Regularity for the stationary Navier-Stokes equations over bumpy boundaries and a local wall law

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Background: Boundary Roughness Effect

We consider viscous incompressible fluids above rough bumpy boundaries $x_2 > \varepsilon\gamma(x_1/\varepsilon)$ with γ Lipschitz and the no-slip boundary condition.

General concern

The effect of wall-roughness on fluid flows.
$\Rightarrow$ The flow may paradoxically be better behaved than flat boundaries.

The Navier wall law

The wall law is a boundary condition on the flat boundary describing an averaged effect from the $O(\varepsilon)$-scale on large scale flows of order $O(1)$. When the boundary is periodic, it gives a slip condition, with $\alpha = \alpha(\gamma)$,

$$u_1 = \varepsilon\alpha\partial_2 u_1\,, \quad u_2 = 0 \quad \text{on} \quad \partial\mathbb{R}^2_+\,.$$

- **Stationary**: Jäger · Mikelić ('01), Gérard-Varet ('09), Gérard-Varet · Masmoudi ('10)
- **Nonstationary**: Mikelić · Nečasová · Neuss-Radu ('13)
- **IBVP**: Higaki ('16)

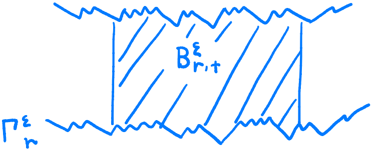
Motivation

To investigate such effects from the point of view of the regularity theory, especially, of the mesoscopic regularity of the steady Navier-Stokes flows.

2D steady Navier-Stokes equations

$$(\mathrm{NS}^\varepsilon) \qquad \begin{cases} -\Delta u^\varepsilon + \nabla p^\varepsilon = -u^\varepsilon \cdot \nabla u^\varepsilon & \text{in } B^\varepsilon_{1,+} \\ \nabla \cdot u^\varepsilon = 0 & \text{in } B^\varepsilon_{1,+} \\ u^\varepsilon = 0 & \text{on } \Gamma^\varepsilon_1 \,. \end{cases}$$

- $u^\varepsilon = (u_1^\varepsilon(x), u_2^\varepsilon(x))^\top$: velocity field
- $p^\varepsilon = p^\varepsilon(x)$: pressure field



For $\varepsilon \in (0,1]$ and $r \in (0,1]$,

$$B^\varepsilon_{r,+} = \{x \in \mathbb{R}^2 \mid x_1 \in (-r,r)\,, \quad \varepsilon\gamma(\frac{x_1}{\varepsilon}) < x_2 < \varepsilon\gamma(\frac{x_1}{\varepsilon}) + r\}\,,$$

$$\Gamma^\varepsilon_r = \{x \in \mathbb{R}^2 \mid x_1 \in (-r,r)\,, \quad x_2 = \varepsilon\gamma(\frac{x_1}{\varepsilon})\}\,.$$

- $\gamma \in W^{1,\infty}$: boundary function, $\gamma(x_1) \in (-1,0)$ for all $x_1 \in \mathbb{R}$

For an open set $\Omega \subset \mathbb{R}^3$ with the Lebesgue measure $|\Omega|$,

$$⨍_\Omega |f| = \frac{1}{|\Omega|}\int_\Omega |f|\,, \qquad (\overline{f})_\Omega = \frac{1}{|\Omega|}\int_\Omega f\,.$$

Regularity Theory

Small Scales ($\lesssim \varepsilon$) $\Rightarrow$ The Schauder theory

The small-scale regularity is determined by the regularity of data.

- Ladyženskaja ('69): Hölder estimate by potential theory
- Giaquinta · Modica ('82): the Campanato spaces

Dependence on the continuity of γ' when the boundary is $x_2 = \gamma(x_1)$.

Large Scales ($\varepsilon \lesssim r \le 1$)

The large-scale regularity is determined by the macroscopic properties.

