Stability of steady states for geometric evolution equations

高坂良史
(神戸大学)

第 45 回偏微分方程式論札幌シンポジウム
– 久保田幸次先生を追悼して –
(2020 年 8 月 17 日-19 日)

Let $\Gamma_t \subset \mathbb{R}^3$ be an evolving surface with respect to time t.
The evolution of Γ_t is governed by the geometric evolution law

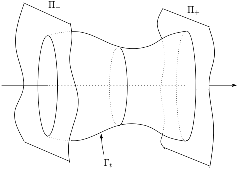
$$V = -\Delta_{\Gamma_t} H \quad \text{(Surface diffusion eq.)},$$

where V is the normal velocity of the surface Γ_t, H is the mean curvature of Γ_t, Δ_{Γ_t} is the Laplace-Beltrami operator on the surface Γ_t.

Variational Structure [Taylor-Cahn (1994)]

The H^{-1}-gradient flow of the area functional.

$\Rightarrow$ These flows decreases the area of the surface Γ_t with respect to time t under the volume constraint.



Considering the capillary energy

$$\text{Area}\,[\boldsymbol{\Gamma_t}] + \boldsymbol{\mu_+}\text{Area}\,[\boldsymbol{\Sigma_{t,+}}] + \boldsymbol{\mu_-}\text{Area}\,[\boldsymbol{\Sigma_{t,-}}],$$

we derive the following problem:

$$\text{(SDB)}\quad \begin{cases} \boldsymbol{V = -\Delta_{\Gamma_t} H} \quad \text{on} \quad \boldsymbol{\Gamma_t}, \\ \boldsymbol{(N, N_{\Pi_\pm})_{\mathbb{R}^3} = \cos\theta_\pm} \quad \text{on} \quad \boldsymbol{\Gamma_t \cap \Pi_\pm}, \\ \boldsymbol{(\nabla_{\Gamma_t} H, \nu_\pm)_{\mathbb{R}^3} = 0} \quad \text{on} \quad \boldsymbol{\Gamma_t \cap \Pi_\pm}, \end{cases}$$

where $\boldsymbol{N}$ and $\boldsymbol{N_{\Pi_\pm}}$ are the outward unit normals to $\boldsymbol{\Gamma_t}$ and $\boldsymbol{\Pi_\pm}$, respectively, and $\boldsymbol{\nu_\pm}$ is the outward unit co-normals to $\boldsymbol{\Gamma_t}$ on $\boldsymbol{\Gamma_t \cap \Pi_\pm}$.

Steady states

Let $\boldsymbol{\Gamma}_*$ be a steady state for (SDB) and $\boldsymbol{H}_*$ be the mean curvature of $\boldsymbol{\Gamma}_*$. Then $\boldsymbol{H}_*$ satisfies

$$\begin{cases} \boldsymbol{\Delta}_{\boldsymbol{\Gamma}_*} \boldsymbol{H}_* = \boldsymbol{0} \quad \text{on} \quad \boldsymbol{\Gamma}_*, \\ (\boldsymbol{\nabla}_{\boldsymbol{\Gamma}_*} \boldsymbol{H}_*, (\boldsymbol{\nu}_*)_\pm)_{\mathbb{R}^3} = \boldsymbol{0} \quad \text{on} \quad \boldsymbol{\Gamma}_* \cap \boldsymbol{\Pi}_\pm, \end{cases}$$

so that

$$\|\boldsymbol{\nabla}_{\boldsymbol{\Gamma}_*} \boldsymbol{H}_*\|^2_{L^2(\boldsymbol{\Gamma}_*)} = \boldsymbol{0} \quad \Leftrightarrow \quad \boldsymbol{H}_* = \text{constant}.$$

Thus we see

steady states $\boldsymbol{\Gamma}_*$ $\Leftrightarrow$ CMC surfaces

In this talk, we only consider axisymmetric case. That is, the steady states are the Delaunay surfaces.

Known results (as the variational problem)

- Athanassenas (1987), Vogel (1987, 1989)

  Between two parallel planes as the variational problem. $\theta_{\pm} = \dfrac{\pi}{2}$.

  cylinders are stable if $\dfrac{d}{r} \leq \pi$ and unstable if $\dfrac{d}{r} > \pi$;

  unduloids are unstable,

  where d and r are the length and the radius of a cylinder, respectively.

