Stability of Lamb dipoles

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Vortex rings

- Dates back to a pioneering work of Helmholtz in 1858
- Developed by Kelvin (formula for speed of Helmholtz's ring, variational principle, conjecture for knotted ring, vortex atom theory...)

Vortex pairs

- Heijst & Flor, *Nature*, 1989
- Symmetric & translating vortex by a constant speed
- Appears as stable vortex structure in large scale flows

2d Euler

$$\partial_t \zeta + v \cdot \nabla \zeta = 0, \ \ v = k * \zeta \quad \text{in } \mathbb{R}^2 \times (0, \infty)$$

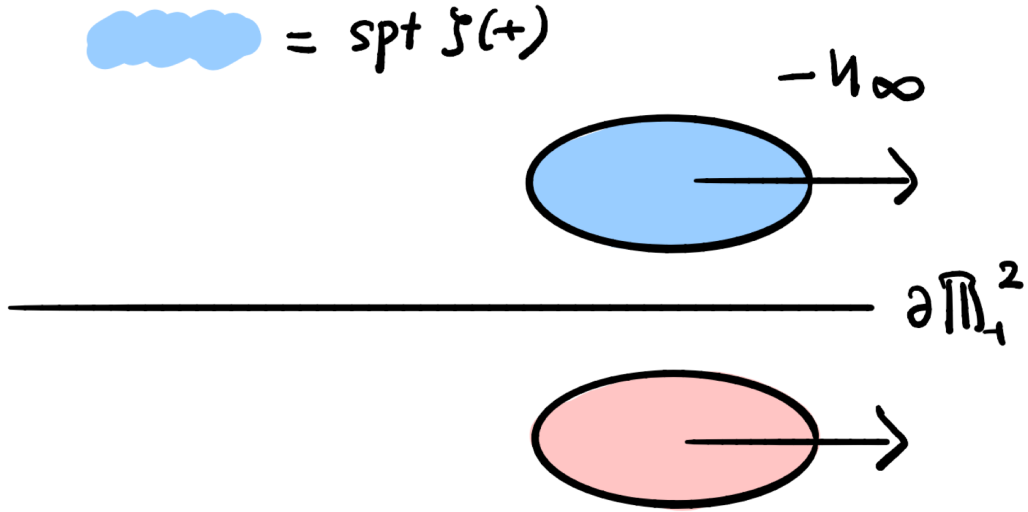
$$\zeta = \zeta_0 \qquad \text{on } \mathbb{R}^2 \times \{t = 0\}$$

$$k(x) = \frac{x^\perp}{2\pi |x|^2}, \quad x^\perp = \ {}^t(-x_2, x_1)$$

Traveling waves

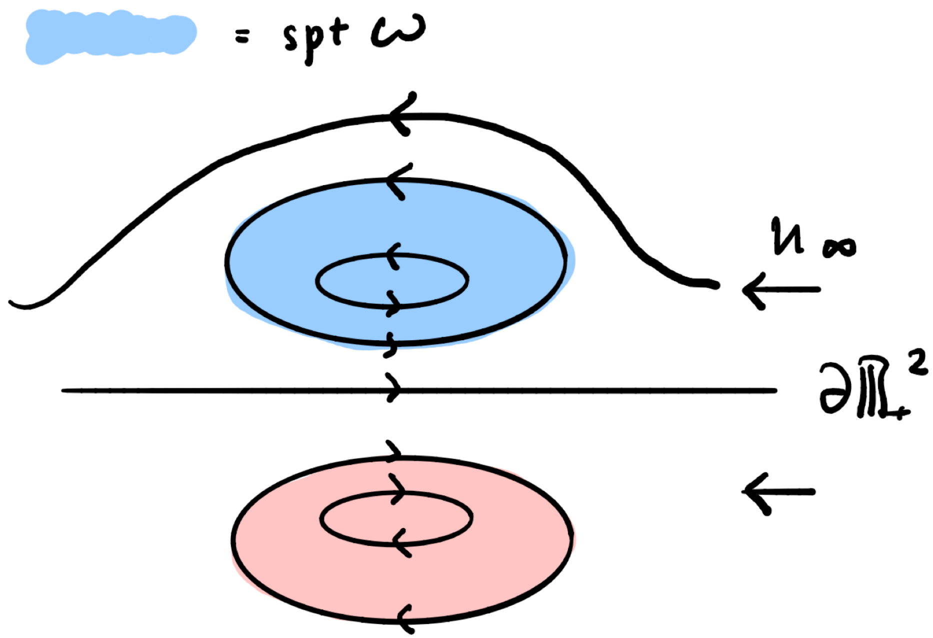
$$v(x,t) = u(x + u_\infty t) - u_\infty$$

