Solvability of doubly nonlinear parabolic equations with p-Laplacian

Shun Uchida (Oita University)

The 45th Sapporo Symposium on Partial Differential Equations
– In memory of Professor Kôji Kubota –

@ Zoom

Aug. 17–19, 2020.

§1. Introduction

We consider the following IBVP of doubly nonlinear parabolic equation:

$$
\text{(P)} \qquad \begin{cases} \partial_t \beta(u(x,t)) - \Delta_p u(x,t) \ni f(x,t) & (x,t) \in Q := \Omega \times (0,T), \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}
$$

- $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$: bounded domain with smooth boundary $\partial\Omega$.
- f : given external force.
- $p \in (1, \infty)$: exponent of p-Laplacian $\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u)$.
- $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a (multi-valued) maximal monotone graph on $\mathbb{R}$ satisfying $0 \in \beta(0)$.

Aim

To show existence of solution to (P) without any assumptions of β except $0 \in \beta(0)$.

$\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is said to be

- *monotone* if β is non-increasing, i.e.,

$$(s_1 - s_2)(\sigma_1 - \sigma_2) \geq 0 \quad \forall \sigma_i \in \beta(s_i) \ \ (i = 1, 2).$$

- *maximal monotone* if β is monotone and there is no monotonic extension of β. This is equivalent to $R(I + \lambda\beta) = \mathbb{R}$ for any $\lambda > 0$.

ex.1 β is continuous (single-valued) non-increasing mapping with $D(\beta) = \mathbb{R}$ or $R(\beta) = \mathbb{R}$, e.g., $\beta(s) = |s|^{r-2}s \ (r > 1), \quad \beta(s) = e^s - 1$.

ex.2 β is possibly multi-valued, e.g.,

$$\beta(s) = \mathrm{sgn}(s) = \begin{cases} -1 & \text{if } s < 0, \\ [-1, 1] & \text{if } s = 0, \\ 1 & \text{if } s > 0. \end{cases} \quad \beta(s) = \mathrm{sgn}^{-1}(s) = \begin{cases} (-\infty, 0] & \text{if } s = -1, \\ 0 & \text{if } s \in (-1, 1), \\ [0, \infty) & \text{if } s = 1, \\ \emptyset & \text{otherwise}. \end{cases}$$

§1. Introduction

(Examples of (P))

ex.1 Let $\beta(s) = |s|^{r-2}s$. Then

$$\partial_t |u|^{r-2}u - \Delta_p u = f.$$

(see, e.g., Raviart 1970, Bamberger 1977, Tsutsumi 1988...)

ex.2 Let $p = 2$ and $\beta(u) = e^u - 1$, $v = e^u$. Then the equation is equivalent to

$$\partial_t v - \Delta \log v = f \qquad v|_{\partial\Omega} \equiv 1.$$

(see, e.g., Berryman–Holland 1982, Esteban–Rodríguez–Vazquez 1988...)

ex.3 Miyoshi–Tsutsumi (2016) derived

$$\partial_t v - \nabla \cdot (|\nabla \log v|^{p-2} \nabla \log v) = f$$

from the singular limit of a generalized Carleman model.

§1. Introduction

⋆ Previous studies of solvability for generalized β:

- Grange–Mignot 1972 : abstract evolution equation $(Au)' + Bu \ni f$.
  - Boundedness condition of A and B ↔ growth condition of β.
  - Standard time discretization technique with properties of subdifferential.
- Barbu 1979 : abstract evolution equation $(Au)' + Bu \ni f$.
  - Hilbert setting.
  - Avoid growth or coerciveness conditions of A by using $(A_\lambda u, Bu)_H \geq 0$.
  - Existence of $\partial_t \beta(u) - \Delta_p u \ni f$ for $p \geq 2$ and $D(\beta) = \mathbb{R}$.
- Alt–Luckhaus 1983 : PDE $\partial_t \beta(u) - \Delta_p u \ni f$.
  - Galerkin's method and convergence argument in L^1.
  - β: single or multi-valued with growth condition of “jump”.

Aim

To show existence of solution to (P) without assumptions, e.g.,
growth condition (boundedness), coerciveness, single-valuedness, or $D(\beta) = \mathbb{R}$.