- Gérard-Varet ('09): $C^{0,\mu}$-est. uniform in ε by a mesoscopic Hölder

$$\left(\fint_{B^\varepsilon_{r,+}} |u^\varepsilon|^2 \right)^{\frac{1}{2}} \le C(\mu) \left(\fint_{B^\varepsilon_{1,+}} |u^\varepsilon|^2 \right)^{\frac{1}{2}} r^\mu \,, \quad \mu \in (0,1) \,,$$

combined with the classical estimates near the boundary $x_2 = \varepsilon\gamma(x_1/\varepsilon)$

- Kenig · Prange ('18): linear elliptic system, mesoscopic Lipschitz
- Zhuge ('20, preprint): mesoscopic Lipschitz, the quantitative method

Main Theorems (CVPDE)

Theorem 1 (mesoscopic Lipschitz)

$\forall\, M \in (0,\infty)$, $\exists\, \varepsilon^{(1)} = \varepsilon^{(1)}(\|\gamma\|_{W^{1,\infty}}, M) \in (0,1)$ s.t.
$\forall\, \varepsilon \in (0,\varepsilon^{(1)}]$, $\forall\, r \in [\varepsilon/\varepsilon^{(1)}, 1]$, any weak solution u^ε to (NS$^\varepsilon$) with

$$(*) \qquad \left(\fint_{B^\varepsilon_{1,+}} |u^\varepsilon|^2 \right)^{\frac{1}{2}} \leq M$$

satisfies

$$\left(\fint_{B^\varepsilon_{r,+}} |u^\varepsilon|^2 \right)^{\frac{1}{2}} \leq C^{(1)}_M r\,,$$

where the constant $C^{(1)}_M$ is independent of ε and r.

Theorem 2 (polynomial approximation)

Fix $M \in (0,\infty)$, $\mu \in (0,1)$. Then, $\exists\, \varepsilon^{(2)} = \varepsilon^{(2)}(\|\gamma\|_{W^{1,\infty}}, M, \mu) \in (0,1)$ s.t. for all weak solutions u^ε to (NS$^\varepsilon$) satisfying $(*)$, the following holds.

(i) $\forall\, \varepsilon \in (0, \varepsilon^{(2)}]$, $\forall\, r \in [\varepsilon/\varepsilon^{(2)}, 1]$, we have

$$\left(\fint_{B^\varepsilon_{r,+}} |u^\varepsilon(x) - c^\varepsilon_r x_2 \mathbf{e}_1|^2 \,\mathrm{d}x \right)^{\frac12} \le C^{(2)}_M \left(r^{1+\mu} + \varepsilon^{\frac12} r^{\frac12} \right),$$

where the coefficient $c^\varepsilon_r = c^\varepsilon_r(\|\gamma\|_{W^{1,\infty}}, M, \mu)$ is a functional of u^ε.

(ii) Let γ be 2π-periodic in addition. Then, $\exists\, \alpha = \alpha(\|\gamma\|_{W^{1,\infty}}) \in \mathbb{R}$ s.t. $\forall\, \varepsilon \in (0, \varepsilon^{(2)}]$, $\forall\, r \in [\varepsilon/\varepsilon^{(2)}, 1]$, we have

$$\left(\fint_{B^\varepsilon_{r,+}} |u^\varepsilon(x) - c^\varepsilon_r (x_2 + \varepsilon\alpha) \mathbf{e}_1|^2 \,\mathrm{d}x \right)^{\frac12} \le \widetilde{C^{(2)}_M} \left(r^{1+\mu} + \varepsilon^{\frac32} r^{-\frac12} \right).$$

Remark

The polynomial approximation requires an analysis of the boundary layer.

Remark (Consequences)

(i) When $r = O(\varepsilon)$, the estimates are no better than the one in Theorem 1. Hence there is no improvement at this scale. On the other hand, if we consider the case $r \in [(\varepsilon/\varepsilon^{(2)})^{\delta}, 1]$ with $\delta \in (0,1)$, then we see that

$$r^{1+\mu} + \varepsilon^{\frac{1}{2}} r^{\frac{1}{2}} \leq (1 + (\varepsilon^{(2)})^{\frac{1}{2}} r^{\frac{1-\delta}{2\delta} - \mu}) r^{1+\mu}\,.$$

Therefore, we call the estimates in Theorem 2 mesoscopic $C^{1,\mu}$-estimates at the scales $r \in [(\varepsilon/\varepsilon^{(2)})^{\delta}, 1]$ with $\delta \in (0, (2\mu+1)^{-1}]$.

