Equivalence of Solutions of Eikonal Equation on Metric Spaces

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Joint work with

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Introduction

- Let $(\mathcal{X}, d)$ be a complete metric space and Ω be a bounded open set of $\mathcal{X}$.

  A special case: $\mathcal{X}$ is a complete **length space**, that is, for any $x, y \in \mathcal{X}$,

$$d(x, y) = \inf\{\text{length of } \xi\text{: } \xi \text{ is a Lipschitz curve joining } x \text{ and } y\}.$$

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  $$u = g \quad \text{on } \partial\Omega,$$

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  ◇ Loss of measure, inner product, and smooth function class

  ◇ Possible lack of local compactness

The eikonal equation

Consider $|\nabla u| = 1$ in $\Omega \subset \mathbb{R}^n$ with $u = g \in C(\partial\Omega)$.

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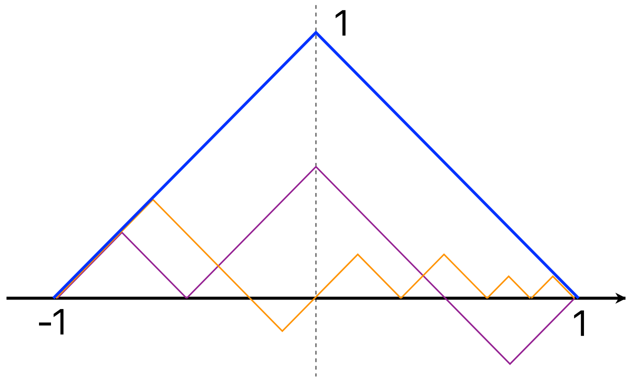

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$\Omega = (-1, 1)$, $g(\pm 1) = 0$

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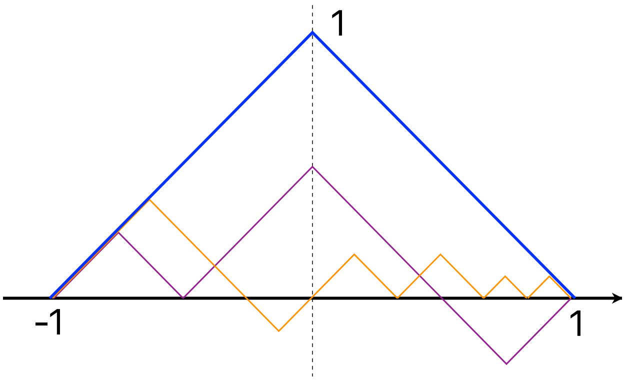

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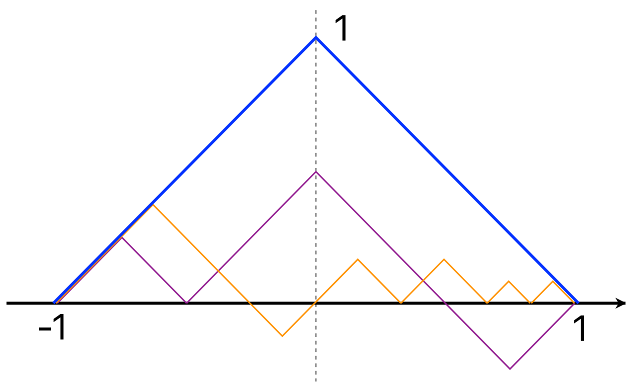

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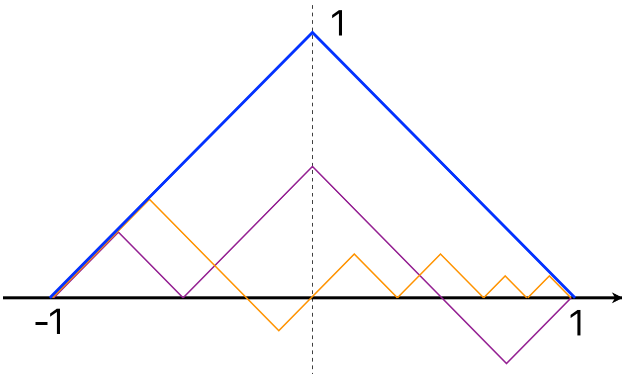

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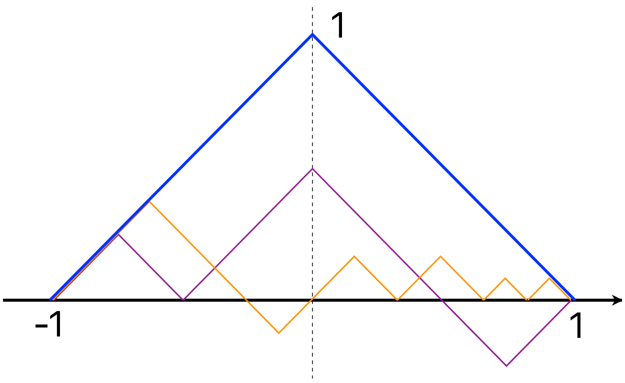