$$\zeta(x,t) = \omega(x + u_\infty t)$$



Steady flows

$$\mathbf{u} \cdot \nabla \omega = 0 \qquad \text{in } \mathbb{R}^2_+$$
$$\mathbf{u} \to \mathbf{u}_\infty \quad \text{as } |x| \to \infty$$

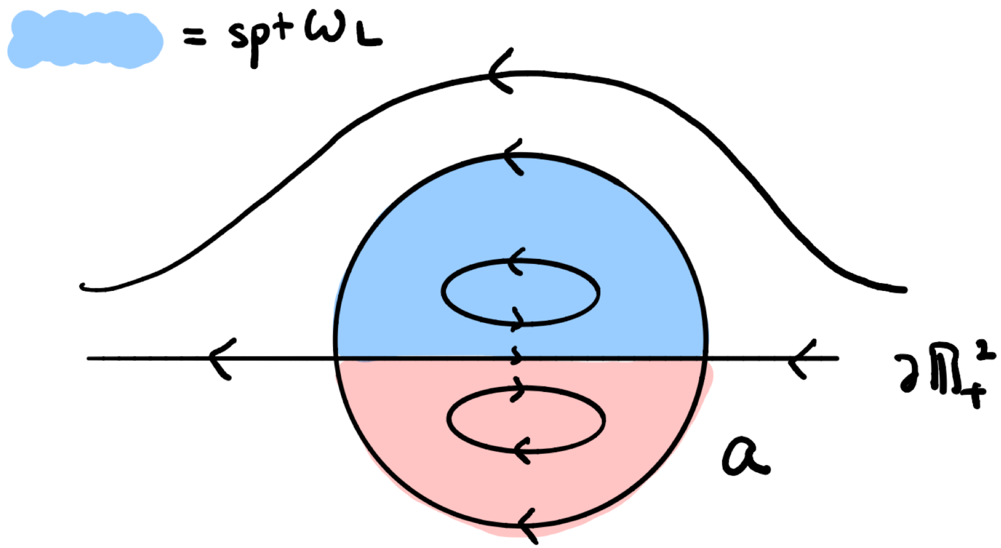


Lamb dipole (Chaplygin-Lamb dipole)

- The only known explicit solution founded by H. Lamb
- First appeared in 1906 at 3rd edition of his "Hydrodynamics"
(2nd edition was published in 1895)
- A non-symmetric dipole was founded by S. A. Chaplygin in 1903
- Rediscovered as "Modon" by Flierl et al. in 1983

$$\omega_L = \lambda \max\{\Psi_L, 0\}, \quad u_L = \nabla^{\perp}\Psi_L$$

$$\Psi_L = \begin{cases} C_L J_1\left(\lambda^{\frac{1}{2}} r\right) \sin\theta, & r \le a \\ -W\left(r - \dfrac{a^2}{r}\right) \sin\theta, & r > a \end{cases}$$