  ⋆Fig.

- Vogel (2006, 2013, 2014) Between two balls. (partial results.)

- Fel and Rubinstein (2013, 2015) The stability criterion. (precise results.)

Aim of this talk

Our aim

To obtain the criteria of the linearized stability for the Delaunay surfaces, which are steady states of (SDB), between two axisymmetric surfaces $\mathbf{\Pi}_{\pm}$.

Strategy

Analyze the eigenvalue problem corresponding to the linearized problem of (SDB) and derive the condition that the sign of eigenvalues changes.

Set $\Gamma_* = \{(x_*(s), y_*(s)\cos\theta, y_*(s)\sin\theta)^T \mid s \in [0, d], \theta \in [0, 2\pi]\}$ where s is the arc-length parameter.

Representation formula by Kenmotsu or Myshkis et al. or Hadzhilazova et al.

For $H_* < 0$, $B \geq 0$,

$$x_*(s) = \int_0^s \frac{1 - B\sin(2H_*(\sigma - \tau))}{\sqrt{1 + B^2 - 2B\sin(2H_*(\sigma - \tau))}}\, d\sigma,$$

$$y_*(s) = -\frac{1}{2H_*}\sqrt{1 + B^2 - 2B\sin(2H_*(s - \tau))}.$$

Γ_* are cylinders if $B = 0$, unduloids if $0 < B < 1$, (series of) spheres if $B = 1$, and nodoids if $B > 1$.

Assume that $\Pi_\pm$ are axisymmetric surfaces. Apply an axisymmetric perturbation from Γ_*, which is the Delaunay surface, and linearize them. Then the corresponding eigenvalue problem is given by

$$\text{(EVP)} \begin{cases} -\Delta_{\Gamma_*} L[w] = \lambda w \quad \text{for} \quad s \in (0,d), \\ \partial_s w \pm (\kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm) w = 0 \quad \text{at} \quad s = 0, d, \\ \partial_s L[w] = 0 \quad \text{at} \quad s = 0, d \end{cases}$$

for w satisfying $\int_0^d w y_* \, ds = 0$, where

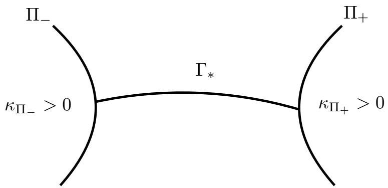
$$\Delta_{\Gamma_*} = (1/y_*)\partial_s(y_*\partial_s), \quad L[v] = \Delta_{\Gamma_*} v + |A_*|^2 v,$$

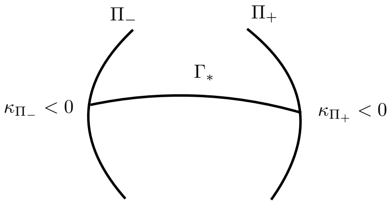
κ_{Γ_*} : the curvature of the generating curve of Γ_*,

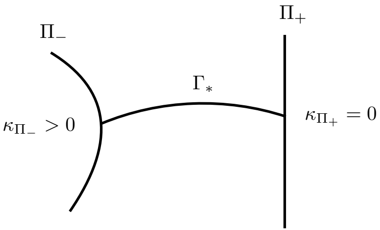
$\kappa_{\Pi_\pm}$: the curvature of $\Pi_\pm$ on $\Gamma_* \cap \Pi_\pm$).

We say that the steady states Γ_* is linearly stable under an axisymmetric perturbation if and only if all of eigenvalues of (EVP) are negative.