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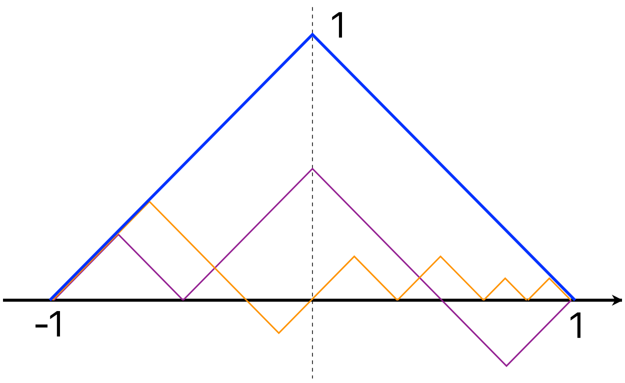

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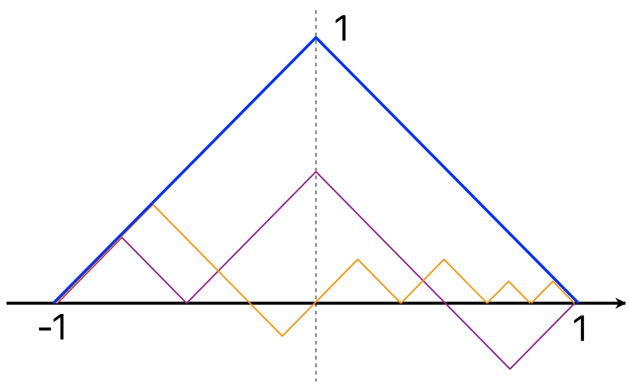

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4) Monge solutions
   (sub-slope) $$\limsup_{y \to x} \frac{[u(x) - u(y)]_+}{|x - y|} = 1, \ \forall x$$

Definitions of metric viscosity solutions

Several notions of metric viscosity solutions will be mentioned:

(1) Curve-based solutions (c-solutions) **using optimal control**

- ◇ [Giga-Hamamuki-Nakayasu '14] [Nakayasu '14]
- ◇ Very weak space structures, strong assumptions on PDEs

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- ◇ Our new notion extends [Newcomb-Su '95] in $\mathbb{R}^n$ to metric spaces
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Curves

- Let $\mathcal{A}_x(\mathbb{R}, \mathcal{X})$ be the set of Lipschitz curves ξ in $\mathcal{X}$ with $\xi(0) = x$ and

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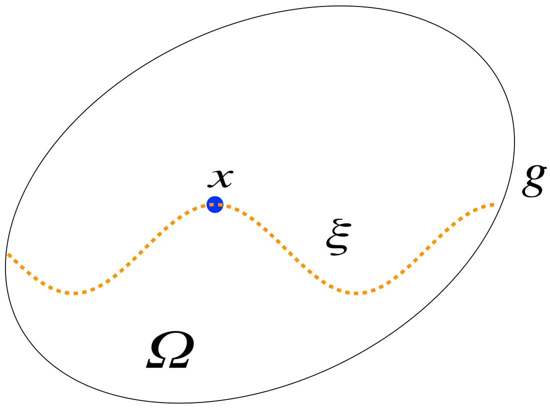

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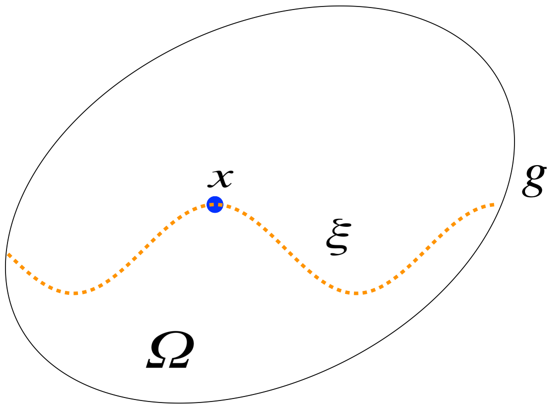

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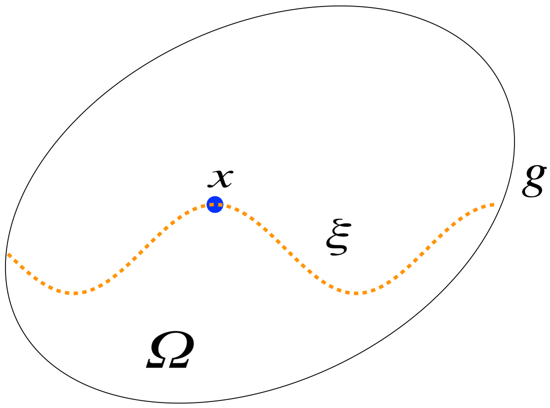

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- At least formally, we can view
$$|\nabla u|(x) = \sup_{\xi \in \mathcal{A}_x(\mathbb{R}, \mathcal{X})} (u \circ \xi)'(0)$$