We consider the following (weak) solution:

Definition

Let (u_0, ξ_0) satisfy $\xi_0(x) \in \beta(u_0(x))$ for a.e. $x \in \Omega$.
Then (u, ξ) is said to be a solution to (P) with the initial data (u_0, ξ_0) if

$$u \in L^\infty(0,T; W_0^{1,p}(\Omega)), \quad \xi \in W^{1,\infty}(0,T; W^{-1,p'}(\Omega)) \cap L^\infty(0,T; L^{p'}(\Omega)),$$

$$\xi(x,t) \in \beta(u(x,t)) \quad \text{for a.e. } (x,t) \in Q,$$

$$\begin{cases} \partial_t \xi(t) - \Delta_p u(t) = f(t) \quad \text{in} \quad W^{-1,p'}(\Omega) \quad \text{for a.e. } t \in (0,T), \\ \xi(\cdot, 0) = \xi_0. \end{cases}$$

Remark

Regularity of (u, ξ) given above leads to for every $t_1, t_2 \in [0,T]$ (see Alt-Luckhaus 1983)

$$\int_\Omega j^*(\xi(x,t_2))dx - \int_\Omega j^*(\xi(x,t_1))dx + \int_{t_1}^{t_2} \|\nabla u(t)\|_{L^p}^p dt = \int_{t_1}^{t_2} \int_\Omega f(x,t)u(x,t)dxdt,$$

where j is the primitive function of β (i.e., $\beta = \partial j$) and j^* is its conjugate.

Remark

Let $f \equiv 0$.

- If $\beta \equiv 0$, we have $j^*(s) = \begin{cases} 0 & s = 0, \\ +\infty & \text{otherwise.} \end{cases}$

  Then (P) has a unique solution for any given $u_0 \in W_0^{1,p}(\Omega)$

  $$u(x,t) = \begin{cases} u_0(x) & t = 0, \\ 0 & t > 0. \end{cases}$$

- If $\beta(s) = \begin{cases} \mathbb{R} & \text{if } s = 0, \\ \emptyset & \text{otherwise} \end{cases}$, we have $j^* \equiv 0$.

  Then (P) has a unique solution $u \equiv 0$ and $\xi \equiv \xi_0$ for any given $\xi_0 \in L^{p'}(\Omega)$.

Theorem 3.1

Let $p \in (1, \infty)$, $q \in [p', \infty]$ and $0 \in \beta(0)$. Then for any $u_0 \in W_0^{1,p}(\Omega)$, $\xi_0 \in L^{p'}(\Omega) \cap L^q(\Omega)$, and $f \in W^{1,p'}(0, T; L^{p'}(\Omega)) \cap L^\infty(0, T; L^q(\Omega))$, there exist at least one solution to (P) with the initial data (u_0, ξ_0) satisfying

$$\sup_{0 \le t \le T} \|\xi(t)\|_{L^{p'}} \le T \sup_{0 \le t \le T} \|f(t)\|_{L^{p'}} + \|\xi_0\|_{L^{p'}},$$

$$\sup_{0 \le t \le T} \|\xi(t)\|_{L^q} \le T \sup_{0 \le t \le T} \|f(t)\|_{L^q} + \|\xi_0\|_{L^q}.$$

Furthermore, it holds that for every $t_1, t_2 \in [0, T]$

$$\int_\Omega j^*(\xi(x, t_2))dx - \int_\Omega j^*(\xi(x, t_1))dx + \int_{t_1}^{t_2} \|\nabla u(t)\|_{L^p}^p dt = \int_{t_1}^{t_2} \int_\Omega f(x, t)u(x, t)dxdt.$$

With additional conditions of initial data, we can obtain a solution which is Lipschitz continuous with respect to t:

Theorem 3.2

In addition to assumptions in Theorem 3.1, let $\Delta_p u_0 \in L^{p'}(\Omega)$. Then there exist at least one solution to (P) with the initial data (u_0, ξ_0) satisfying

$$\sup_{0\le t\le T} \|\xi(t)\|_{L^{p'}} \le T \sup_{0\le t\le T} \|f(t)\|_{L^{p'}} + \|\xi_0\|_{L^{p'}},$$

$$\sup_{0\le t\le T} \|\xi(t)\|_{L^q} \le T \sup_{0\le t\le T} \|f(t)\|_{L^q} + \|\xi_0\|_{L^q},$$

$$\|\xi(t_1) - \xi(t_2)\|_{L^1} \le C|t_1 - t_2| \quad \forall t_1, t_2 \in [0, T].$$

