begin{cases} 2-2u, & u > \frac{1}{2}; \\ 2u, & u \le \frac{1}{2}; \end{cases}$$

$$\int_{\mathbb{R}} K^+(y)dy = \int_{\mathbb{R}} K^-(y)dy \le \frac{2}{3}$$



Main results

$$(TW)\begin{cases} c\phi'(\xi) = d\phi''(\xi) + K * \phi(\xi) - \alpha\phi(\xi) + f(\phi(\xi)), \ \xi \in \mathbb{R} \\ \phi(-\infty) = 0, \ \phi(+\infty) = 1. \end{cases}$$

$$C_b^k(X) := \{\phi \in C^k(X) \mid \sup_{\xi \in X} |\phi^{(j)}(\xi)| < \infty \ (j = 0, 1, \ldots, k)\}, \ (X \subset \mathbb{R}, \ k = 0, 1, 2, \ldots)$$

Theorem 3.4 [S.-I. Ei, J.-S. Guo, H. I., C.-C. Wu (2020,JMAA), H. I. (Master's thesis)]

Let Assumption 3.2 be enforced.

Then for any $c > c_R$ there exists the wave profile $\phi > 0$ satisfying

$$\begin{cases} c\phi'(\xi) = d\phi''(\xi) + K * \phi(\xi) - \alpha\phi(\xi) + f(\phi(\xi)), \ \xi \in \mathbb{R} \\ \phi(-\infty) = 0, \ \liminf_{\xi \to +\infty} \phi(\xi) > 0. \end{cases}$$

Moreover, if $\phi \in C_b^2(\mathbb{R})$ and $c > \max\{c_R, c_K\}$, then $\phi(+\infty) = 1$.

Here,

$$c_K := \sqrt{\left(\int_{\mathbb{R}} |K(y)|dy\right)\left(\int_{\mathbb{R}} y^2|K(y)|dy\right)}.$$

Remark $d > 0 \Rightarrow \phi \in C_b^2(\mathbb{R}), \quad d = 0, \ f \in C^1(\mathbb{R}) \Rightarrow \phi \in C_b^2(\mathbb{R})$

Remark This result does not need the stability of $u = 1$ in the sense of PDE.

Outline of proof (The existence of the positive wave profile)

$$(TW1)\ c\phi'(\xi) = d\phi''(\xi) + K * \phi(\xi) - \alpha\phi(\xi) + f(\phi(\xi)),\ \xi \in \mathbb{R}$$

Definition 4.1

Given a constant $c > 0$. A pair of continuous functions $\{\overline{\phi}, \underline{\phi}\}$ are upper and lower solutions of $(TW1)$ if $\overline{\phi}, \underline{\phi} \in C^2(\mathbb{R} \setminus A)$,

$$c\overline{\phi}'(\xi) \geq d\overline{\phi}''(\xi) + (K^+ * \overline{\phi})(\xi) - (K^- * \underline{\phi})(\xi) - \alpha\overline{\phi}(\xi) + f(\overline{\phi}(\xi)),\ \forall \xi \in \mathbb{R} \setminus A,$$
$$c\underline{\phi}'(\xi) \leq d\underline{\phi}''(\xi) + (K^+ * \underline{\phi})(\xi) - (K^- * \overline{\phi})(\xi) - \alpha\underline{\phi}(\xi) + f(\underline{\phi}(\xi)),\ \forall \xi \in \mathbb{R} \setminus A,$$

for some finite set $A \subset \mathbb{R}$.

Outline of proof (The existence of the positive wave profile)

$$(TW1)\ c\phi'(\xi) = d\phi''(\xi) + K * \phi(\xi) - \alpha\phi(\xi) + f(\phi(\xi)),\ \xi \in \mathbb{R}$$

$$\begin{cases} c\overline{\phi}'(\xi) \geq d\overline{\phi}''(\xi) + (K^+ * \overline{\phi})(\xi) - \underline{(K^- * \underline{\phi})(\xi)} - \alpha\overline{\phi}(\xi) + f(\overline{\phi}(\xi)), & \xi \in \mathbb{R} \setminus A, \\ c\underline{\phi}'(\xi) \leq d\underline{\phi}''(\xi) + (K^+ * \underline{\phi})(\xi) - \underline{(K^- * \overline{\phi})(\xi)} - \alpha\underline{\phi}(\xi) + f(\underline{\phi}(\xi)), & \xi \in \mathbb{R} \setminus A, \end{cases}$$