(ii) A comparison between two estimates in Theorem 2 highlights the regularity improvement coming from the boundary periodicity: in fact,

$$\varepsilon^{\frac{3}{2}} r^{-\frac{1}{2}} < \varepsilon^{\frac{1}{2}} r^{\frac{1}{2}}\,, \quad r \in (\varepsilon, 1]\,.$$

Remark (Relation with the wall law)

(i) Let us define a polynomial P_N^ε by

$$P_N^\varepsilon(x) = (x_2 + \varepsilon\alpha)\mathbf{e}_1 \,.$$

Then P_N^ε is an explicit (shear flow) solution to

$$(\mathrm{NS}_N^\varepsilon) \qquad \begin{cases} -\Delta u_N^\varepsilon + \nabla p_N^\varepsilon = -u_N^\varepsilon \cdot \nabla u_N^\varepsilon & \text{in } \mathbb{R}^2_+ \\ \nabla \cdot u_N^\varepsilon = 0 & \text{in } \mathbb{R}^2_+ \\ u_{N,1} = \varepsilon\alpha\partial_2 u_{N,1} \,, \quad u_{N,2} = 0 & \text{on } \partial\mathbb{R}^2 \end{cases}$$

with a trivial pressure $p_N^\varepsilon = 0$.

(ii) The second estimate reads as follows: any weak solution u^ε to (NS$^\varepsilon$) can be approximated, at mesoscopic scales, by the Navier polynomial P_N^ε multiplied by a constant depending on u^ε (*a local Navier wall law*).

Strategy

We apply a compactness argument originating from the works by Avellaneda · Lin ('87, '89) on uniform estimates in homogenization.

Compactness

The mesoscopic regularity is inherited from the limit system when $\varepsilon \to 0$ posed in a domain with a flat boundary. Here no regularity is needed for the original boundary, beyond the boundedness of γ and of its gradient.

We use such regularity in order to verify the boundary layer expansion

$$\begin{cases} u^\varepsilon(x) = (\overline{\partial_2 u_1^\varepsilon})_{B^\varepsilon_{r,+}} \left(x_2 \mathbf{e}_1 + \varepsilon v(\frac{x}{\varepsilon}) \right) + o(r) \ \text{ in } \left(\fint_{B^\varepsilon_{r,+}} |\cdot|^2 \right)^{\frac{1}{2}}, \\ p^\varepsilon(x) = (\overline{\partial_2 u_1^\varepsilon})_{B^\varepsilon_{r,+}} \, q(\frac{x}{\varepsilon}) \, . \end{cases}$$

The strategies are summarized as

- Construction of the boundary layer corrector (v, q)
- Mesoscopic regularity by compactness
- Iteration of the compactness argument

Boundary Layer Corrector

The expansion $u^\varepsilon(x) \sim \varepsilon v(x/\varepsilon)$ and $p^\varepsilon(x) \sim q(x/\varepsilon)$ leads to

$$\text{(BL)} \qquad \begin{cases} -\Delta v + \nabla q = 0\,, & y \in \Omega^{\mathrm{bl}} \\ \nabla \cdot v = 0\,, & y \in \Omega^{\mathrm{bl}} \\ v(y', \gamma(y')) = -\gamma(y')\mathbf{e}_1\,, \end{cases}$$

where $\Omega^{\mathrm{bl}} = \{y \in \mathbb{R}^2 \mid y_2 > \gamma(y_1)\}$.