§3. Sketch of proof

We adopt the standard time discretization technique (Raviart 1970, Grange–Mignot 1972).
Let $N \in \mathbb{N}$ and $\tau := T/N$. Then define $u_\tau = \{u_\tau^0, u_\tau^1, \dots, u_\tau^N\}$ and $\xi_\tau = \{\xi_\tau^0, \xi_\tau^1, \dots, \xi_\tau^N\}$ by

$$\begin{cases} \dfrac{\xi_\tau^{n+1}(x) - \xi_\tau^n(x)}{\tau} - \Delta_p u_\tau^{n+1}(x) = f_\tau^n(x) & x \in \Omega, \\ \xi_\tau^{n+1}(x) \in \beta(u_\tau^{n+1}(x)) & x \in \Omega, \\ u_\tau^{n+1}(x) = 0 & x \in \partial\Omega, \end{cases}$$

where $u_\tau^0 := u_0$, $\xi_\tau^0 := \xi_0$, and $f_\tau^n := \frac{1}{\tau} \int_{n\tau}^{(n+1)\tau} f(\cdot, s)ds$.
We next connect $u_\tau = \{u_\tau^0, u_\tau^1, \dots, u_\tau^N\}$ and $\xi_\tau = \{\xi_\tau^0, \xi_\tau^1, \dots, \xi_\tau^N\}$ between $[0, T]$ by

$$\Pi_\tau u_\tau(t) := \begin{cases} u_\tau^{n+1} & \text{if } t \in (n\tau, (n+1)\tau], \\ u_\tau^0 & \text{if } t = 0, \end{cases}$$

$$\Lambda_\tau \xi_\tau(t) := \frac{\xi_\tau^{n+1} - \xi_\tau^n}{\tau}(t - n\tau) + \xi_\tau^n \quad \text{if } t \in [n\tau, (n+1)\tau].$$

By discussing the limits as $\tau \to 0$, we observe the convergence $\Pi_\tau u_\tau$ and $\Lambda_\tau \xi_\tau$ to a desired solution to (P).

Therefore to prove Theorem 3.1, we have to assure the solvability of

$$
\text{(E)} \quad \begin{cases} \xi(x) - \Delta_p u(x) = h(x) & x \in \Omega, \\ \xi(x) \in \beta(u(x)) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega. \end{cases}
$$

Theorem 3.3

Let $p \in (1,\infty)$, $q \in (1,\infty]$, and $0 \in \beta(0)$. Then for every $h \in L^{p'}(\Omega) \cap L^q(\Omega)$, (E) possesses a unique solution $u \in W_0^{1,p}(\Omega)$ such that $\xi, \Delta_p u \in L^{p'}(\Omega) \cap L^q(\Omega)$ and

$$
\|\xi\|_{L^{p'}} \leq \|h\|_{L^{p'}}, \quad \|\xi\|_{L^q} \leq \|h\|_{L^q}.
$$

Formally, we can interpret the inequalities above as the result of integration by parts:

$$
\int_\Omega |\beta(u)|^{q-2}\beta(u)\Delta_p u dx = -(q-1)\int_\Omega |\beta(u)|^{q-2}\beta'(u)|\nabla u|^p dx \leq 0,
$$

where $\beta' \geq 0$ since β is monotone.

§3. Sketch of proof

- If $h \in W^{-1,p'}(\Omega)$, the following functional possesses a minimizer:

$$I(u) := \psi(u) + \frac{1}{p}\|\nabla u\|_{L^p}^p - \int_\Omega h(x)u(x)dx,$$

where $\psi(u) := \displaystyle\int_\Omega j(u(x))dx \geq 0$ and j is a primitive function of β (i.e., $\beta = \partial j$).

- When we deal with the functional I on $W_0^{1,p}(\Omega)$, we have $\partial_{W_0^{1,p}} I(u) = \partial_{W_0^{1,p}}\psi(u) - \Delta_p u - h$ and the minimizer satisfies $0 \in \partial_{W_0^{1,p}} I(u)$.
However, $\xi \in \partial_{W_0^{1,p}}\psi(u)$ may NOT satisfy $\xi(x) \in \beta(u(x))$ a.e. Ω unless $D(\beta) = \mathbb{R}$ (cf. Brézis 1972).

§3. Sketch of proof

- Then we first consider

$$I_\lambda(u) := \int_\Omega j_\lambda(u(x))dx + \frac{1}{p}\|\nabla u\|_{L^p}^p - \int_\Omega h(x)u(x)dx$$

  in $L^p(\Omega)$. Then it holds that $\partial_{L^p}\psi_\lambda(u) = \widetilde{\beta_\lambda}(u)$
  (β_λ: Yosida approximation of β in $\mathbb{R}$, j_λ: Moreau–Yosida regularization of j in $\mathbb{R}$,
  $\widetilde{\beta_\lambda}$: realization of β_λ in $L^p(\Omega) \times L^{p'}(\Omega)$).
- Remark that $\displaystyle\int_\Omega j_\lambda(u)dx$ may NOT coincide with
  $\displaystyle\psi_\lambda(u) := \inf_{v \in L^p(\Omega)} \left\{ \frac{\|u-v\|_{L^p}^2}{2\lambda} + \int_\Omega j(v)dx \right\}$ unless $p = 2$. Moreover,
  - $\widetilde{\beta_\lambda}$ is Lipschitz continuous $L^p \to L^p$, but NOT $L^p \to L^{p'}$ if $1 < p < 2$.
  - The domain of $\widetilde{\beta_\lambda}$ dose NOT coincide with $L^p(\Omega)$ when $1 < p < 2$ (e.g., $\beta = Id$).