The sign convention of $\kappa_{\Pi_\pm}$







▸ stablization-1

Now we obtain the following properties:

- $\{\lambda_n\} \subset \mathbb{R}$ (i.e. Linearized operator is self-adjoint in the suitable space) with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$. That is, if $\lambda_1 < 0$, Γ_* are linearly stable.
- The eigenvalues depend continuously on $\kappa_{\Pi_\pm}$, κ_{Γ_*} and d, and are monotone decreasing in $\kappa_{\Pi_\pm}$.
- $\lambda_1 \leq 0$ if d is small enough and $\kappa_{\Pi_\pm}$ are positive and large enough.
- 0 is an eigenvalue of (EVP) if and only if the parameters $\kappa_{\Pi_\pm}$, H_*, B, d, τ, $\theta_\pm$ fulfill

$$A^w(H_*, B, d, \tau)\kappa_{\Pi_-}\kappa_{\Pi_+} + B^w_-(H_*, B, d, \tau, \theta_+)\kappa_{\Pi_-} + B^w_+(H_*, B, d, \tau, \theta_-)\kappa_{\Pi_+} + C^w(H_*, B, d, \tau, \theta_\pm) = 0. \quad \cdots (\star)$$

Here $A^w, B^w_\pm, C^w$ depend on the configuration of Γ_* and satisfy $B^w_- B^w_+ - A^w C^w \geq 0$.

▸ ⋆precise form

Case I: $A^w \neq 0$ and $B^w_- B^w_+ - A^w C^w \neq 0$.

$$(\star) \quad \Leftrightarrow \quad \kappa_{\Pi_+} = -\frac{B^w_-}{A^w} + \frac{\dfrac{B^w_- B^w_+ - A^w C^w}{(A^w)^2}}{\kappa_{\Pi_-} - \left(-\dfrac{B^w_+}{A^w}\right)}.$$

Case II: $A^w \neq 0$ and $B^w_- B^w_+ - A^w C^w = 0$.

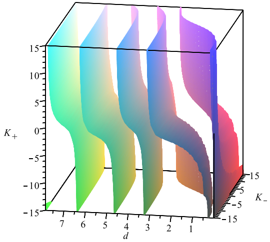
$$(\star) \quad \Leftrightarrow \quad \left\{\kappa_{\Pi_-} - \left(-\frac{B^w_+}{A^w}\right)\right\}\left\{\kappa_{\Pi_+} - \left(-\frac{B^w_-}{A^w}\right)\right\} = 0.$$

Case III: $A^w = 0$.

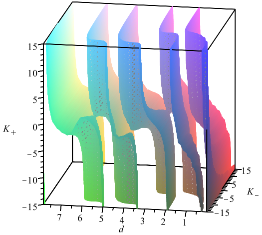
$$(\star) \quad \Leftrightarrow \quad B^w_- \kappa_{\Pi_-} + B^w_+ \kappa_{\Pi_+} + C^w = 0.$$

Stability of steady states

Combining these hyperbolas and straight lines, we have the following surfaces.



(a) Case I & Case III

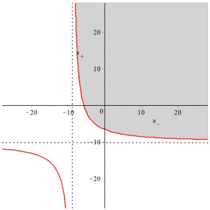


(b) Case I & Case II & Case III

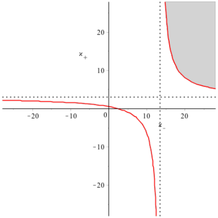
We must check the following:

- Is $B_-^w B_+^w - A^w C^w$ non-negative or non-positive ?
- Are there $d > 0$ satisfying $A^w = 0$?
- Are there $d > 0$ satisfying $B_-^w B_+^w - A^w C^w = 0$? (Today we do not care.)

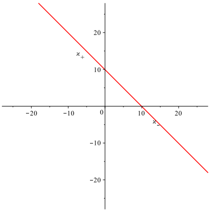
Criteria of the linearized stabilty



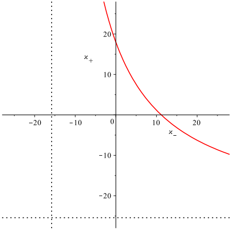
$0 < d < d_1$



$0 < d < d_1$



$d = d_1$

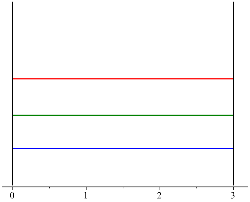


$d_1 < d < d_2$

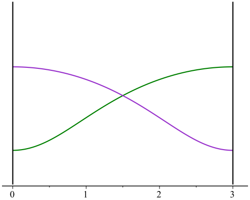
d_1 (> 0) is the 1st zero-point of A^w.