Proposition 1

$\exists!\, v \in H^1_{loc}(\overline{\Omega^{\mathrm{bl}}})$ to (BL) satisfying

$$\sup_{\eta \in \mathbb{Z}} \int_\eta^{\eta+1} \int_{\gamma(y')}^\infty |\nabla v(y_1, y_2)|^2 \, \mathrm{d}y_2 \, \mathrm{d}y_1 \leq C(\|\gamma\|_{W^{1,\infty}})\,.$$

(Outlined Proof) Gérard-Varet · Masmoudi ('10), Kenig · Prange ('18).

- Equivalent problem on a strip with the Dirichlet-to-Neumann op. DN
- Estimates for DN in $H^{\frac{1}{2}}_{uloc}$ (Note that $W^{1,\infty} \hookrightarrow H^{\frac{1}{2}}_{uloc}$)
- The Saint-Venant energy estimate controlling the nonlocality

Compactness Argument

(MNS$^\varepsilon$)
$$\begin{cases} -\Delta U^\varepsilon + \nabla P^\varepsilon = -\nabla\cdot(U^\varepsilon\otimes b^\varepsilon + b^\varepsilon\otimes U^\varepsilon) \\ \qquad\qquad\qquad\qquad -\lambda^\varepsilon U^\varepsilon\cdot\nabla U^\varepsilon + \nabla\cdot F^\varepsilon \quad \text{in } B^\varepsilon_{1,+} \\ \nabla\cdot U^\varepsilon = 0 \quad \text{in } B^\varepsilon_{1,+}\,, \qquad U^\varepsilon = 0 \quad \text{on } \Gamma^\varepsilon_1\,, \end{cases}$$

$$b^\varepsilon(x) = C^\varepsilon\Big(x_2\mathbf{e}_1 + \varepsilon v(\frac{x}{\varepsilon})\Big)\,, \quad x\in B^\varepsilon_{1,+}\,.$$

Note that $\nabla\cdot b^\varepsilon = 0$ in $B^\varepsilon_{1,+}$ and $b^\varepsilon = 0$ on Γ^ε_1.

The Caccioppoli inequality

$\exists\, K_0\in(0,\infty)$ depending only on $\|\gamma\|_{W^{1,\infty}}$ s.t. $\forall\,\theta\in(0,1)$, we have

$$\begin{aligned}\|\nabla U^\varepsilon\|^2_{L^2(B^\varepsilon_{\theta,+})} \le K_0\Big(&(1-\theta)^{-2}\|U^\varepsilon\|^2_{L^2(B^\varepsilon_{1,+})} \\ &+ \big(|C^\varepsilon|^4 + (1-\theta)^{-\frac{4}{3}}|C^\varepsilon|^{\frac{4}{3}}\big)\|U^\varepsilon\|^2_{L^2(B^\varepsilon_{1,+})} \\ &+ (\lambda^\varepsilon)^4(1-\theta)^{-4}\|U^\varepsilon\|^6_{L^2(B^\varepsilon_{1,+})} + \|F^\varepsilon\|^2_{L^2(B^\varepsilon_{1,+})}\Big)\,.\end{aligned}$$

Lemma 1

$\forall\,\beta \in (0,\infty)$, $\forall\,M \in (0,\infty)$, $\forall\,\mu \in (0,1)$, $\exists\,\theta_0 = \theta_0(M,\mu) \in (0,\frac{1}{8})$ s.t.
$\forall\,\gamma$ with $\|\gamma\|_{W^{1,\infty}} \le \beta$, $\forall\,\theta \in (0,\theta_0]$, $\exists\,\varepsilon_\mu = \varepsilon_\mu(\beta,M,\mu,\theta) \in (0,1)$ s.t.
$\forall\,\varepsilon \in (0,\varepsilon_\mu]$, $\forall\,(\lambda^\varepsilon, C^\varepsilon) \in [-1,1]^2$, $\forall\,F^\varepsilon \in L^2(B^\varepsilon_{1,+})^{3\times 3}$ with