Curve-based solutions of $|\nabla u| = 1$

Definition [Giga-Hamamuki-Nakayasu '14]

- $u \in USC(\Omega)$ is called a c-subsolution if for any $x \in \Omega$ and $\xi \in \mathcal{A}_x(\mathbb{R}, \mathcal{X})$, $u \circ \xi$ satisfies $|(u \circ \xi)'(0)| \leq 1$ in the viscosity sense, i.e.,

$$|\phi'(0)| \leq 1$$

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- u is called a local c-supersolution if $|w'| \geq 1 - \varepsilon$ in $(-\delta, \delta)$ for $\delta = \delta(x) > 0$ small instead of (T^-, T^+) . We can accordingly define local c-solutions.

Slopes and test classes

Let Ω be an open set of a **length space** $\mathcal{X}$. For $u \in \mathrm{Lip}_{loc}(\Omega)$ and $x \in \Omega$, let

- local slope $$|\nabla u|(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{d(x, y)}$$

  (usc envelope) $$|\nabla u|^*(x) = \limsup_{y \to x} |\nabla u|(y)$$

- super/sub-slope $$|\nabla^{\pm} u|(x) = \limsup_{y \to x} \frac{[u(y) - u(x)]_{\pm}}{d(x, y)} \qquad ([a]_{\pm} := \max\{\pm a, 0\})$$

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Test classes:

$$\overline{\mathcal{C}}(\Omega) := \left\{u \in \mathrm{Lip}_{loc}(\Omega) \,:\, |\nabla^{+} u| = |\nabla u| \text{ and } |\nabla u| \text{ is continuous in } \Omega\right\} \qquad \bigvee$$

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Definition [Gangbo-Święch '15]

Let Ω be an open subset of a complete **length space** $(\mathcal{X}, d)$.

- $u \in USC(\Omega)$ is called an s-subsolution if

$$|\nabla \psi_1|(x) - |\nabla \psi_2|^*(x) \leq 1$$

  holds for any $\psi_1 \in \underline{\mathcal{C}}(\Omega)$ and $\psi_2 \in \mathrm{Lip}_{loc}(\Omega)$ such that $u - \psi_1 - \psi_2$ attains a local maximum at $x \in \Omega$.

- $u \in LSC(\Omega)$ is called an s-supersolution if

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- $u \in C(\Omega)$ is called an s-solution if it is an s-sub and an s-supersolution.

Monge solutions of $|\nabla u| = 1$

Definition (Generalization of [Newcomb-Su '95] in length spaces)

Let Ω be an open subset of a complete **length space** $(\mathcal{X}, d)$.

- $u \in \mathrm{Lip}_{loc}(\Omega)$ is called a Monge subsolution if, at any $x \in \Omega$,

$$|\nabla^- u|(x) \left(= \limsup_{y \to x} \frac{[u(x) - u(y)]_+}{d(x, y)} \right) \leq 1.$$

- $u \in \mathrm{Lip}_{loc}(\Omega)$ is called a Monge supersolution if, at any $x \in \Omega$,

$$|\nabla^- u|(x) \geq 1.$$

- $u \in \mathrm{Lip}_{loc}(\Omega)$ is called a Monge solution if it is both a Monge sub- and supersolution, i.e., at any $x \in \Omega$,

$$|\nabla^- u|(x) = 1.$$

Main result

Consider the eikonal equation:

$$|\nabla u| = 1 \quad \text{in } \Omega. \tag{1}$$

Theorem (Equivalence of solutions)

Let Ω be an open set of a complete **length space** $(\mathcal{X}, d)$. Let $u \in C(\Omega)$. Then the following statements are equivalent:

(a) u is a local c-solution of (1);

(b) u is a locally uniformly continuous s-solution of (1);

(c) u is a Monge solution of (1).

In addition, if any of (a)–(c) holds, then u is locally Lipschitz with

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- The local uniform continuity can be dropped if $\mathcal{X}$ is locally compact.
- It turns out that $u \in \underline{C}(\Omega)$ (certain weak concavity).

Remarks

- Under the same assumptions, we can generalize our result for

$$|\nabla u| = f(x) \quad \text{in } \Omega,$$

if $f > 0$ is locally uniformly continuous (or continuous if $\mathcal{X}$ is loc. cpt.).

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- If $(\mathcal{X}, d)$ is not a length space but only a rectifiably connected space, take

$$\tilde{d}(x, y) = \inf \{\text{length of } \xi : \xi \text{ is a rectifiable curve connecting } x \text{ and } y\}.$$

  We can still apply our result, since $(\mathcal{X}, \tilde{d})$ is a **length space**.

Conclusion and future problems

- The known metric viscosity solution are equivalent in rectifiably connected metric spaces, especially in length spaces.
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$$\partial_t u + H(x, |\nabla u|) = 0 \quad \text{in } \mathcal{X} \times (0, \infty).$$

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Thank you for your kind attention!