Proposition 4.3

Suppose that there exists a pair of upper-lower-solution $\{\overline{\phi}, \underline{\phi}\}$ such that

$$(1) \quad \underline{\phi},\ \overline{\phi}\ :\ \mathbb{R} \to [0, u^+],$$

$$(2) \quad \underline{\phi}(\xi) \leq \overline{\phi}(\xi)\ (\forall \xi \in \mathbb{R}),$$

$$(3) \quad \overline{\phi}'(z-) \geq \overline{\phi}'(z+),\ \underline{\phi}'(z-) \leq \underline{\phi}'(z+),\ (\forall z \in A).$$

Then $(TW1)$ has a solution ϕ such that $\underline{\phi} \leq \phi \leq \overline{\phi}$ in $\mathbb{R}$.

Idea of Proof : [S. Ma (2001)]

$$\Gamma := \{\psi \in C(\mathbb{R}) \mid \underline{\phi} \leq \psi \leq \overline{\phi}\}$$

The integral operator $P_d^c : \Gamma \to \Gamma$ $\quad \exists (c, \phi)$ satisfying $(TW1) \Leftrightarrow \phi = P_d^c[\phi]$ (p.77)

To use Schauder's fixed point theorem, we check the property of P_d^c.

Outline of proof (The existence of the positive wave profile)

$$(TW1)\ c\phi'(\xi) = d\phi''(\xi) + K * \phi(\xi) - \alpha\phi(\xi) + f(\phi(\xi)),\ \xi \in \mathbb{R}$$

$$\begin{cases} c\overline{\phi}'(\xi) \geq d\overline{\phi}''(\xi) + (K^+ * \overline{\phi})(\xi) - (K^- * \underline{\phi})(\xi) - \alpha\overline{\phi}(\xi) + f(\overline{\phi}(\xi)), & \xi \in \mathbb{R} \setminus A, \\ c\underline{\phi}'(\xi) \leq d\underline{\phi}''(\xi) + (K^+ * \underline{\phi})(\xi) - (K^- * \overline{\phi})(\xi) - \alpha\underline{\phi}(\xi) + f(\underline{\phi}(\xi)), & \xi \in \mathbb{R} \setminus A, \end{cases}$$

Lemma 4.2

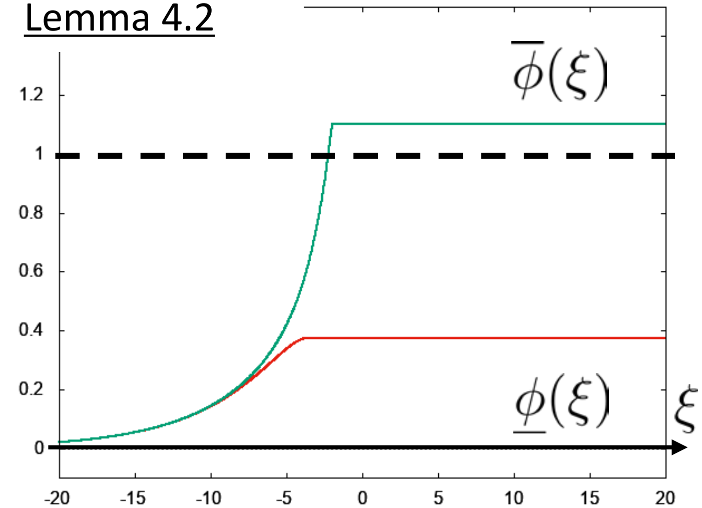
Let Assumption 3.2 be enforced. For a given $c > c_R$, we set

$$\overline{\phi}(\xi) = \min\{e^{\lambda\xi} + he^{\nu\lambda\xi}, \delta\} \quad \underline{\phi}(\xi) = \begin{cases} e^{\lambda\xi} - he^{\nu\lambda\xi}, & \xi \leq \xi_M, \\ \eta, & \xi \geq \xi_M. \end{cases}$$

for suitable constants $\lambda,\ \nu,\ h,\ \xi_M$.