$$\|F^\varepsilon\|_{L^2(B^\varepsilon_{1,+})} \le M\varepsilon_\mu\,,$$

any weak solution U^ε to (MNS$^\varepsilon$) with

$$(**) \qquad \fint_{B^\varepsilon_{1,+}} |U^\varepsilon|^2 \le M^2$$

satisfies

$$\fint_{B^\varepsilon_{\theta,+}} \left|U^\varepsilon(x) - (\overline{\partial_2 U^\varepsilon_1})_{B^\varepsilon_{\theta,+}}\big(x_2\mathbf{e}_1 + \varepsilon v(\frac{x}{\varepsilon})\big)\right|^2 \mathrm{d}x \le M^2\theta^{2+2\mu}\,.$$

Remark

We can choose the scale parameter θ freely as long as $\theta \in (0,\theta_0]$.

Lemma 2

Fix $\beta \in (0,\infty)$, $M \in (0,\infty)$, and $\mu \in (0,1)$. Let $\theta_0 \in (0,\frac{1}{8})$ be the constant in Lemma 1. Choose $\theta = \theta(M,\mu) \in (0,\theta_0]$ small to satisfy

$$4(1-\theta)^{\frac{3}{2}}\big(C_1(1-\theta^{\mu})^{-1}(6+2^8M^4)^{\frac{1}{2}}M\theta^{\frac{1}{2}}\big)^4 \le 1$$
$$\text{and}\quad C_1(1-\theta^{\mu})^{-2}(6+2^8M^4)M\theta \le 1\,,$$

where C_1 is a numerical constant. Moreover, let $\varepsilon_\mu = \varepsilon_\mu(\theta) \in (0,1)$ be the corresponding constant for θ in Lemma 1. Then, $\forall\, k \in \mathbb{N}$, $\forall\, \varepsilon \in (0, \theta^{k-1}(\theta^{2(2+\mu)}\varepsilon_\mu^2)]$, any weak sol. u^ε to (NS$^\varepsilon$) with $(*)$ satisfies

$$\fint_{B^{\varepsilon}_{\theta^k,+}} \big|u^\varepsilon(x) - a_k^\varepsilon\big(x_2\mathbf{e}_1 + \varepsilon v(\frac{x}{\varepsilon})\big)\big|^2\,\mathrm{d}x \le M^2\theta^{(2+2\mu)k}\,,$$

$$|a_k^\varepsilon| \le C_2\theta^{-\frac{3}{2}}(1-\theta)^{-1}\big(6+2^6(1-\theta)^{-2}M^4\big)^{\frac{1}{2}}M\sum_{l=1}^{k}\theta^{\mu(l-1)}\,.$$

Basic idea

Induction on $k \in \mathbb{N}$ using compactness (Lemma 1) at each step.

Difficulty

Nonlinearity and lack of smallness.

Let the estimates hold for $k \in \mathbb{N}$ and let $\varepsilon \in (0, \theta^{k+2(2+\mu)}\varepsilon_\mu^2]$.
We define $U^{\varepsilon/\theta^k} = U^{\varepsilon/\theta^k}(y)$ and $P^{\varepsilon/\theta^k} = P^{\varepsilon/\theta^k}(y)$ by

$$U^{\varepsilon/\theta^k}(y) = \frac{1}{\theta^{(1+\mu)k}}\Big(u^\varepsilon(\theta^k y) - \theta^k a_k^\varepsilon\big(y_2\mathbf{e}_1 + \frac{\varepsilon}{\theta^k}v(\frac{\theta^k y}{\varepsilon})\big)\Big),$$

$$P^{\varepsilon/\theta^k}(y) = \frac{1}{\theta^{\mu k}}\Big(p^\varepsilon(\theta^k y) - a_k^\varepsilon q(\frac{\theta^k y}{\varepsilon})\Big).$$

Then we see that, by the induction assumption,

$$\fint_{B_{1,+}^{\varepsilon/\theta^k}} |U^{\varepsilon/\theta^k}|^2 \le M^2$$

and that $(U^{\varepsilon/\theta^k}, P^{\varepsilon/\theta^k})$ is a weak solution to ...