If $d \geq d_1$, there are no pairs $(\kappa_{\Pi_-}, \kappa_{\Pi_+})$ such that Γ_* is stable.

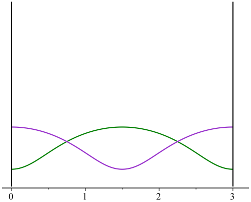
Consider the case that $\Pi_{\pm}$ are the parallel planes and $\theta_{\pm} = \frac{\pi}{2}$.



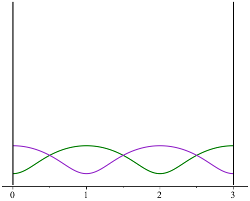
cylinder



unduloid $\left(\hat{H}_* d = \frac{\pi}{2}\right)$



unduloid $(\hat{H}_* d = \pi)$



unduloid $\left(\hat{H}_* d = \frac{3\pi}{2}\right)$

$\hat{H}_* = -H_*$

green: $\tau = \pi/(4\hat{H}_*)$, $d_1 = \pi/\hat{H}_*$

purple: $\tau = -\pi/(4\hat{H}_*)$,

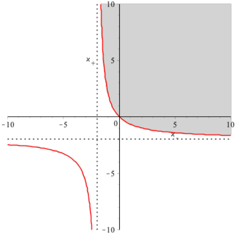
$$d_1 = \begin{cases} \pi/\hat{H}_* & (B \in (0, B_c]) \\ \hat{d}(B, \hat{H}_*) & (B \in (B_c, 1)) \end{cases}$$

$$(\hat{d}(B, \hat{H}_*) \in (\pi/2, \pi))$$

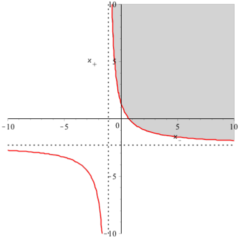
★Known Results

Stability of unduloids with $\theta_\pm = \dfrac{\pi}{2}$ and $\tau = \dfrac{\pi}{4\hat{H}_*}$

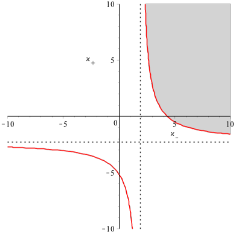
Also, we can prove the monotonicity of axes with respect to B for $\hat{H}_* d = \dfrac{\pi}{2}$.



$B = 0$ (cylinder)



$B = 0.25$

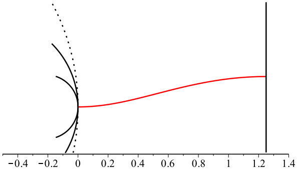


$B = 0.55$

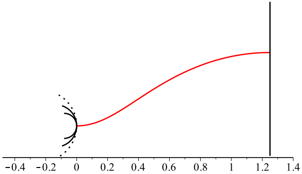
Consequently, unduloids in this setting are unstable.

stablization-1 stablization-2

Stabilization of steady states



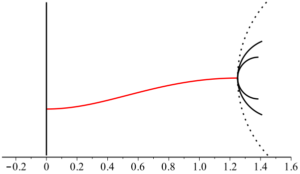
(a) $B = 0.25$. If $\kappa_{\Pi\_} > \kappa^0_{\Pi\_} \approx 0.68$, unduloids are linearly stable.



(b) $B = 0.55$. If $\kappa_{\Pi\_} > \kappa^0_{\Pi\_} \approx 4.31$, unduloids are linearly stable.

▸ sign convention ▸ criteria-unduloids

Stabilization of steady states



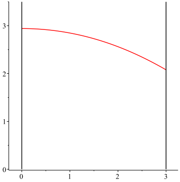
(c) $\boldsymbol{B = 0.25}$. If $\boldsymbol{\kappa_{\Pi_+} > \kappa^0_{\Pi_+} \approx 1.34}$, unduloids are linearly stable.

criteria-unduloids

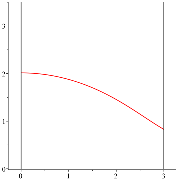
Let us consider the case of $\theta_- = \frac{\pi}{2}$, $\theta_+ = \frac{\pi}{3}$ and

$$\Pi_- = \{(0, \eta)^T \mid \eta \in \mathbb{R}^2\}, \quad \Pi_+ = \{(3, \eta)^T \mid \eta \in \mathbb{R}^2\}.$$