Then the functions $\{\overline{\phi}, \underline{\phi}\}$ are upper and lower solutions of $(TW1)$.

Lemma 4.2



$$\begin{cases} c\phi'(\xi) = d\phi''(\xi) + K * \phi(\xi) - \alpha\phi(\xi) + f(\phi(\xi)),\ \xi \in \mathbb{R} \\ \phi(-\infty) = 0,\ \liminf_{\xi \to +\infty} \phi(\xi) > 0. \end{cases}$$

Assumption 3.2

(1) $f(u) = f'(0)u$ for $u \in [0, \eta]$.

(2) $f'(0) > \alpha$.

(3) $\int_{\mathbb{R}} K^-(y)dy \leq \min\left\{\frac{-f(\delta)}{\delta}, \frac{(f'(0) - \alpha)\eta}{\delta}\right\}$

Outline of proof (Asymptotic behavior)

$$(TW1)\ c\phi'(\xi) = d\phi''(\xi) + K*\phi(\xi) - \alpha\phi(\xi) + f(\phi(\xi)),\ \xi \in \mathbb{R}$$

$$c_K := \sqrt{\left(\int_{\mathbb{R}} |K(y)|dy\right)\left(\int_{\mathbb{R}} y^2|K(y)|dy\right)}.$$

Using L^2 estimates based on [M.Alfalo and J.Coville (2012)], we can gain this lemma.

Lemma 4.4

$$c > c_K \text{ and } \phi \in C_b^2(\mathbb{R}) \Rightarrow \phi' \in L^2(\mathbb{R}) \text{ and } \phi'(+\infty) = 0.$$

Idea of proof

We multiply $(TW1)$ by ϕ' and then integrate from $-q < 0$ to $p > 0$ to get

$$cI_{p,q} \leq \sqrt{I_{p,q}}\left(\int_{-q}^{p} \{(K*\phi - \alpha\phi)(\xi)\}^2 d\xi\right)^{1/2} + C_1 \leq c_K\sqrt{I_{p,q}^2 + C_2 I_{p,q}} + C_1,$$

where $I_{p,q} = \int_{-q}^{p} (\phi')^2 dx$ and $C_1,\ C_2$ are positive constants.

$$\int_{-q}^{p} \{K*\phi(\xi) - \alpha\phi(\xi)\}^2 d\xi = \int_{-q}^{p} \left\{\int_{\mathbb{R}}\int_0^1 K(\xi - y)(y-\xi)\phi'(\xi + s(y-\xi))dsdy\right\}^2 d\xi \leq c_K^2(I_{p,q} + C_2),$$

Outline of proof (Asymptotic behavior)

$$(TW1)\ c\phi'(\xi) = d\phi''(\xi) + K * \phi(\xi) - \alpha\phi(\xi) + f(\phi(\xi)),\ \xi \in \mathbb{R}$$

$$c_K := \sqrt{\left(\int_{\mathbb{R}} |K(y)|dy\right)\left(\int_{\mathbb{R}} y^2|K(y)|dy\right)}.$$

Using L^2 estimates based on [M.Alfalo and J.Coville (2012)], we can gain this lemma.

Lemma 4.4

$$c > c_K \text{ and } \phi \in C_b^2(\mathbb{R}) \Rightarrow \phi' \in L^2(\mathbb{R}) \text{ and } \phi'(+\infty) = 0.$$