§3. Sketch of proof

Let $1 < p < 2$ and consider

$$I_\lambda(u) := \frac{\lambda}{2}\|u\|_{L^2}^2 + \int_\Omega j_\lambda(u) + \frac{1}{p}\|\nabla u\|_{L^p}^p - \int_\Omega h(x)u(x)dx$$

in L^2. Then the minimizer $u_\lambda \in L^2(\Omega) \cap W_0^{1,p}(\Omega)$ of I_λ satisfies $\lambda u_\lambda + \beta_\lambda(u_\lambda) - \Delta_p u_\lambda = h$ in L^2. To establish uniform boundedness, multiply it by $k_m^q(\beta_\lambda(u_\lambda))$ for $q < 2$ or $K_M^q(\beta_\lambda(u_\lambda))$ for $q \geq 2$, where

$$k_m^q(s) := \begin{cases} |s|^{q-2}s & \text{if } |s| \geq m, \\ m^{q-2}s & \text{if } |s| \leq m, \end{cases} \quad K_M^q(s) := \begin{cases} |s|^{q-2}s & \text{if } |s| \leq M, \\ M^{q-1}\,\mathrm{sgn}(s) & \text{if } |s| \geq M. \end{cases}$$

Letting $m \to 0$ or $M \to \infty$, we obtain $\|\beta_\lambda(u_\lambda)\|_{L^q} \leq \|h\|_{L^q}$.

By using this estimate and letting $\lambda \to 0$, we can assure the solvability of (E).

Applying Theorem 3.3 to

$$(\mathrm{P})_\tau^{n+1} \quad \begin{cases} \dfrac{\xi_\tau^{n+1}(x) - \xi_\tau^n(x)}{\tau} - \Delta_p u_\tau^{n+1}(x) = f_\tau^n(x) & x \in \Omega, \\ \xi_\tau^{n+1}(x) \in \beta(u_\tau^{n+1}(x)) & x \in \Omega, \\ u_\tau^{n+1}(x) = 0 & x \in \partial\Omega, \end{cases}$$

we have

$$\|\xi_\tau^{n+1}\|_{L^q} \le \|\xi_\tau^n + \tau f_\tau^n\|_{L^q} \le \|\xi_\tau^n\|_{L^q} + \tau \sup_{0 \le t \le T} \|f(t)\|_{L^q}.$$

which leads to (remark $\tau = T/N$)

$$\sup_{n=1,2,\ldots,N} \|\xi_\tau^n\|_{L^q} \le \|\xi_0\|_{L^q} + T \sup_{0 \le t \le T} \|f(t)\|_{L^q}$$

i.e.,

$$\sup_{0 \le t \le T} \|\Lambda_\tau \xi_\tau(t)\|_{L^q} \le \|\xi_0\|_{L^q} + T \sup_{0 \le t \le T} \|f(t)\|_{L^q} \quad \to \quad \sup_{0 \le t \le T} \|\xi(t)\|_{L^q} \le \|\xi_0\|_{L^q} + T \sup_{0 \le t \le T} \|f(t)\|_{L^q}$$

as $\tau \to 0$.

To prove Theorem 3.2, we need the following lemma:

Lemma

Let $h_i \in L^{p'}(\Omega)$ and (u_i, ξ_i) be the unique solution to

$$
(\mathrm{E})_i \quad \begin{cases} \xi_i(x) - \Delta_p u_i(x) = h_i(x) & x \in \Omega, \\ \xi_i(x) \in \beta(u_i(x)) & x \in \Omega, \\ u_i(x) = 0 & x \in \partial\Omega, \end{cases}
$$

such that $\xi_i, \Delta_p u_i \in L^{p'}(\Omega)$, where $i = 1, 2$. Then

$$
\|\xi_1 - \xi_2\|_{L^1} \leq \|h_1 - h_2\|_{L^1}.
$$

($\because$) Let $u_{\lambda i}$ $(i = 1, 2)$ be a unique solution to

$$
(\mathrm{E})_{\lambda i} \quad \begin{cases} \beta_\lambda(u_{\lambda i}(x)) - \Delta_p u_{\lambda i}(x) = h_i(x) & x \in \Omega, \\ u_{\lambda i}(x) = 0 & x \in \partial\Omega. \end{cases}
$$

Remark that $\|\beta_\lambda(u_{\lambda i}(x))\|_{L^{p'}}$ is uniformly bounded
and $(u_{\lambda i}, \beta_\lambda(u_{\lambda i}(x)))$ converges to (u_i, ξ_i) (unique solution to $(\mathrm{E})_i$) as $\lambda \to 0$.