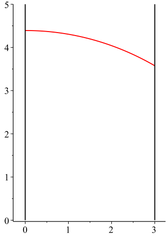
For $0 < d < d_1$, where d_1 is the first zero point of A^w, the possible configurations of the steady states are, for example, the following:



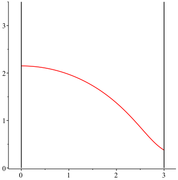
(i) The unduloid with
$B = 0.55$
$\hat{H}_* d \approx 0.83$
$\tau \approx -2.98$
$(\hat{H}_* d_1 \approx 1.90)$



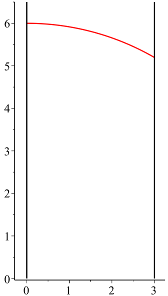
(ii) The unduloid with
$B = 0.55$
$\hat{H}_* d \approx 1.26$
$\tau \approx -2.04$
$(\hat{H}_* d_1 \approx 1.90)$



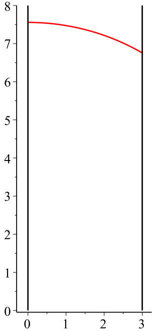
(iii) The unduloid with
$B = 0.75$
$\hat{H}_* d \approx 0.67$
$\tau \approx -3.94$
$(\hat{H}_* d_1 \approx 1.66)$



(iv) The unduloid with
$B = 0.75$
$\hat{H}_* d \approx 1.47$
$\tau \approx -1.93$
$(\hat{H}_* d_1 \approx 1.66)$



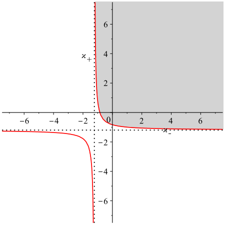
(v) The part of sphere
$B = 1.00$
$\hat{H}_* d \approx 0.52$
$\tau \approx -4.71$
$(\hat{H}_* d_1 \approx 1.67)$



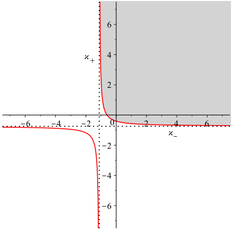
(vi) The nodoid with
$B = 1.25$
$\hat{H}_* d \approx 0.47$
$\tau \approx -5.28$
$(\hat{H}_* d_1 \approx 1.63)$

Stability of steady states with $\theta_- = \dfrac{\pi}{2}$ and $\theta_+ = \dfrac{\pi}{3}$

$\boldsymbol{B = 0.55}$

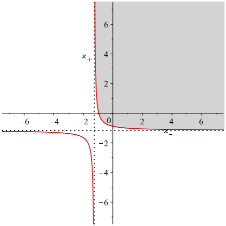


The unduloid for (i) is stable.

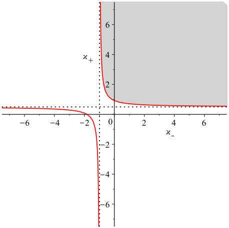


The unduloid for (ii) is stable.

$\boldsymbol{B = 0.75}$



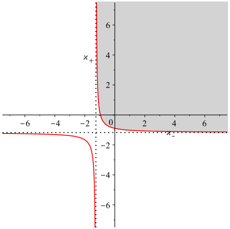
The unduloid for (iii) is stable.



The unduloid for (iv) is unstable.

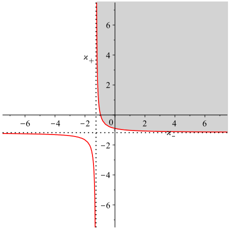
Stability of steady states with $\theta_- = \frac{\pi}{2}$ and $\theta_+ = \frac{\pi}{3}$

$\boldsymbol{B = 1.00}$



The part of sphere for (v) is stable.