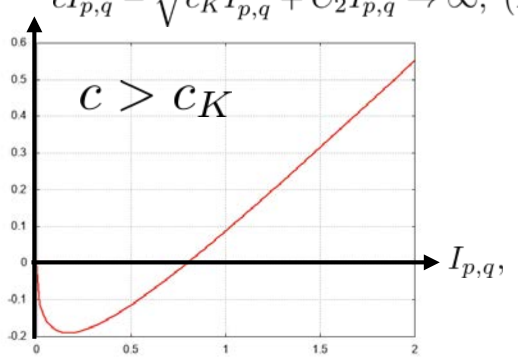
Idea of proof

We multiply $(TW1)$ by ϕ' and then integrate from $-q < 0$ to $p > 0$ to get

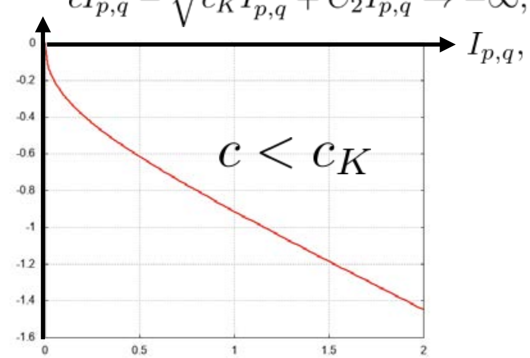
$$cI_{p,q} \leq \sqrt{I_{p,q}}\left(\int_{-q}^{p} \{(K * \phi - \alpha\phi)(\xi)\}^2 d\xi\right)^{1/2} + C_1 \leq c_K\sqrt{I_{p,q}^2 + C_2 I_{p,q}} + C_1,$$

where $I_{p,q} = \int_{-q}^{p} (\phi')^2 dx$ and $C_1,\ C_2$ are positive constants.

$$cI_{p,q} - \sqrt{c_K I_{p,q}^2 + C_2 I_{p,q}} \to \infty,\ (I_{p,q} \to \infty)$$



$$cI_{p,q} - \sqrt{c_K I_{p,q}^2 + C_2 I_{p,q}} \to -\infty,\ (I_{p,q} \to \infty)$$



Outline of proof (Asymptotic behavior)

$$(TW1)\ c\phi'(\xi) = d\phi''(\xi) + K * \phi(\xi) - \alpha\phi(\xi) + f(\phi(\xi)),\ \xi \in \mathbb{R}$$

$$c_K := \sqrt{\left(\int_{\mathbb{R}} |K(y)|dy\right)\left(\int_{\mathbb{R}} y^2|K(y)|dy\right)}.$$

Using L^2 estimates based on [M.Alfalo and J.Coville (2012)], we can gain this lemma.

Lemma 4.4

$$c > c_K \text{ and } \phi \in C_b^2(\mathbb{R}) \Rightarrow \underline{\phi' \in L^2(\mathbb{R}) \text{ and } \phi'(+\infty) = 0.}$$

Proposition 4.5

$$c > c_K \text{ and } \phi \in C_b^2(\mathbb{R}) \Rightarrow \underline{\phi(+\infty) \text{ exists and satisfies } f(\phi(+\infty)) = 0.}$$

Summary

1. When the kernel has small negative parts and the nonlinear term satisfies suitable condition, we could prove the existence of traveling wave solutions with positive wave profile.

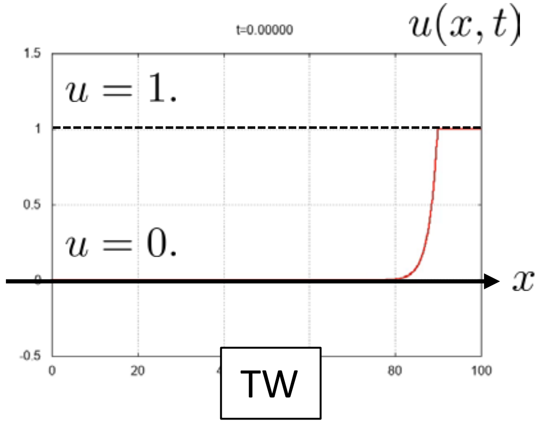
$$\begin{cases} c\phi'(\xi) = d\phi''(\xi) + K * \phi(\xi) - \alpha\phi(\xi) + f(\phi(\xi)), \ \xi \in \mathbb{R} \\ \phi(-\infty) = 0, \ \liminf_{\xi \to +\infty} \phi(\xi) > 0. \end{cases}$$

2. In particular, the sufficiently rapid traveling waves connect two constant states.

Future work

- Stability
- Minimal speed
- Uniqueness
- Pulsating front

Initial data: exponential decaying function



Thank you for your attention