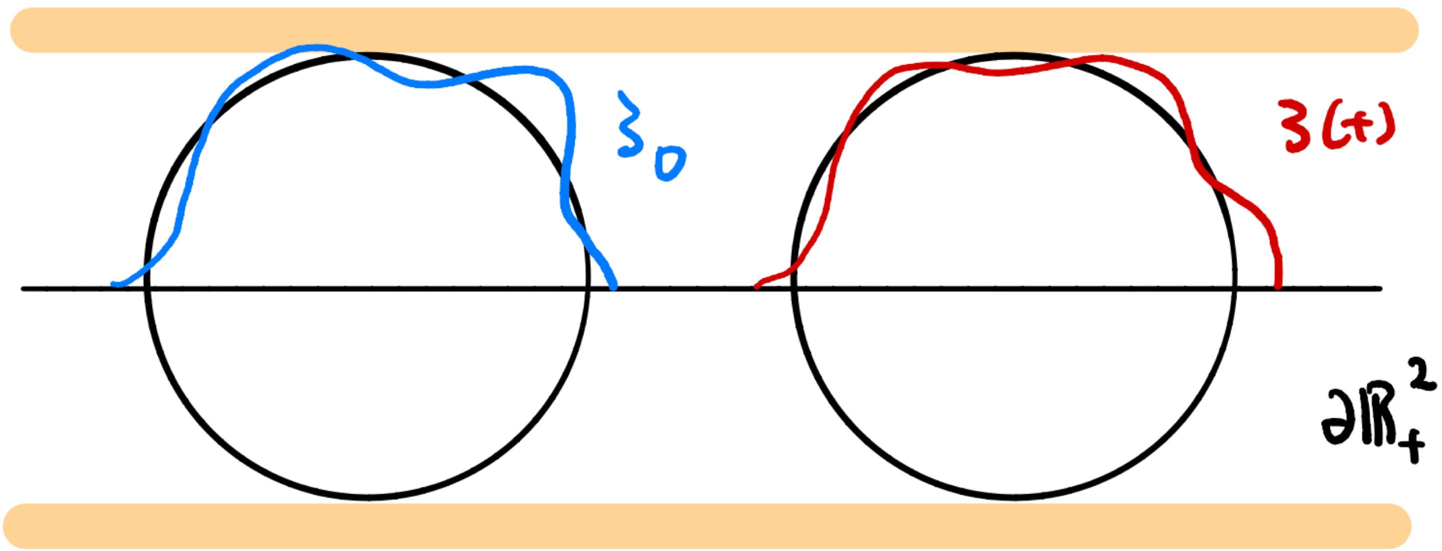
- $W > 0, \lambda > 0$: given
- C_L, a: determined by W, λ
- (r, θ) : polar coordinate
- J_1: 1st order Bessel function of 1st kind

Orbital stability

- Geffena & Heijst, *Fluid Dynamics Research*, 1998
- Experimental & numerical works suggest stability
- Lamb dipole is multi-signed & traveling wave

A distance from orbit

$$d(\zeta_0, \omega_L) = inf_{y \in \partial \mathbb{R}^2_+} \{ ||\zeta_0 - \omega_L(\cdot + y)||_2 + ||x_2(\zeta_0 - \omega_L(\cdot + y))||_1 \}$$



Stability of Lamb dipole

Thm(A & Choi, arXiv)

Let $0 < \lambda, W < \infty$. For $\forall \nu > 0, \forall \varepsilon > 0, \exists \delta > 0\ \ s.t.$
$\forall \zeta_0 \in L^2 \cap L^1(\mathbb{R}^2_+)$ satisfying $x_2 \zeta_0 \in L^1(\mathbb{R}^2_+), \zeta_0 \geq 0, ||\zeta_0||_1 \leq \nu$,

$$d(\zeta_0, \omega_L) \leq \delta \ \Rightarrow\ \exists \zeta(t)\ \ s.t.\ d(\zeta(t), \omega_L) \leq \epsilon,\ \ \forall t > 0$$