Modified Navier-Stokes equations

$$\begin{cases} -\Delta_y U^{\varepsilon/\theta^k} + \nabla_y P^{\varepsilon/\theta^k} = -\nabla_y \cdot \left(U^{\varepsilon/\theta^k} \otimes (\theta^k b^{\varepsilon/\theta^k}) \right. \\ \qquad\qquad\qquad\qquad\qquad \left. +(\theta^k b^{\varepsilon/\theta^k}) \otimes U^{\varepsilon/\theta^k} \right) \\ \qquad\qquad\qquad\qquad\qquad -\theta^{(2+\mu)k} U^{\varepsilon/\theta^k} \cdot \nabla_y U^{\varepsilon/\theta^k} \\ \qquad\qquad\qquad\qquad\qquad + \nabla_y \cdot F^{\varepsilon/\theta^k} \quad \text{in } B_{1,+}^{\varepsilon/\theta^k} \\ \nabla_y \cdot U^{\varepsilon/\theta^k} = 0 \quad \text{in } B_{1,+}^{\varepsilon/\theta^k} \qquad U^{\varepsilon/\theta^k} = 0 \quad \text{on } \Gamma_1^{\varepsilon/\theta^k}, \end{cases}$$

where

$$b^{\varepsilon/\theta^k}(y) = C_k^\varepsilon \big(y_2 \mathbf{e}_1 + \frac{\varepsilon}{\theta^k} v(\frac{\theta^k y}{\varepsilon}) \big), \qquad C_k^\varepsilon = \theta^k a_k^\varepsilon,$$

$$F^{\varepsilon/\theta^k}(y) = -\theta^{-\mu k} \big(b^{\varepsilon/\theta^k}(y) \otimes b^{\varepsilon/\theta^k}(y) - \big(C_k^\varepsilon y_2 \mathbf{e}_1 \big) \otimes \big(C_k^\varepsilon y_2 \mathbf{e}_1 \big) \big).$$

Key ingredient

A suitable choice of the scale parameter $\theta = \theta(M)$ controlling the coefficients of M. This is done in the spirit of the Newton shooting.

Proof of Theorem 2 (i)

Fix $\mu \in (0,1)$ and set $\varepsilon^{(2)} = \theta^{2(2+\mu)}\varepsilon_\mu^2$. We take $\varepsilon \in (0, \varepsilon^{(2)}]$.
Since every $r \in [\varepsilon/\varepsilon^{(2)}, \theta]$ satisfies $r \in (\theta^k, \theta^{k-1}]$ with some $2 < k \in \mathbb{N}$,

$$\begin{aligned}
&\left(\fint_{B^\varepsilon_{r,+}} |u^\varepsilon(x) - a^\varepsilon_k x_2 \mathbf{e}_1|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \left(\theta^{-3}\fint_{B^\varepsilon_{\theta^{k-1},+}} |u^\varepsilon(x) - a^\varepsilon_k x_2 \mathbf{e}_1|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq M\theta^{(1+\mu)(k-1)-\frac{3}{2}} + \theta^{-\frac{3}{2}}|a^\varepsilon_k|\,\varepsilon\left(\fint_{B^\varepsilon_{\theta^{k-1},+}} \left|v(\frac{x}{\varepsilon})\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq M\theta^{(1+\mu)(k-1)-\frac{3}{2}} + \left(\theta^{-\frac{3}{2}} \sup_{k\in\mathbb{N}} |a^\varepsilon_k|\right) \varepsilon^{\frac{1}{2}} (\theta^{k-1})^{\frac{1}{2}}\,.
\end{aligned}$$

Then, from $\theta^{k-1} \in (0, \theta^{-1} r)$,

$$\left(\fint_{B^\varepsilon_{r,+}} |u^\varepsilon(x) - a^\varepsilon_k x_2 \mathbf{e}_1|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \leq C^{(2)}(M,\mu,\theta)(r^{1+\mu} + \varepsilon^{\frac{1}{2}} r^{\frac{1}{2}})\,.$$