$\boldsymbol{B = 1.25}$



The nodoid for (vi) is stable.

$$\hat{H}_* \tau = \frac{1}{2}\mathrm{Arcsin}\left(\frac{\cos^2\theta_- - \sqrt{B^2 - \cos^2\theta_-}\,\sin\theta_-}{B}\right)$$

(i), (iii), (v), (vi)

$$\hat{H}_*(d-\tau) = \frac{1}{2}\left\{\pi - \frac{1}{2}\mathrm{Arcsin}\left(\frac{-\cos^2\theta_+ + \sqrt{B^2 - \cos^2\theta_+}\,\sin\theta_+}{B}\right)\right\}$$

(ii), (iv)

$$\hat{H}_*(d-\tau) = \frac{1}{2}\left\{\pi - \frac{1}{2}\mathrm{Arcsin}\left(\frac{-\cos^2\theta_+ - \sqrt{B^2 - \cos^2\theta_+}\,\sin\theta_+}{B}\right)\right\}$$

Remark: $B \geq \max\{|\cos\theta_-|, |\cos\theta_+|\}$

Thank you for your attention !

By the help with Maple, we can confirm

$$
\begin{aligned}
&A^w(\hat{H}_*, B, d, \tau) \\
&= \frac{1}{8(\hat{H}_*)^3 Q(0)Q(d)}\Big[(\hat{H}_*)^2(1-B^2)^2 I_1^2 \cos(2\hat{H}_*\tau)\cos(2\hat{H}_*(d-\tau)) \\
&\quad - 4(\hat{H}_*)^2(1+B^2)I_1 I_2 \cos(2\hat{H}_*\tau)\cos(2\hat{H}_*(d-\tau)) \\
&\quad + 3(\hat{H}_*)^2 I_2^2 \cos(2\hat{H}_*\tau)\cos(2\hat{H}_*(d-\tau)) \\
&\quad + 2\hat{H}_*(1+B^2)I_1\{Q(0)\sin(2\hat{H}_*\tau)\cos(2\hat{H}_*(d-\tau)) \\
&\qquad\qquad + Q(d)\cos(2\hat{H}_*\tau)\sin(2\hat{H}_*(d-\tau))\} \\
&\quad - 4\hat{H}_* B I_1\{Q(0)\cos(2\hat{H}_*(d-\tau)) - Q(d)\cos(2\hat{H}_*\tau)\} \\
&\quad - 4\hat{H}_* I_2\{Q(0)\sin(2\hat{H}_*\tau)\cos(2\hat{H}_*(d-\tau)) \\
&\qquad\qquad + Q(d)\cos(2\hat{H}_*\tau)\sin(2\hat{H}_*(d-\tau))\} \\
&\quad + 2Q(0)Q(d)\{1+\sin(2\hat{H}_*\tau)\sin(2\hat{H}_*(d-\tau))\} \\
&\quad - ((Q(0))^2 + (Q(d))^2)\cos(2\hat{H}_*\tau)\cos(2\hat{H}_*(d-\tau))\Big].
\end{aligned}
$$

Precise form of A^w

$$B_+^w B_-^w - A^w C^w$$

$$= \frac{1}{16H_*^4 Q(0)Q(d)}\Big[-\hat{H}_*\{(1+B^2)(1+\sin(2\hat{H}_*\tau)\sin(2\hat{H}_*(d-\tau)))$$

$$-((Q(0))^2+(Q(d))^2)\}I_1 - \hat{H}_*\big(3-\sin(2\hat{H}_*(d-\tau))\sin(2\hat{H}_*\tau)\big)I_2$$

$$+Q(0)\cos(2\hat{H}_*\tau)\sin(2\hat{H}_*(d-\tau))$$

$$+Q(d)\sin(2\hat{H}_*\tau)\cos(2\hat{H}_*(d-\tau))\Big]^2$$

$$\geq 0,$$

where $\hat{H}_* = -H_* > 0$, $Q(s) = \sqrt{1+B^2+2B\sin(2\hat{H}_*(s-\tau))}$, and

$$I_1 := \int_0^d \frac{1}{\sqrt{1+B^2+2B\sin(2\hat{H}_*(\sigma-\tau))}}\,d\sigma,$$

$$I_2 := \int_0^d \sqrt{1+B^2+2B\sin(2\hat{H}_*(\sigma-\tau))}\,d\sigma.$$

▶ ⋆EV