§3. Sketch of proof

Testing (E)$_{\lambda 1}$−(E)$_{\lambda 2}$ by $\mathrm{sgn}^\circ(u_{\lambda 1} - u_{\lambda 2})$ (sgn°: minimal section of sgn), we have

$$\int_{\{x\in\Omega;\ u_{\lambda 1}(x)\neq u_{\lambda 2}(x)\}} |\beta_\lambda(u_{\lambda 1}(x)) - \beta_\lambda(u_{\lambda 2}(x))|dx \le \|h_1 - h_2\|_{L^1},$$

$$\Rightarrow \ \|\beta_\lambda(u_{\lambda 1}) - \beta_\lambda(u_{\lambda 2})\|_{L^1} \le \|h_1 - h_2\|_{L^1}.$$

Applying Dunford-Pettis's theorem, we have $\|\xi_1 - \xi_2\|_{L^1} \le \|h_1 - h_2\|_{L^1}$. □

Recall that $u_\tau = \{u_\tau^0, u_\tau^1, \dots, u_\tau^N\}$ and $\xi_\tau = \{\xi_\tau^0, \xi_\tau^1, \dots, \xi_\tau^N\}$ are defined by

$$(\mathrm{P})_\tau^{n+1} \quad \begin{cases} \dfrac{\xi_\tau^{n+1}(x) - \xi_\tau^n(x)}{\tau} - \Delta_p u_\tau^{n+1}(x) = f_\tau^n(x) & x \in \Omega, \\ \xi_\tau^{n+1}(x) \in \beta(u_\tau^{n+1}(x)) & x \in \Omega, \\ u_\tau^{n+1}(x) = 0 & x \in \partial\Omega, \end{cases}$$

where $u_\tau^0 := u_0$, $\xi_\tau^0 := \xi_0$, $f_\tau^n := \frac{1}{\tau}\int_{n\tau}^{(n+1)\tau} f(\cdot, s)ds$.
Moreover, $\xi_\tau^{-1} := \xi_0 - \tau\Delta_p u_0 - \tau f_\tau^0, \quad f_\tau^{-1} \equiv f(\cdot, 0)$.

Applying Lemma to $(\mathrm{P})_\tau^{n+1}$ and $(\mathrm{P})_\tau^{n}$, we get

$$\|\xi_\tau^{n+1} - \xi_\tau^{n}\|_{L^1} \leq \tau \| f_\tau^{n} - f_\tau^{n-1}\|_{L^1} + \|\xi_\tau^{n} - \xi_\tau^{n-1}\|_{L^1}$$

for any $n = 0, 1, \ldots, N-1$. Since $\sum_{n=0}^{N-1} \| f_\tau^{n} - f_\tau^{n-1}\|_{L^1} \leq \int_{-\tau}^{T} \left\| \frac{df}{dt} \right\|_{L^1} dt$, we obtain

$$\begin{aligned}\left\| \frac{\xi_\tau^{n+1} - \xi_\tau^{n}}{\tau} \right\|_{L^1} &\leq \sum_{n=0}^{N-1} \| f_\tau^{n} - f_\tau^{n-1}\|_{L^1} + \left\| \frac{\xi_\tau^{0} - \xi_\tau^{-1}}{\tau} \right\|_{L^1} \\ &\leq \int_{-\tau}^{T} \left\| \frac{df}{dt} \right\|_{L^1} dt + \|\Delta_p u_0 + f_\tau^0\|_{L^1}\end{aligned}$$

(we need $\Delta_p u_0 \in L^{p'}(\Omega)$ in order to apply Lemma to $(\mathrm{P})_\tau^{0}$), which leads to

$$\|\Lambda_\tau \xi_\tau(t_1) - \Lambda_\tau \xi_\tau(t_2)\|_{L^1} \leq C|t_1 - t_2|.$$

Therefore by letting $\tau \to 0$, we can assure that the solution constructed in the proof of Theorem 3.1 satisfy $\|\xi(t_1) - \xi(t_2)\|_{L^1} \leq C|t_1 - t_2|$ when $\Delta_p u_0 \in L^1(\Omega)$.